Final report on the Simons Institute program
"Algorithms and Complexity in Algebraic Geometry"

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Overall objectives and assessment of the program

The central goal of the Simons Institute program on *Algorithms and Complexity in Algebraic Geometry* was to increase exchange and collaboration between algebraic geometers on the one hand and computer scientists on the other. Two developments over the past few years made such a program timely. First, advances in computer science have spawned the field of *computational algebraic geometry*, which has led to the development and implementation of new, efficient algorithms for algebraic and numerical problems. Second, algebraic geometry has been used to prove complexity lower bounds and shows promise to do much more; indeed, arguably the most viable current approach to tackling the most central problems in complexity theory, such as P versus NP, is the *Geometric Complexity Theory (GCT)* program pioneered by Ketan Mulmuley and collaborators, and has algebraic geometry as its cornerstone. Additional goals included making progress on certain specific problems (some of which are discussed below), and attracting new researchers to the use of algebraic geometry in complexity theory and algorithm design.

One and a half years after its completion, it is already evident that the program was spectacularly successful in attracting new researchers to the area, establishing collaborations between algebraic geometers and theoretical computer scientists, and making significant progress on the major questions in the field.

Major progress and outcomes

Geometric Complexity Theory (GCT). The program proposal included the major goal of clarifying the role of the GCT program with regard to the fundamental lower bound questions in theoretical computer science, at least as far as algebraic models of computation are concerned. This goal was achieved in a very strong sense! Several insights were obtained successively during and after the program that will necessitate a major rethinking of the GCT program; in particular, it is now clear that a considerably finer methodology than that originally proposed will be required for the GCT program to be successful.

Christian Ikenmeyer, a Research Fellow in the program, and Greta Panova, a young combinatorialist, started a collaboration during their stay at the Simons Institute and achieved a major breakthough in [88]: they proved that the vanishing of rectangular Kronecker coefficients *cannot* be used to prove super-polynomial determinantal complexity lower bounds for the permanent. Very recent follow-up work by Bürgisser, Ikenmeyer and Panova [40] goes a step further and shows that the permanent versus determinant problem cannot be resolved using occurrence obstructions. Although this is an impossibility result, it should be seen as a spectacular success of the program because it forces a major revision of the GCT program.

The necessity of a better understanding of Kronecker coefficients was stressed throughout the program. These natural quantities are the tensor product multiplicities of symmetric group representations. The article [87] by Ikenmeyer, Mulmuley and Walter disproves a conjecture in GCT by showing that deciding the positivity of Kronecker coefficients is NP-hard, by building a surprisingly constructive link between algebraic combinatorics and computational complexity. On the other hand, the asymptotic version of the Kronecker positivity problem (and its natural generalization to multiplicities of Lie group representations) was shown to be in NP and coNP and therefore is likely to be polynomial time solvable [37]. This is very surprising in view of the intricate polyhedral structure of the moment cones. The asymptotic positivity problem has close links to quantum information theory, so it is not surprising that the latter results were obtained in collaboration with physicists Matthias Christandl and Michael Walter.

The paper by Landsberg and Ressayre [105] on Valiant's conjecture assuming symmetry is an instance of an outcome of the program in several stages. First, Alpert, Bogart and Velasco answered a question posed by Ressayre at the first program workshop, pointing towards a new direction for investigating Valiant's conjecture. The paper [105] is the first step in that direction, proving Valiant's conjecture in a restricted model.

The method of shifted partial derivatives for proving complexity lower bounds has received great attention over many years. During the program, Klim Efremenko, Joseph Landsberg, Hal Schenk and Jerzy Weyman [58] showed that this method *cannot* be used to prove a significant new separation of the permanent and determinant, but at the same time identified generalizations of the method that hold promise for proving new lower complexity bounds.

Here is another fine example of a fruitful interaction between areas of pure math (invariant theory), algorithms and complexity theory, and quantum computation, that was triggered by the program. In his talk at the first workshop, Avi Wigderson posed the problem of computing the rank of a matrix of linear forms over the free skew field (the non-commutative rank problem). This problem turns out to be intimately connected to a problem in invariant theory, namely bounding the generating degree of the invariant ring of tuples of matrices with simultaneous left/right SL-action. Following this, Gábor Ivanyos, Youming Qiao, and K.V. Subrahmanyam [89,90] studied the invariant-theoretic problems for this invariant ring and proved exponential degree bounds. After the preprints [89,90] had appeared, Wigderson and his collaborators, Ankit Garg, Leonid Gurvits and Rafael Oliveira, showed that an existing algorithm of Gurvits can be used to give a polynomial-time algorithm for the non-commutative rank problem. This led to progress in derandomization, a core problem of theoretical computer science, namely to a polynomial time algorithm for non-commutative rational identity of formulas (allowing also divisions). Finally, Harm Derksen, an invariant theorist, with his student Visu Makam, discovered that a clever use of one lemma in [90] even yields a polynomial bound for the invariant ring under question, thus greatly clarifying the picture.

Polynomial Equation Solving. Smale's 17th problem, which is one of the leading problems in the field from the complexity point of view, was completely solved in 2015 by postdoc Pierre Lairez using a clever derandomization idea [100]. While Lairez was not a participant of the program, he was directly influenced by it and was invited to present his breakthrough result at the program's reunion meeting.

It may sound surprising that it was unknown whether there is a numerically stable algorithm for computing eigenvalue-eigenvector pairs for complex matrices that is provably polynomial time. In [6], Diego Armentano, Carlos Beltrán, Peter Bürgisser, Felipe Cucker and Michael Shub resolved this long-standing open problem in numerical linear algebra that was originally posed by James Demmel.

Further seminal progress in the complexity of solving systems of polynomial equations is the article [51] by Cucker, Krick and Shub, which describes and analyzes a numerically stable algorithm for computing the homology of real projective varieties. This work deepens our understanding of the complexity framework when solving systems of polynomial equations of positive dimension.

On the more practical side, Jon Hauenstein and his collaborators made progress in numerical algebraic geometry in various directions. In a project with Oeding, Ottaviani and Sommese [78], methods were developed to compute low-rank decompositions of tensors using the software Bertini, and this was used to derive new cases of generic identifiability. Recent work with Brake and

Vinzant [28] offers an exciting connection to tropical geometry, demonstrating how the numerical monodromy approach can be used to compute the tropicalization of real and complex curves.

Exponential families are fundamental to statistics and machine learning. In the work [119] by Mateusz Michalek, Bernd Sturmfels, Caroline Uhler and Piotr Zwiernik, this theory is embedded into the context of algebraic geometry. The theory of exponential varieties, developed here for the first time, can be seen as a conceptual generalization of toric geometry, which arises from the very special case of discrete exponential families.

In [134], Research Fellow Cynthia Vinzant resolved a widely circulated conjecture in frame theory known as the 4M-4 conjecture, due to A. Bandeira, J. Cahill, D. Mixon and A. Nelson. The resolution uses methods from algebraic geometry, as extensively discussed at the program.

Tensors and Multilinear Algebra. The program advanced the study of tensors and their decompositions from both a practical and theoretical perspective.

Tensor decomposition consists of writing a tensor as a sum of simpler (indecomposable) ones. In many cases of interest this decomposition is unique and gives a canonical form for the tensor, which is called identifiable. Since tensors are used in mathematical modeling in many areas, from signal processing and topic search on the web to other engineering applications, this has become a hot topic. During the program, researchers in algebraic geometry, numerical analysis and complexity theory began to collaborate together on tensors and their applications. The work [78] is a fine example of this interaction, and originated from discussions between Giorgio Ottaviani, Luke Oeding and Jon Hauenstein, all long-term participants of the program, around the number of decompositions of a generic tensor on complex numbers. Andrew Sommese and Jon Hauenstein, founders of the software Bertini, realized that with homotopic techniques it is possible to predict with high probability the number of decompositions, starting from a generic one and repeating different loops until the number of new solutions stabilizes. As a result, two new cases of identifiability were discovered numerically. Following this discovery, vector bundle techniques from algebraic geometry actually yielded a proof that in these cases we actually have identifiability.

In a different direction, Luke Oeding's work with Robeva and Sturmfels [123] forged a brand new connection between tensors in algebraic geometry and finite frame theory in functional analysis.

The topic of eigenvalues and singular values of tensors received considerable attention during the program; for some outcomes see [2,29,30], which deal with structural, probabilistic, and algorithmic aspects. This is related to the topic of orthogonally decomposable tensors, which is currently receiving much attention in the scientific computing and theoretical CS literature. Boralevi, Draisma, Horobet and Robeva [24] provided for the first time an intrinsic characterization of those tensors.

Further Highlights. Research Fellow Ben Rossman, in a collaboration with Li-Yang Tan (a Fellow in the companion program on Algorithmic Spectral Graph Theory) and Rocco Servedio of Columbia University, solved a 30-year old open problem in complexity theory by proving that the polynomial hierachy is infinite relative to a random oracle [132]. This paper received the Best Paper Award in the *56th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, 2015.

Digest of selected feedback from participants

As hoped for, the program managed to attract extremely talented young algebraic geometers, who were previously not very familiar with complexity theory, to work on questions in complexity. An illustrative example is the wide range of collaborations established by Research Fellow Mateusz Michalek, who worked with no fewer than eleven co-authors on seven published research projects (with more still in the pipeline). In their feedback, several of the algebraic geometers at the program explicitly mentioned how much they valued the opportunity to exchange the ideas between participants with different ranges of expertise, and in particular with researchers from algorithms and complexity theory.

The program was enormously useful for getting graduate students and postdoctoral Fellows involved in state-of-the-art research, as it allowed for direct interaction with many of the world's experts. It is clear that the program has helped to shape the scientific future of several young and promising researchers. In her feedback, senior algebraic geometer Teresa Krick wrote that she especially appreciated meeting a lot of new people, mostly younger researchers who are just starting their career, many of whom were women.

Finally we note that many intriguing connections emerged during the semester between this program and its sister program Algorithmic Spectral Graph Theory (fueled in part by shared senior participants such as Pablo Parillo from MIT, and collaborations between Fellows from the two programs such as that between Rossman and Tan mentioned earlier). Indeed, the connections were strong enough that the reunion workshops of the two programs were scheduled with some overlap, allowing participants to spend time at both. One interesting example of such a connection was provided by Lek-Heng Lim in the following remark in his research report: "The mysterious Grothendieck constant, which plays an important role in the Unique Games Conjecture and Semidefinite Programming approximations of NP-hard problems, and the exponent of matrix multiplication, which plays an important role in Algebraic Computational Complexity Theory, are intimately related. The former is the spectral norm of the structure tensor of matrix-matrix product whereas the latter is its rank." This and other connections will continue to be explored in the future.

Program activities

Workshops and Boot Camp. The program began with the *Algebraic Geometry Boot Camp*, which featured introductory lectures around the major themes of the semester. Particularly exciting was a detailed exposition of the Coppersmith-Wingrad method for bounding the exponent of matrix multiplication, along with very recent improvements, by V. Vassilevska Williams. Another highlight were the homework sessions, where participants were given the opportunity to actually apply the theoretical material presented in the lectures to concrete computational problems.

The workshop on *Geometric Complexity Theory* faced daunting challenges: both the algebraic geometry and theoretical computer science required to approach the core questions of GCT require considerable background, and few participants had both. To meet this challenge, the workshop had an extensive focus on tutorials given by experts in the respective areas. This overall structure turned out to be quite successful; for example, several prominent invariant theorists attending the Boot Camp commented that they were surprised by the deep connections between their field and complexity theory that have been revealed by the GCT program.

The workshop on *Solving Systems of Polynomial Equations* focused on recent algorithmic advances, both numerical and symbolic, on novel domains of application, and on fundamental issues of complexity in algebraic geometry. The range of topics was broad and included the role of condition in numerical equation solving (Smale's 17th problem), understanding and solving systems of sparse polynomials (fewnomials) and its recently discovered fascinating link to fundamental complexity lower bound questions (the real tau-conjecture), convex algebraic geometry (fusing convex optimization theory and real algebraic geometry), and current software tools such as Bertini, Singular and Macaulay2. It is remarkable that many of the senior participants of this workshop had first met at a program at MSRI in Berkeley during the Fall of 1998, but had not gathered as a group since then. The workshop helped to present, discuss and celebrate the substantial advances in the field since that time, as well as to welcome a spectacular group of younger researchers to the field and set the agenda for future developments.

The workshop on *Tensors and Multilinear Algebra* was centered around tensor rank and its associated decomposition. The workshop was received with enthusiasm and saw a constant stream of participants from the Algorithmic Spectral Graph Theory program as well as faculty, postdocs and graduate students from the Math and EECS Departments attending the talks. This is not surprising given the growing interest in tensors across a range of disciplines; indeed, it is notable that not only did tensors play a prominent role in all four workshops in the Algebraic Geometry program, but also featured in two of the three workshops in the parallel program on Algorithmic Spectral Graph Theory.

The fourth and final workshop on *Symbolic and Numerical Methods for Tensors and Representation Theory* was a tutorial workshop, sponsored jointly with MSRI, aimed at enabling junior researchers (especially graduate students) to gain familiarity with the computer algebra software Macaulay2, to learn some of the main research questions and themes of the semester, and to experience first-hand how computational techniques contribute to this research.

Regular Seminars and Other Activities. The program also featured a diverse array of regular weekly seminars hosted by long-term participants and attended by program visitors as well as campus faculty and students.

Research fellows Michael Forbes and Klim Efremenko co-organized a lively seminar on *Algebraic Complexity*, which was important for introducing algebraic geometers to basic problems and concepts from theoretical computer science (black box and white box derandomization, hitting sets, polynomial identity testing, shallow circuits, etc.).

Jonathan Hauenstein (University of Notre Dame) and Gregorio Malajovich (Federal University of Rio de Janeiro) organized a seminar on *Polynomial Equation Solving*, which focused on various aspects of solving polynomial equations, both from a theoretical complexity point-of-view as well as in the context of practical applications.

Bernd Sturmfels (UC Berkeley) ran a weekly seminar on *Computational Algebraic Geometry* that connected PhD students from UC Berkeley with the visitors of the Simons Institute program. This inspiring seminar featured expositions by senior Institute visitors, as well as research talks by postdocs and gradudate students both from Berkeley and elsewhere.

Finally, Joseph Landsberg (Texas A&M University) taught a lively and very well attended graduate course on *Geometry and Complexity Theory*, covering in depth the two most central topics in the program: the complexity of matrix multiplication, and permanent vs determinant problems. The notes from Landsberg's class are available on the Simons Institute webpage and will be be published soon as a graduate text by Cambridge University Press [103].

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