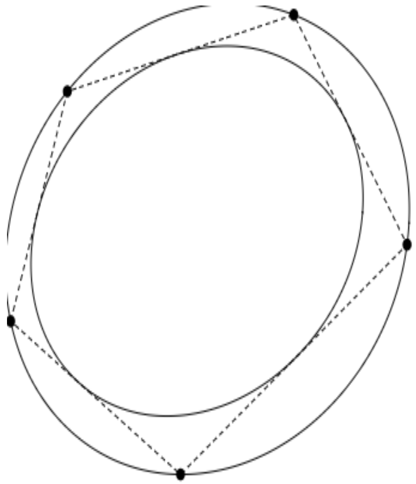
Provable Real-Time Learning with applications to Robotics



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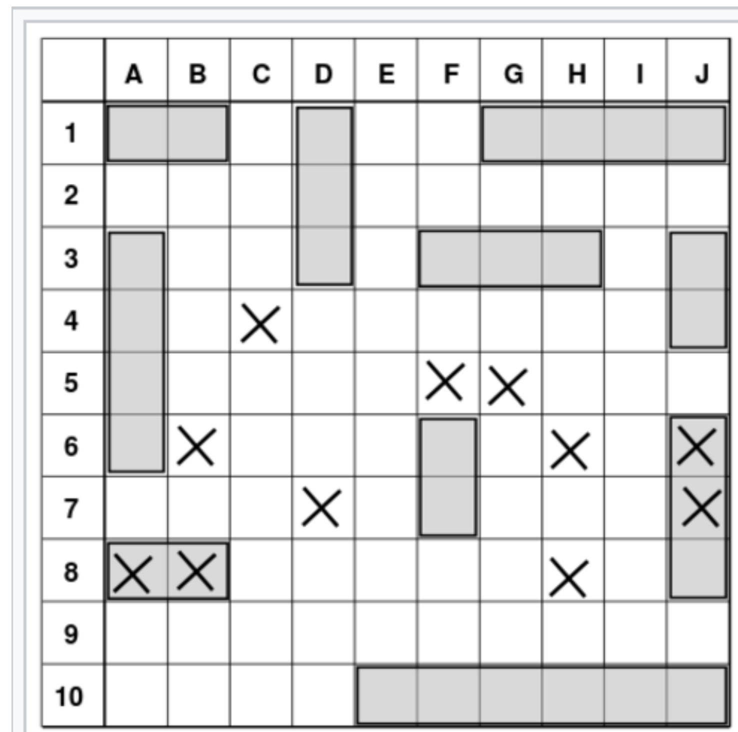
How to find a battleship

- A "sea" of *M* squares which contains
  (at some unknown location) a “battleship” of *K* squares.
- Both the sea and the battleship are rectangular shape.
- Find the battleship by probing at least one of its squares.



Path Planning in the Dark

- In control space we know start & destination configurations
- Can only ask Boolean queries regarding feasible positions
- As in Battleships (game), Piano Mover,
- or Drones in a crowded supermarket

obstacles.mp4

RamiLevy.mp4

NanoDrone (NanoDrone.mp4)

Big Data

- Volume: huge amount n of data points
- Variety: high dimensional d space
- Velocity: data arrive in real-time

Need to support:

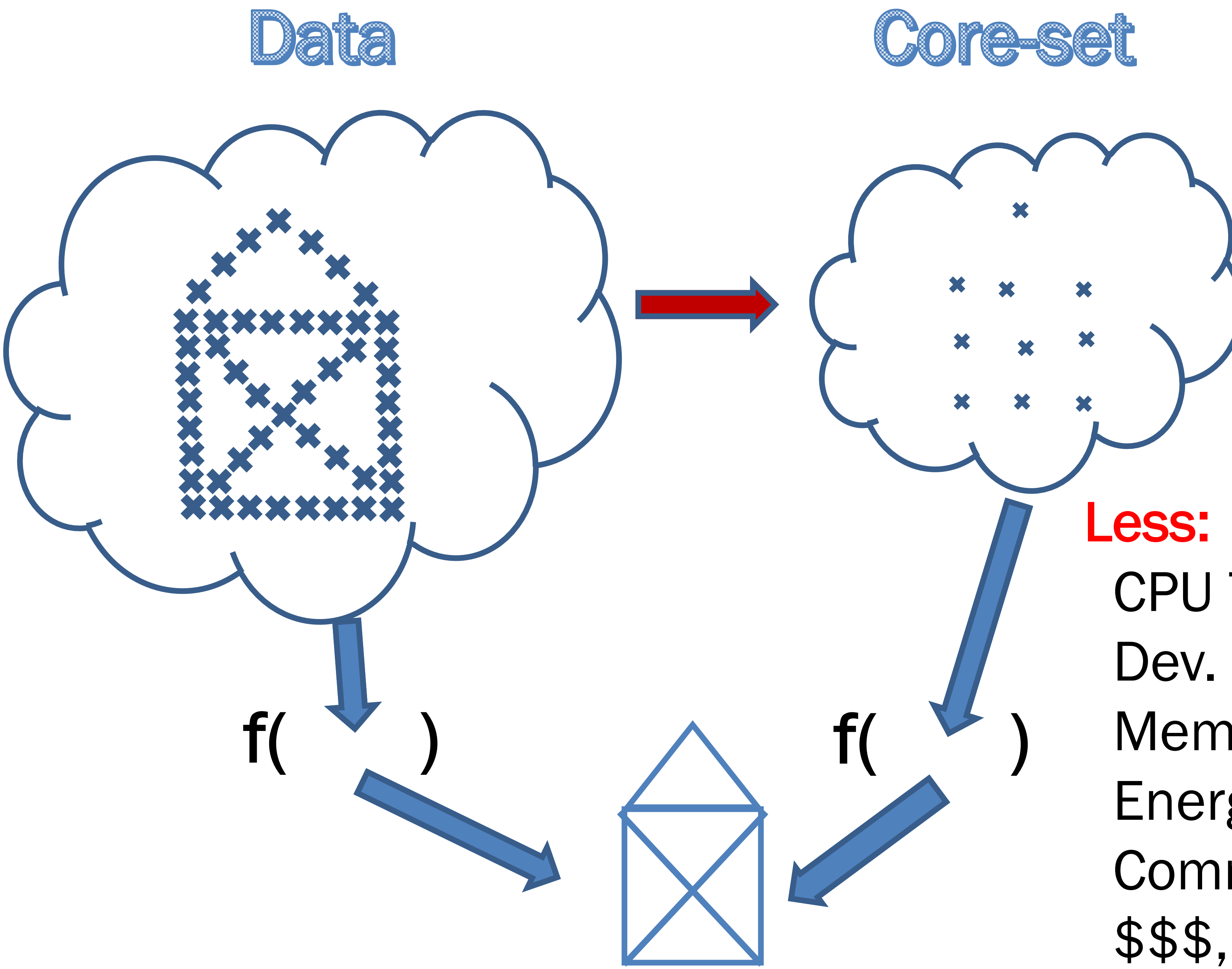
- Streaming (one pass, logarithmic memory)
- Distributed data (on cloud)
- Simple computations (embarrassingly parallel)
- No assumption on order of points

Big Data Computation model

- = Streaming + Parallel computation
- Input: infinite stream of vectors
- $n =$ vectors seen so far
- $\sim\log n$ memory
- M processors
- *~log (n)/M* insertion time per point

(Embarrassingly parallel)

Focus on ~~optimization~~ summarization



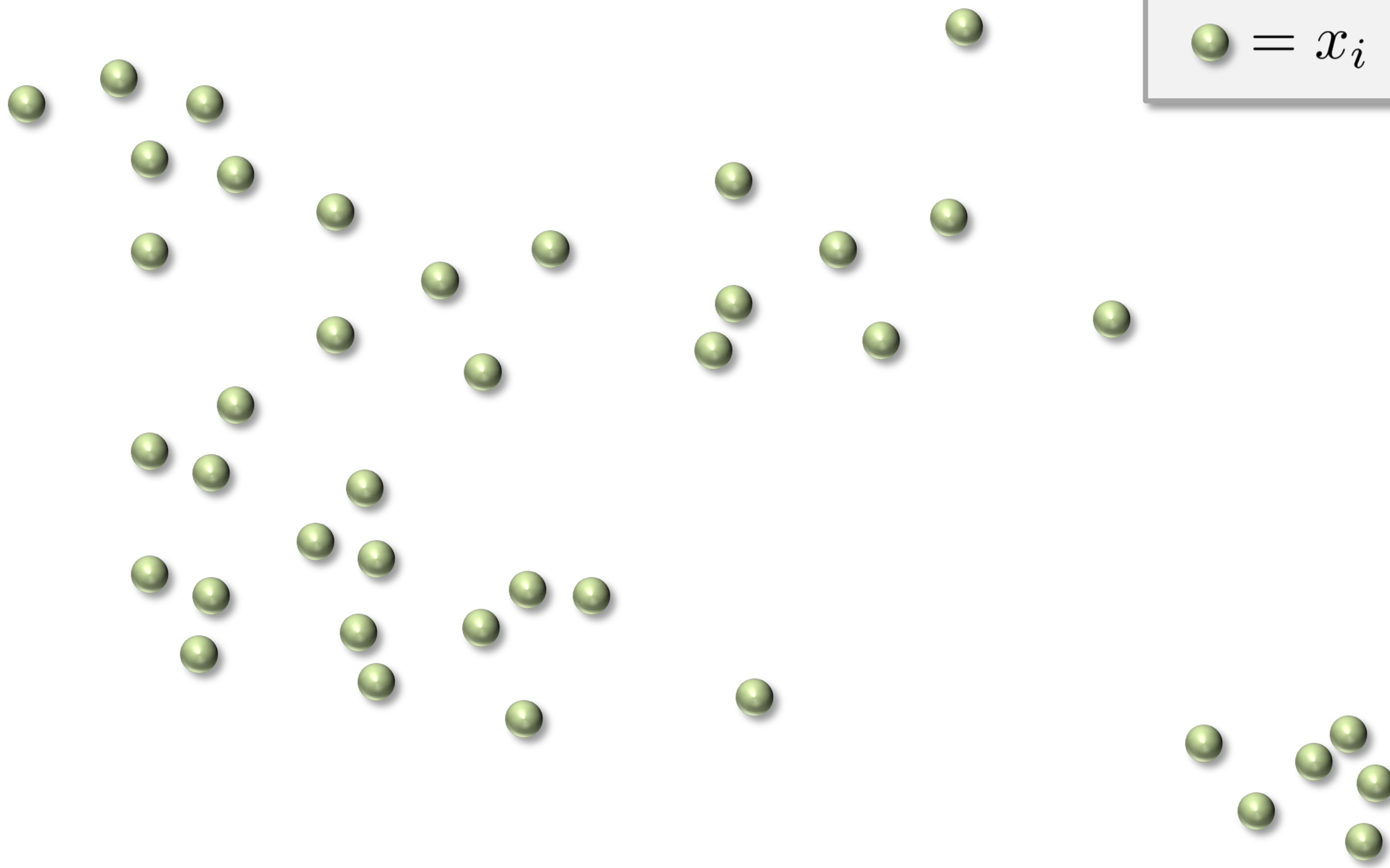
Less:

CPU Time
Dev. Time
Memory
Energy
Comm.
$$$, ...

Example Coresets

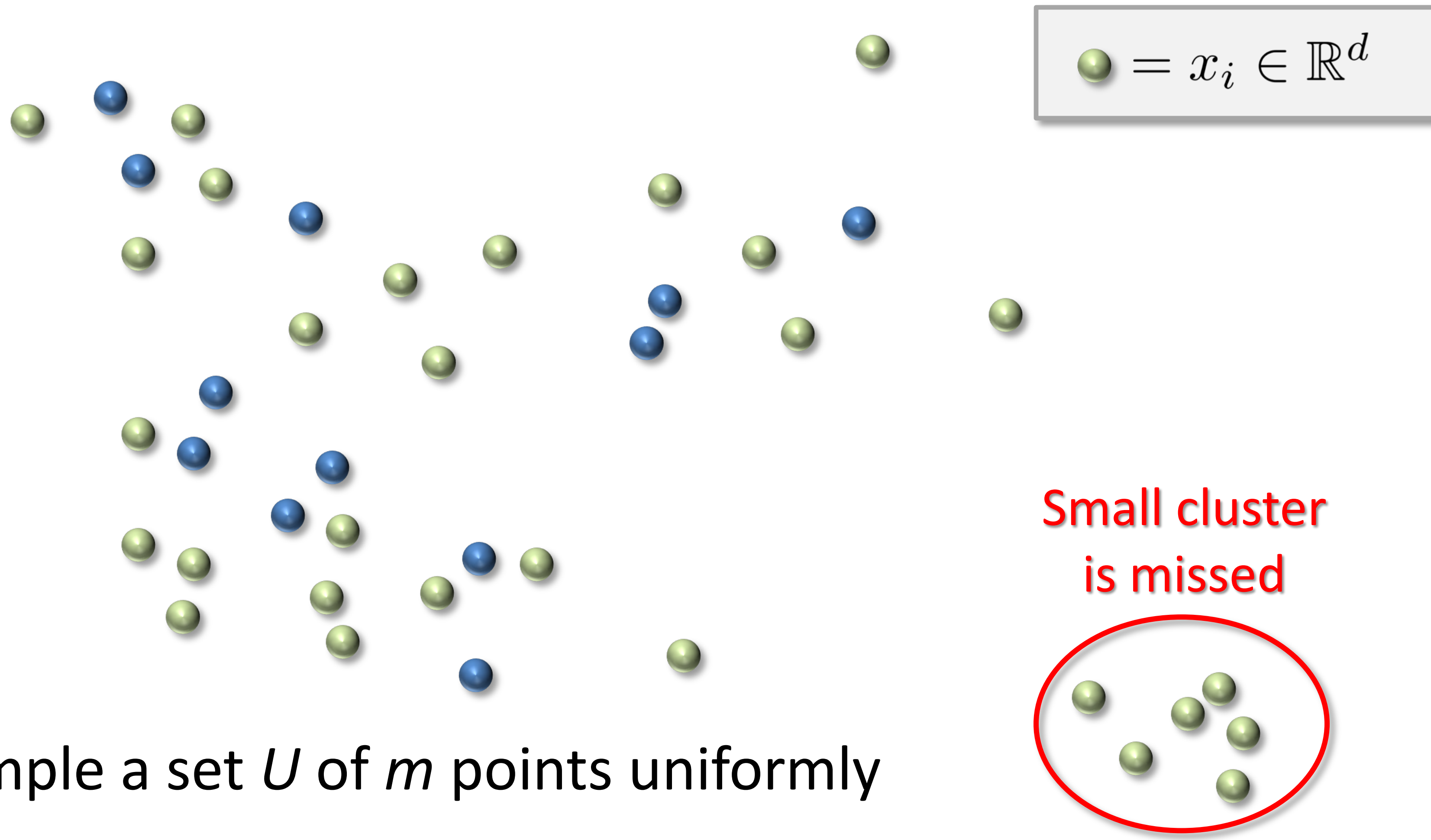
- Deep Learning [F, Tukan, Kener, To appear]
- Graph Summarization [F, Sedat, Rus, ICML'17]
- Mixture of Gaussians [F, Krause, etc JMLR'17]
- LSA/PCA/SVD [F, Rus, and Volkob, NIPS'16]
- k-Means [F, Barger, SDM'16]
- Non-Negative Matrix Factorization [F, Tassa, KDD15]
- Robots Localization [F, Cindy, Rus, ICRA'15]
- Robots Coverage [F, Gil, Rus, ICRA'13]
- Segmentation [F, Rosman, Rus, Volkob, NIPS'14]
- ….
- k-Line Means [F, Fiat, Sharir, FOCS'06]

Naïve Uniform Sampling



$= x_i \in \mathbb{R}^d$

Naïve Uniform Sampling



Sample a set *U* of *m* points uniformly

← High variance

Simplest coreset definition

Let

- P be a set, called *point set*
- X be a set, called *query set*
- $\mathrm{cost}(P, x)$: maps every query $x \in X$ into a non-negative number

For a given $\epsilon > 0$, the set $C \subseteq P$ is a *core-set* if for every $x \in X$ we have

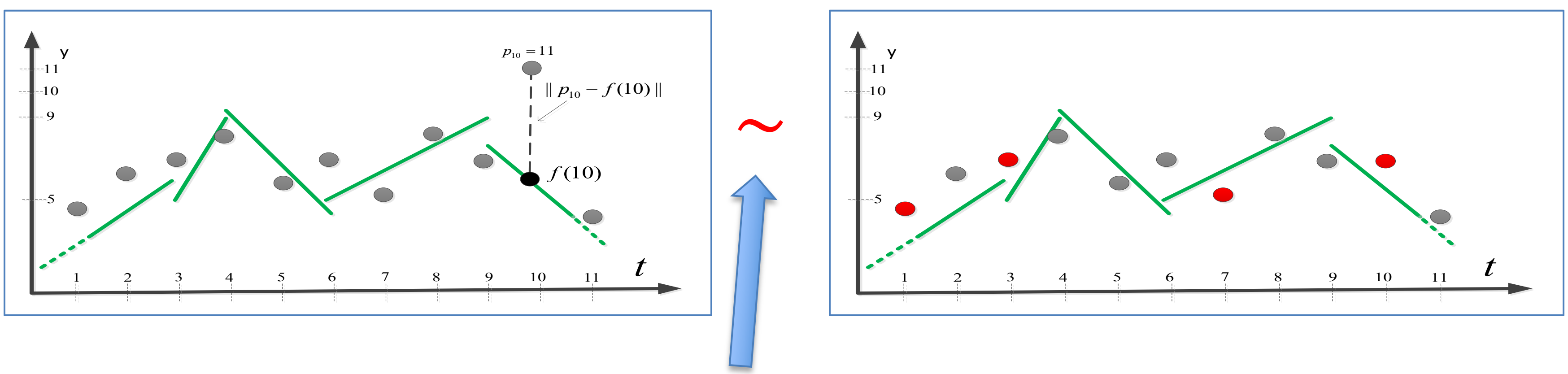
$$\mathrm{cost}(P, x) \sim cost(C, x)$$

up to $(1 \pm \epsilon)$ approximation factor

From Big Data to Small Data

Suppose that we can compute such a corset C of size $\frac{1}{\epsilon}$ for every set P of n points

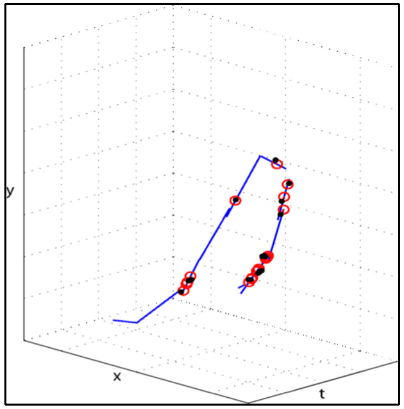
- in time n^3,
- off-line, non-parallel, non-streaming algorithm



$(1 \pm \epsilon)$

Read the first $\frac{2}{\epsilon}$ streaming points and reduce them into $\frac{1}{\epsilon}$ weighted points in time $\left(\frac{2}{\epsilon}\right)^5$

$1+\epsilon$ corset for P_1



Read the next $\frac{2}{\epsilon}$ streaming point and reduce them into $\frac{1}{\epsilon}$ weighted points in time $\left(\frac{2}{\epsilon}\right)^5$

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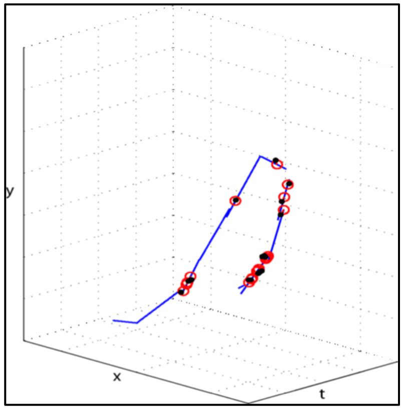
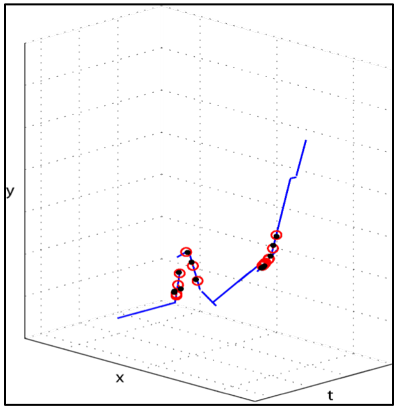
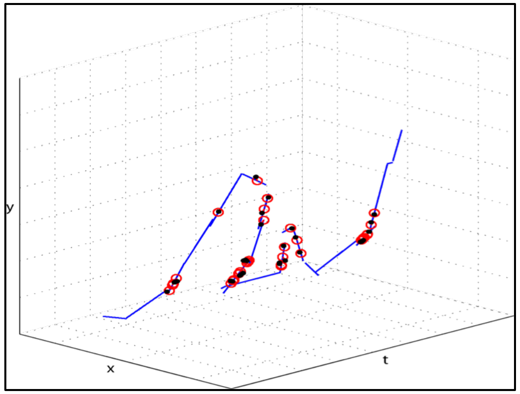


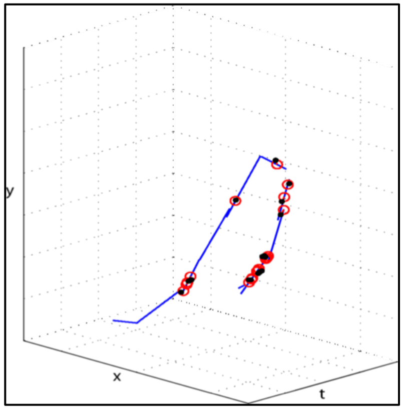
$1+\epsilon$ corset for P_2

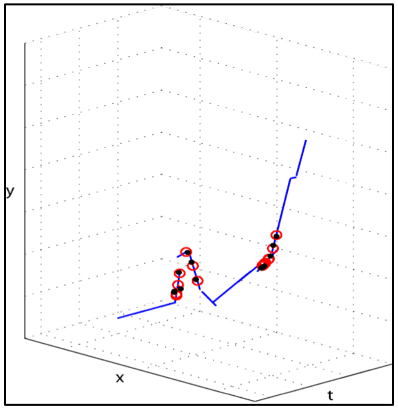


Merge the pair of ϵ-coresets into an ϵ-corset of $\frac{2}{\epsilon}$ weighted points

$1+\epsilon$-corset for $P_1 \cup P_2$

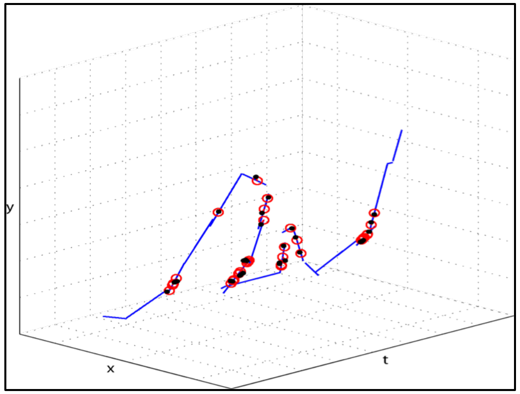


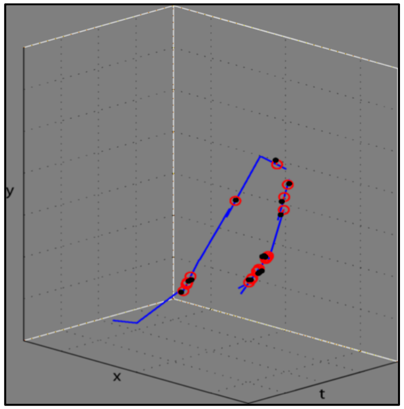


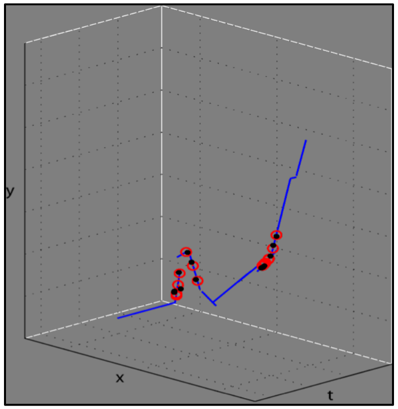


Delete the pair of original coresets from memory

$1+\epsilon$-corset for $P_1 \cup P_2$



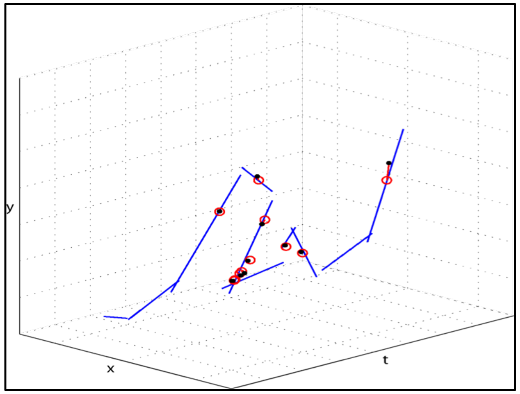


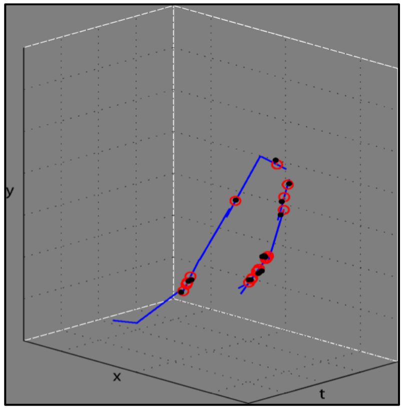


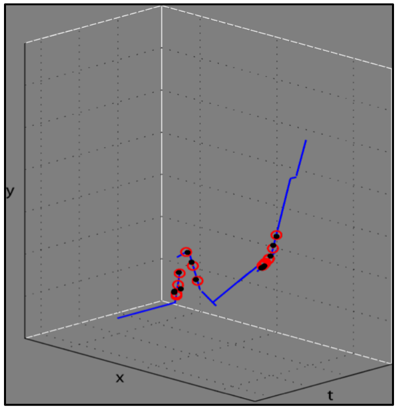
Reduce the $\frac{2}{\epsilon}$ weighted points into $\frac{1}{\epsilon}$ weighted points by constructing their coreset

$1+\epsilon$-corset for

$1+\epsilon$-corset for $P_1 \cup P_2$



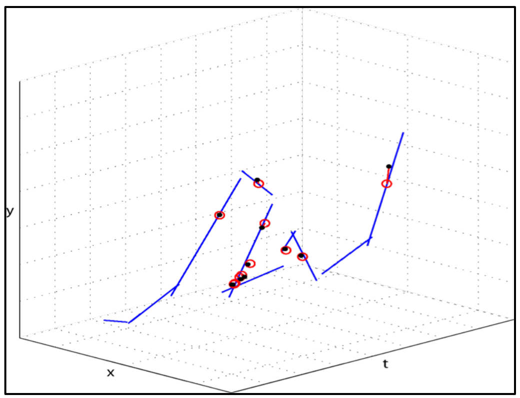




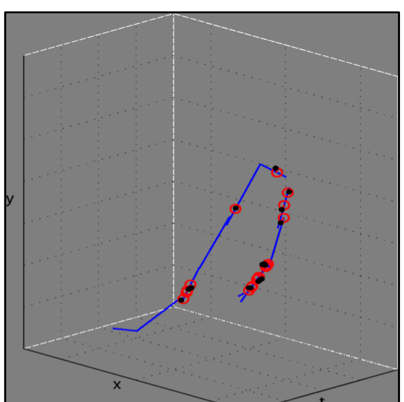
Reduce the $\frac{2}{\epsilon}$ weighted points into $\frac{1}{\epsilon}$ weighted points by constructing their coreset

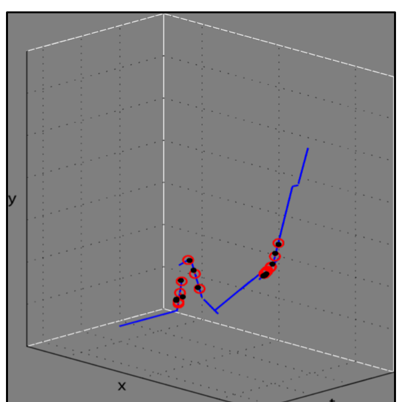
$1 + \epsilon$-corset for

$1 + \epsilon$-corset for $P_1 \cup P_2$

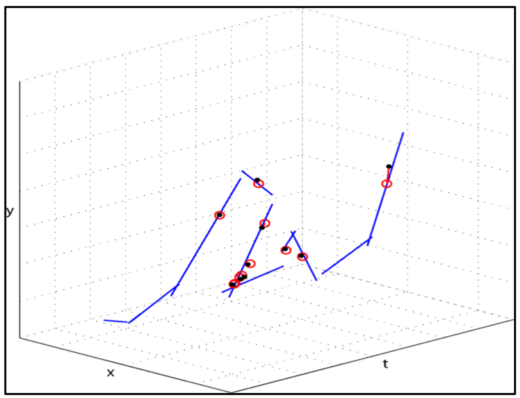


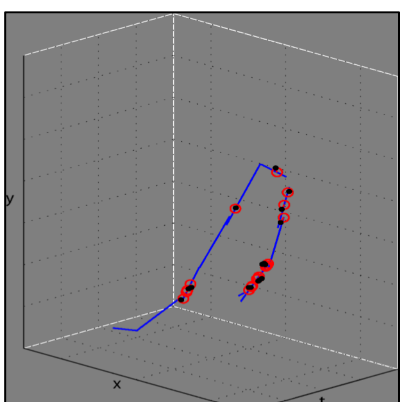
$=(1 + \epsilon)^2$-corset for $P_1 \cup P_2$

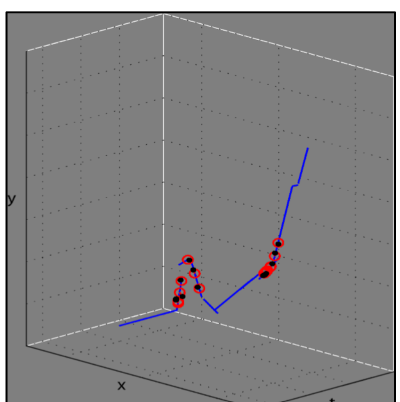




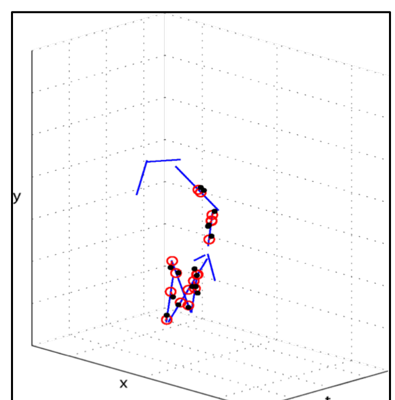
$(1+\epsilon)^2$-corset for $P_1 \cup P_2$



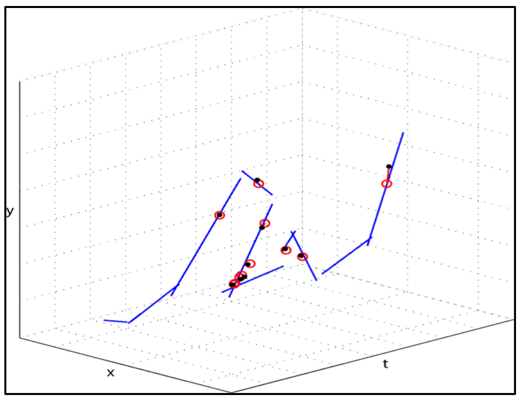


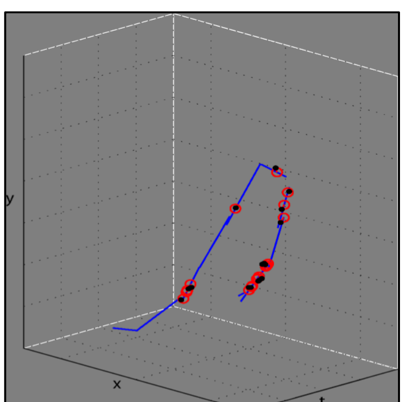


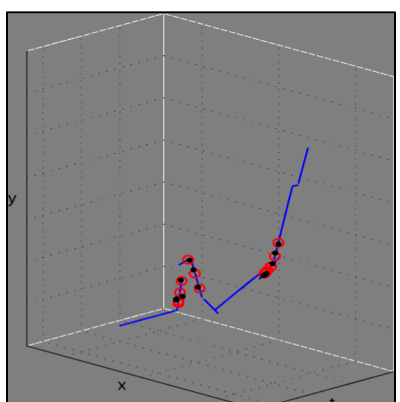
$(1+\epsilon)$-corset for P_3



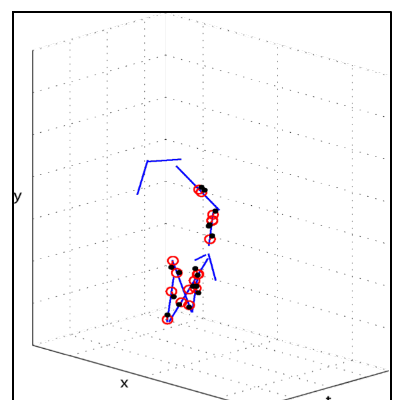
$(1+\epsilon)^2$-corset for $P_1 \cup P_2$



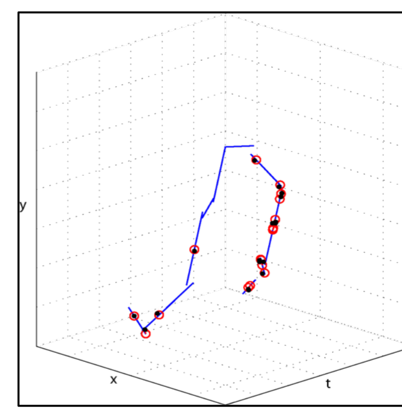




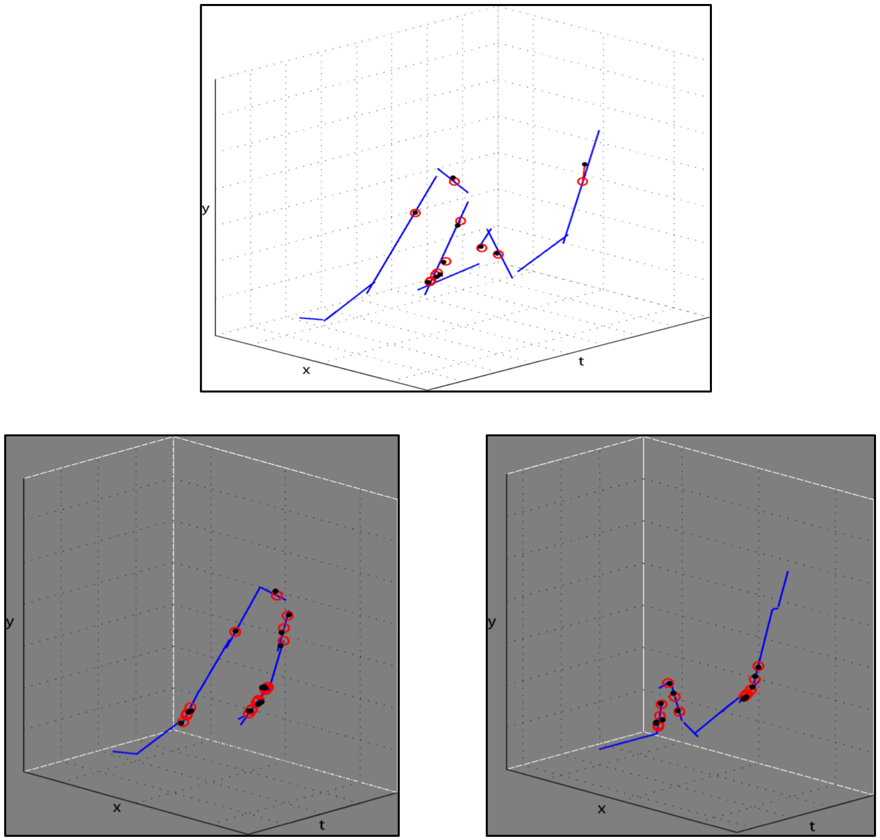
$(1+\epsilon)$-corset for P_3



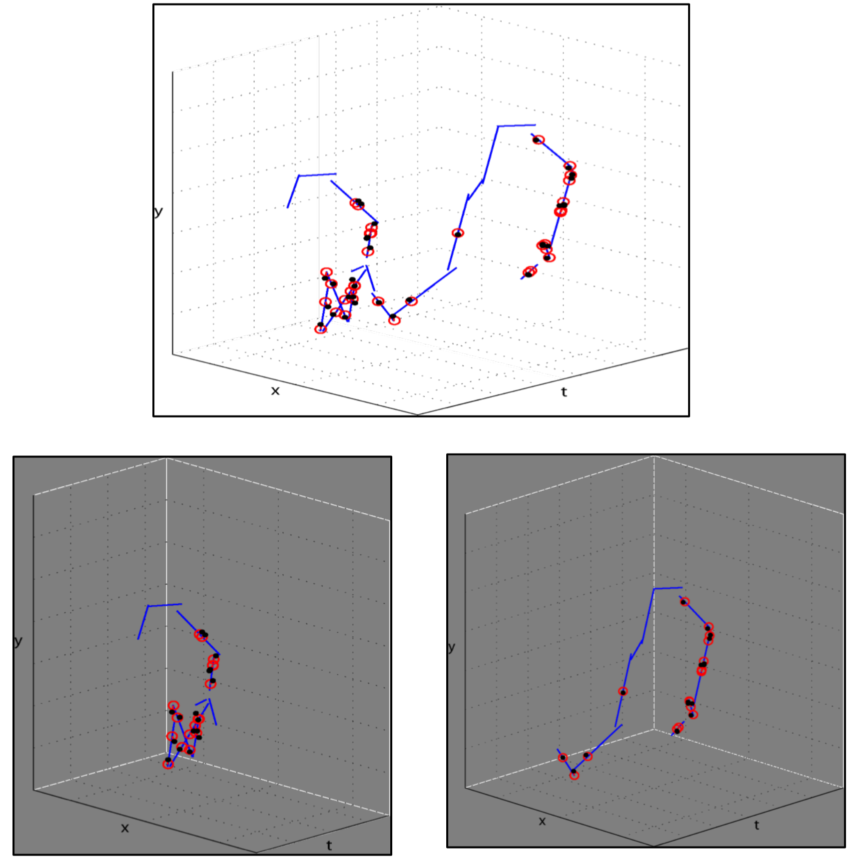
$(1+\epsilon)$-corset for P_4



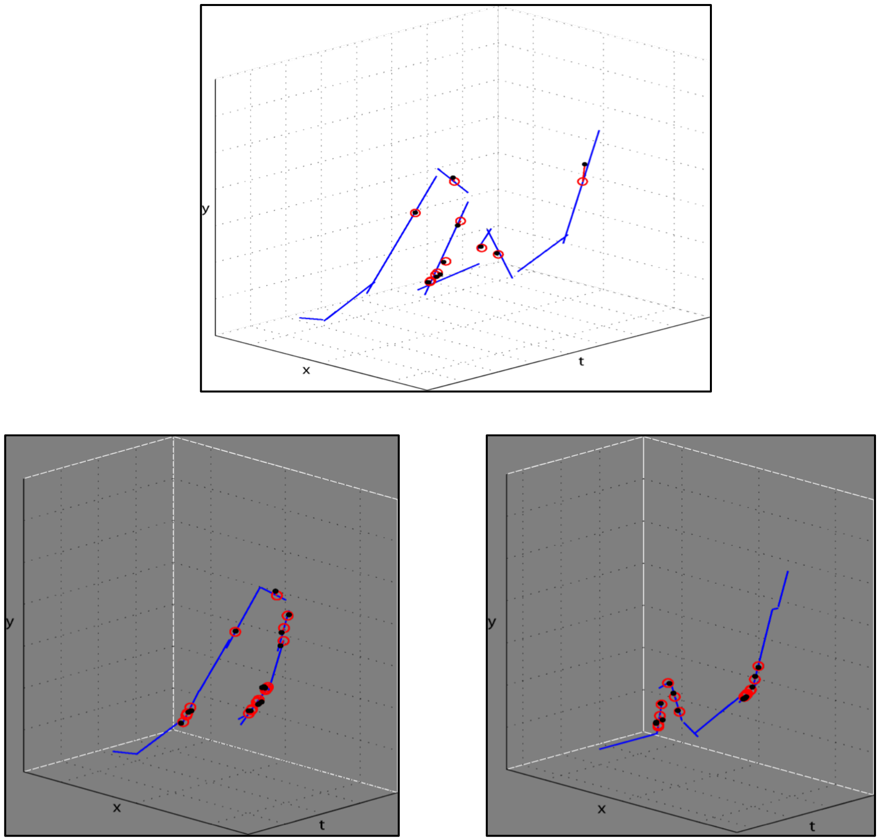
$(1+\epsilon)^2$-corset for $P_1 \cup P_2$



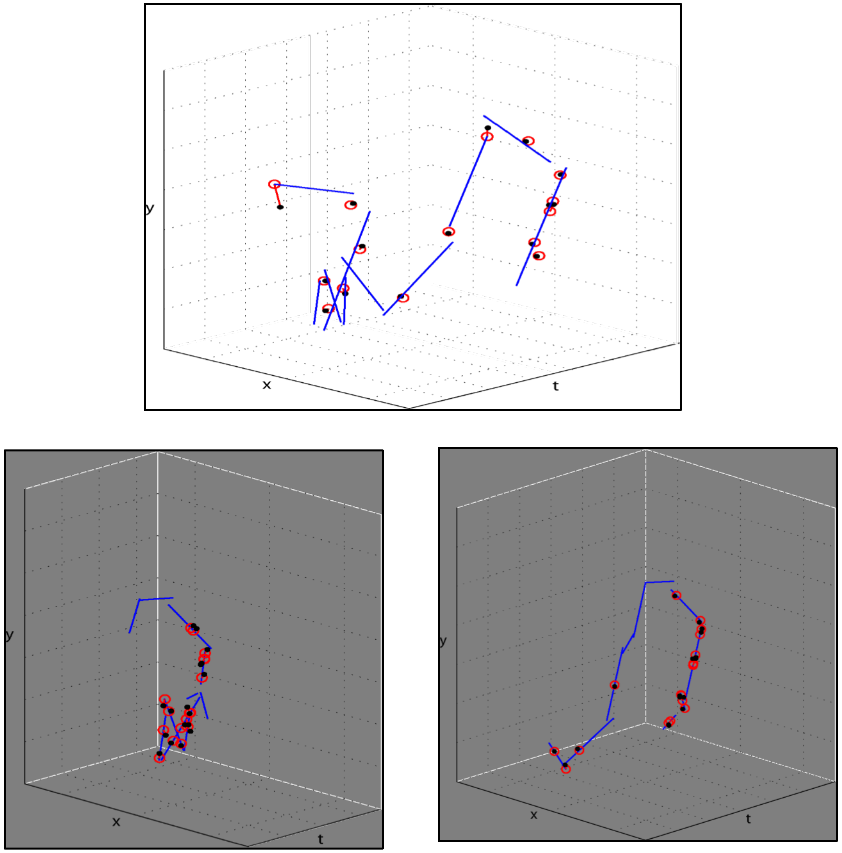
$(1+\epsilon)$-corset for $P_3 \cup P_4$

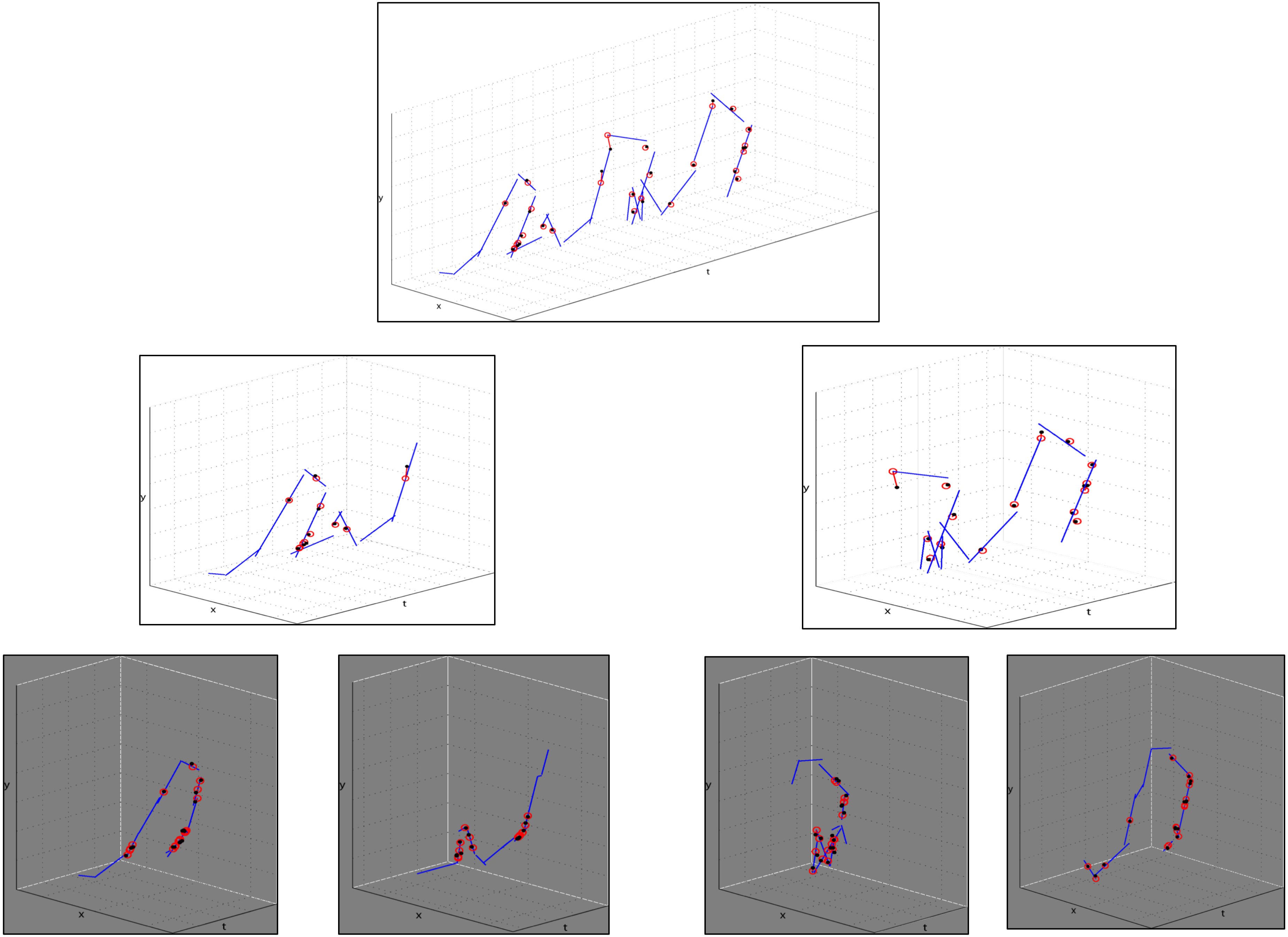


$(1+\epsilon)^2$-corset for $P_1 \cup P_2$



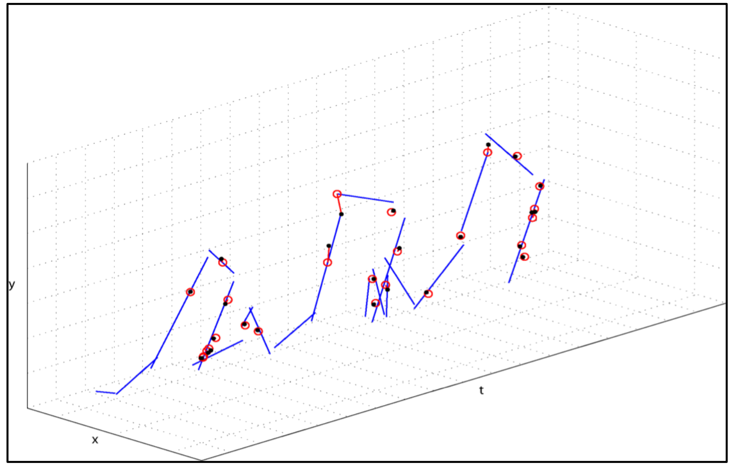
$(1+\epsilon)^2$-corset for $P_3 \cup P_4$

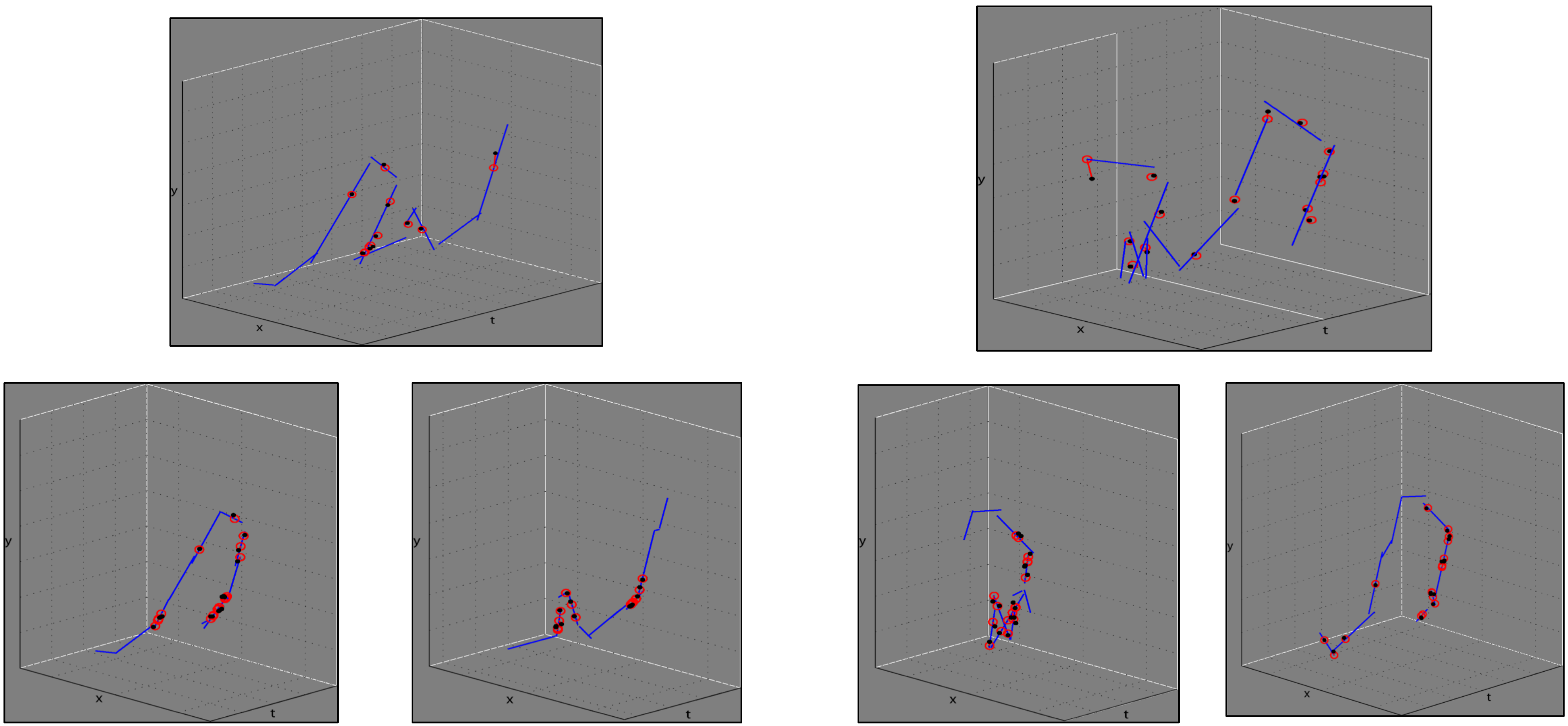



y
x
t

$(1+\epsilon)^2$-coreset for

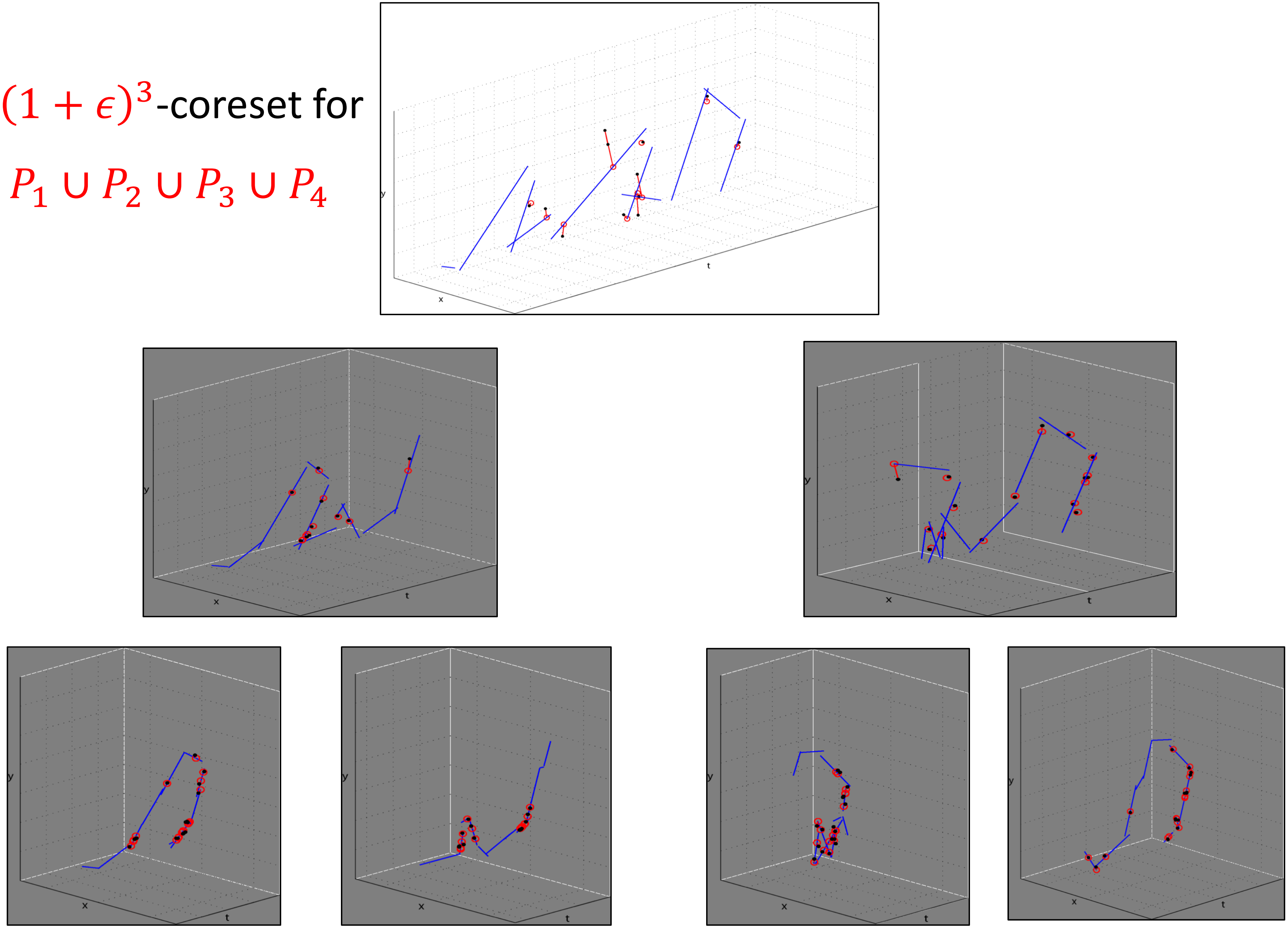
$P_1 \cup P_2 \cup P_3 \cup P_4$

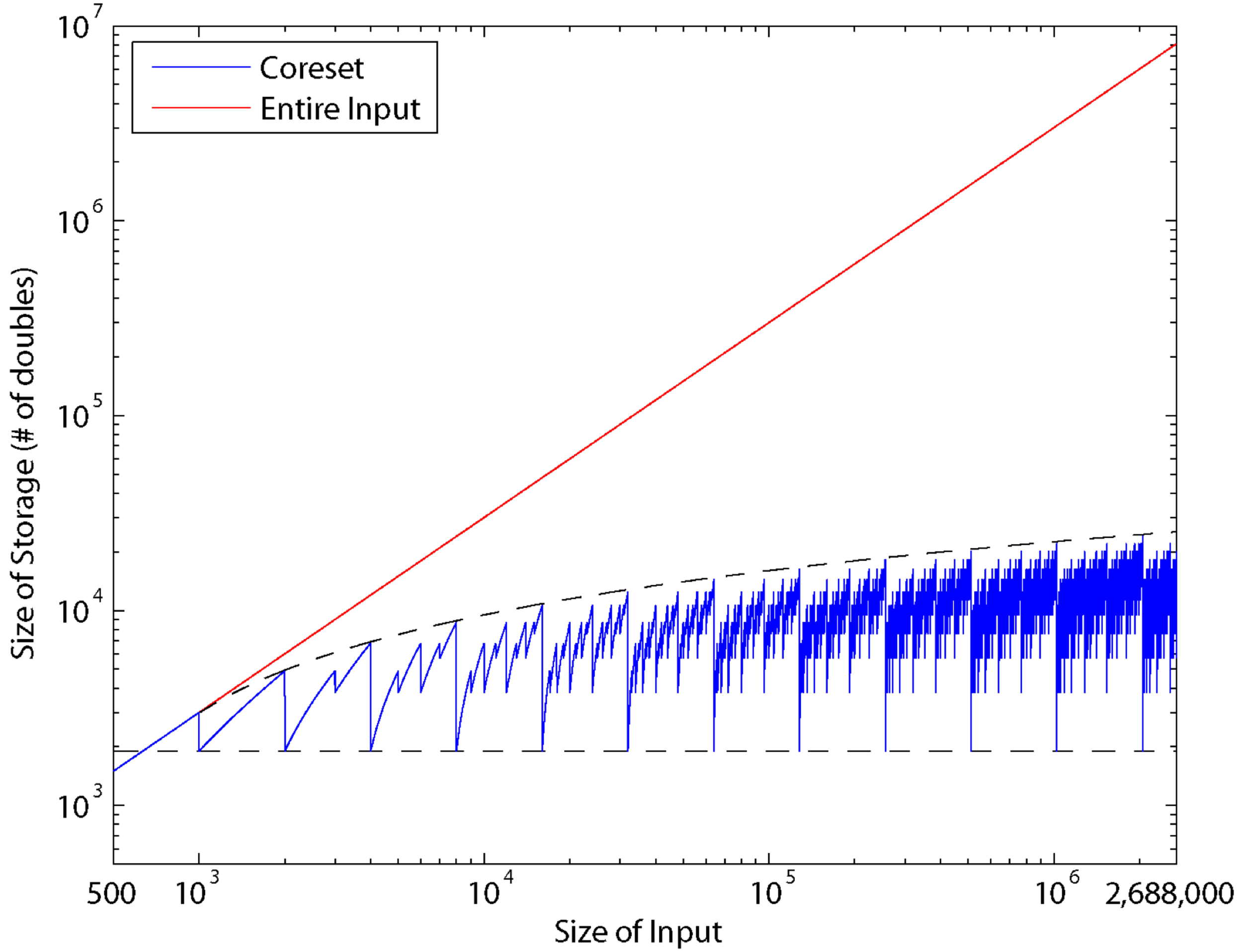




$(1+\epsilon)^3$-coreset for

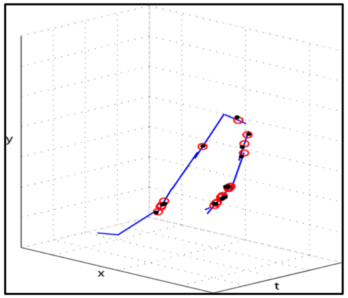
$P_1 \cup P_2 \cup P_3 \cup P_4$

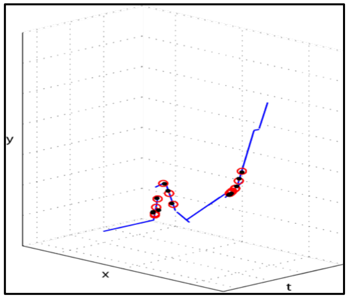


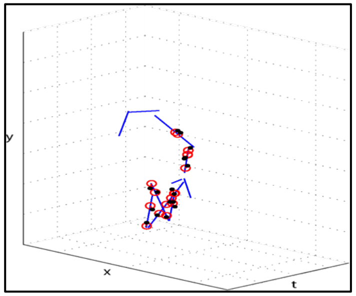


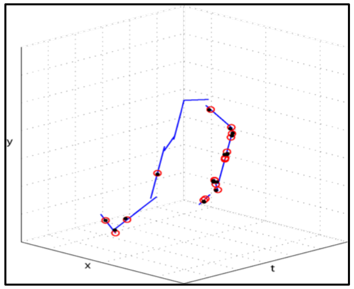
Coreset
Entire Input
Size of Storage (# of doubles)
10^7
10^6
10^5
10^4
10^3
500
10^3
10^4
10^5
10^6
2,688,000
Size of Input

Parallel Computation

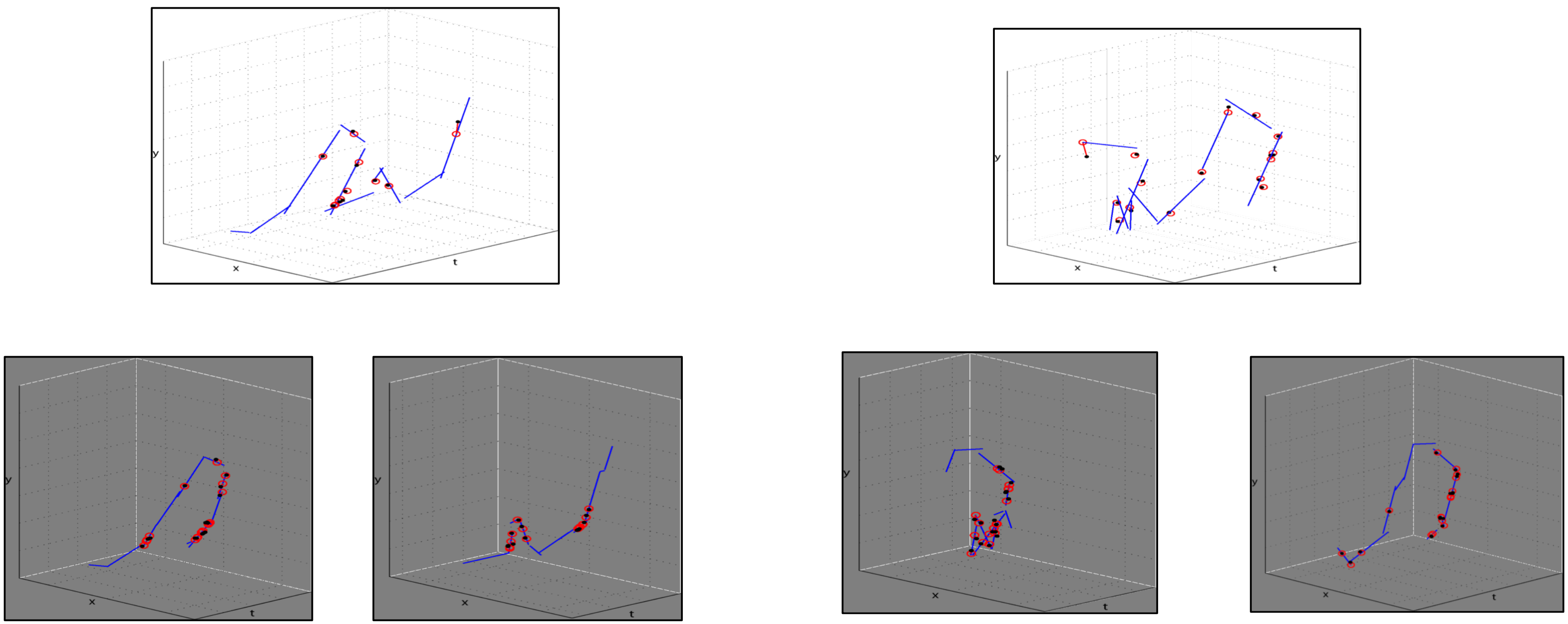






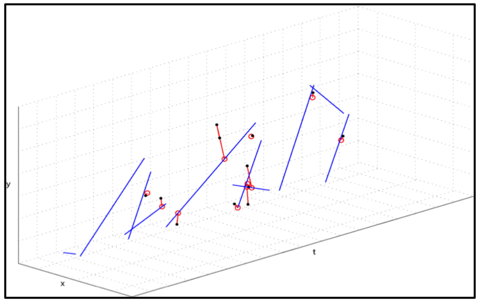


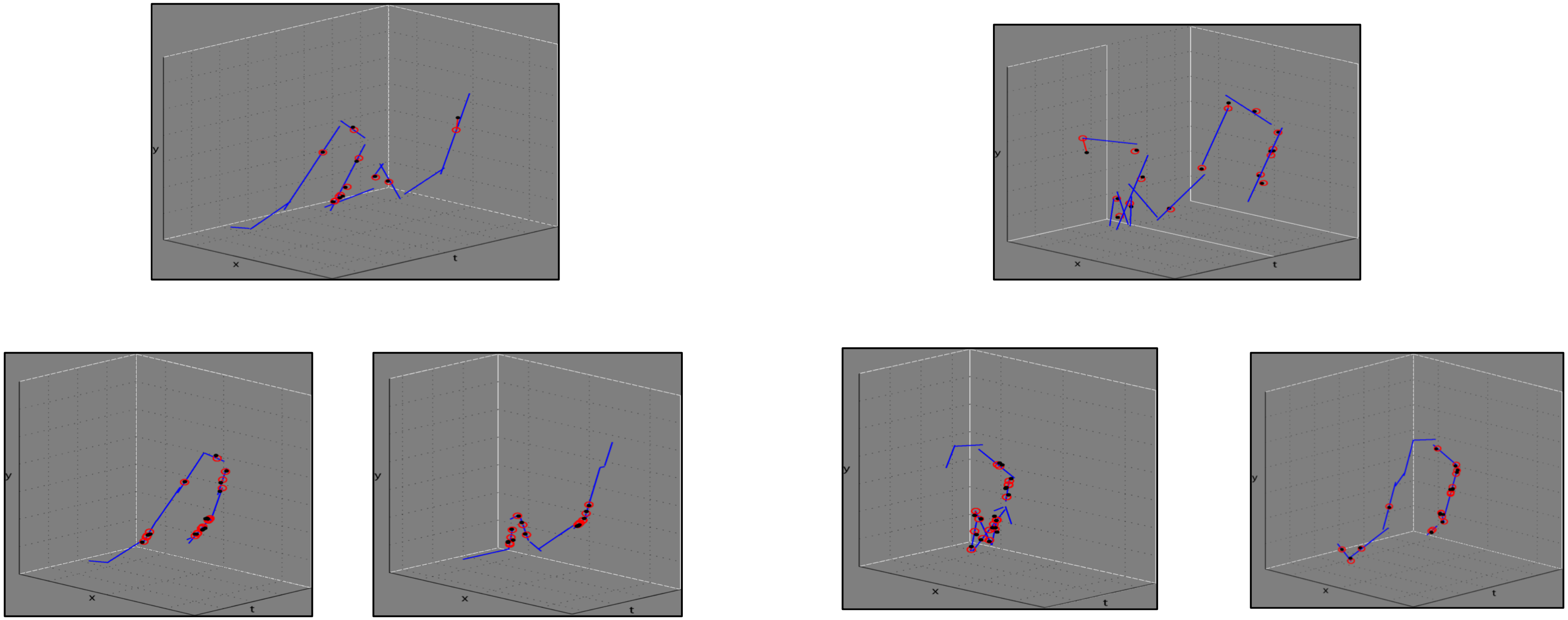
Parallel Computation



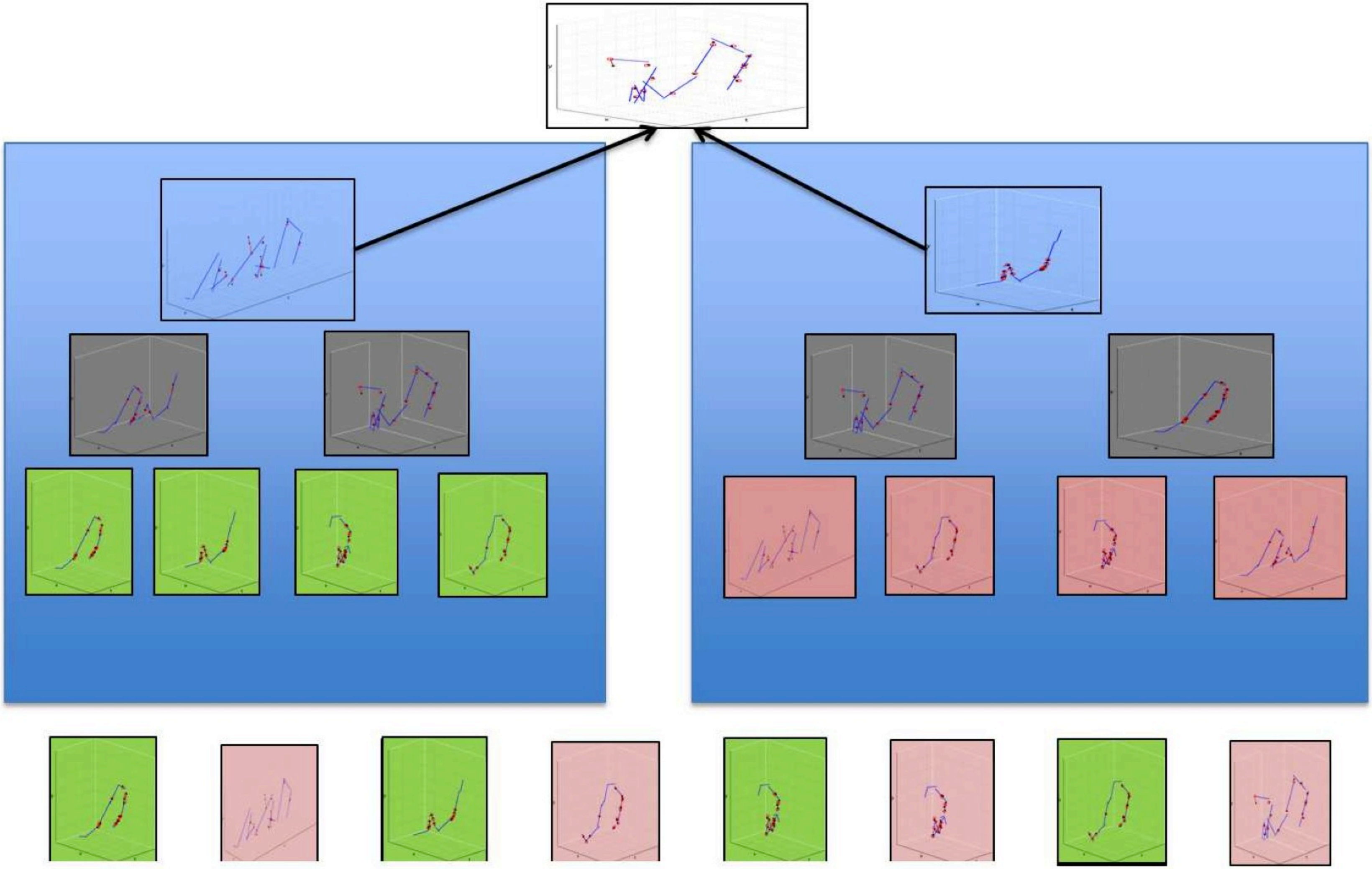
Parallel Computation

Run off-line
algorithm
on corset
using single
computer





Parallel+ Streaming Computation



Coresets for convex optimization

- A generic framework for learning kernel
- E.g: Logistic regression,
  - PCA/SVD with outliers,
  - Numerous kernels in Machine learning

Main tool:

generic-SVD via coreset for John Ellipsoid

- Relation to obstacle detection and path planning

Related Work

- Clarkson (SODA'2005)
  - Approximation for L_1 regression using weak coreset (only for off-line optimization)
- A. Dasgupta, P.Drineas, B. Harb, R. Kumar, M. Mahoney (SODA'2008)

  Weak coreset for L_p regression
- LaValle & Kuffmer, RRT trees (1998)

  Heuristics for path planning using sampling

Theorem [Feldman, Langberg, STOC'11]

[F., Langberg]

Suppose that

$$\mathrm{cost}(P,x) := \sum_{p\in P} w(p)k(p,x)$$

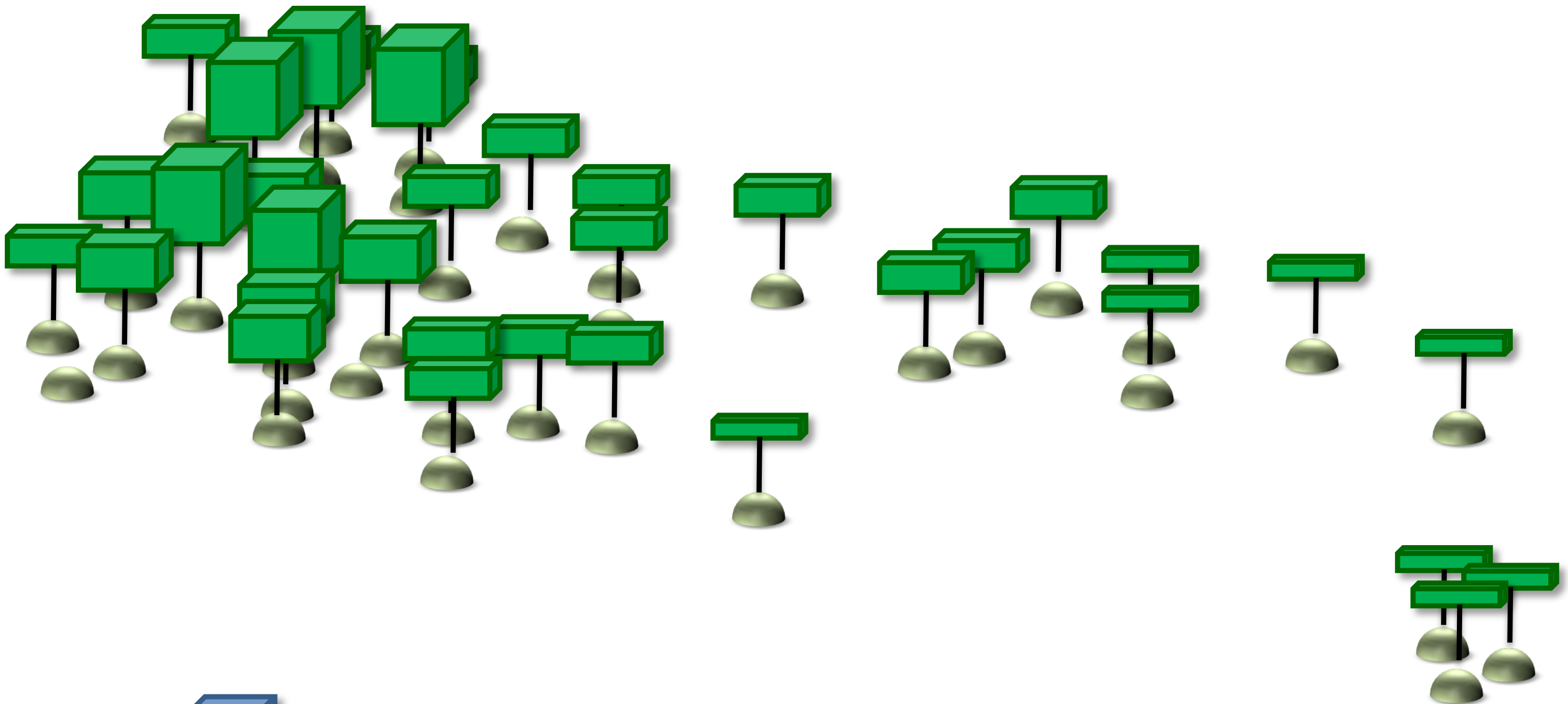
where $k: P\times X \to [0,\infty)$.

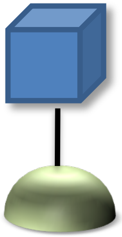
A sample $C \subseteq P$ from the distribution

$$\mathrm{sensitivity(p)} = \max_{x\in X} \frac{k(p,x)}{\sum_{p'} k(p',x)}$$

is a coreset if $|C| \sim \frac{\text{dimension of } X}{\epsilon} \cdot \sum_p \text{sensitibity}(p)$

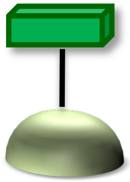
Importance Weights





$Sensitivity(p)$

Sampling distribution



$$\frac{1}{Sensitivity(p)}$$

Weights

Sensitivity for convex optimization

- We want to minimize/estimate

$$f(x) \sim cost(P, x) = \sum_{p \in P} k(p, x)$$

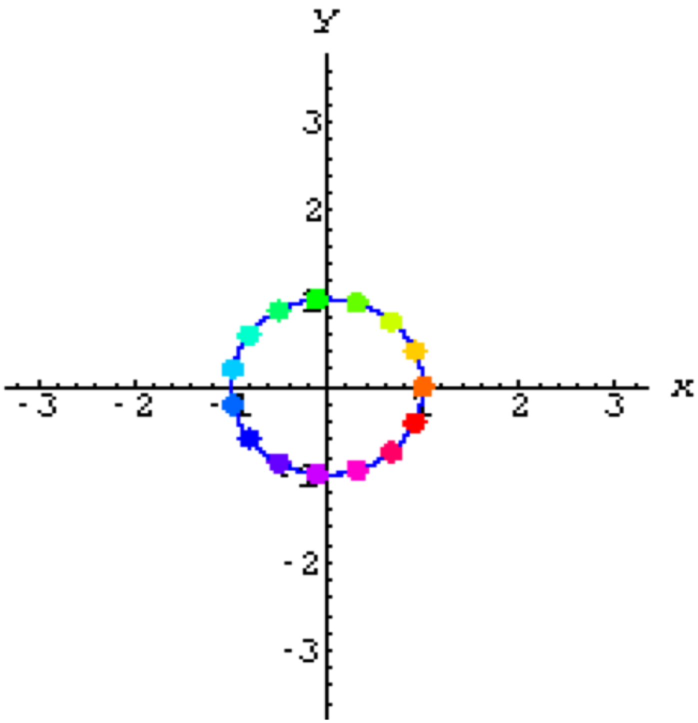
over $x \in X = \mathbb{R}^d$,

where f is convex

Query space as a convex shape

- Example: $k(p, x) = |px|^2$

$$f(x) = ||Px||^2,$$



Every unit vector x
is mapped to $x \cdot f(x)$

Gif by Todd Will

Query space as a convex shape

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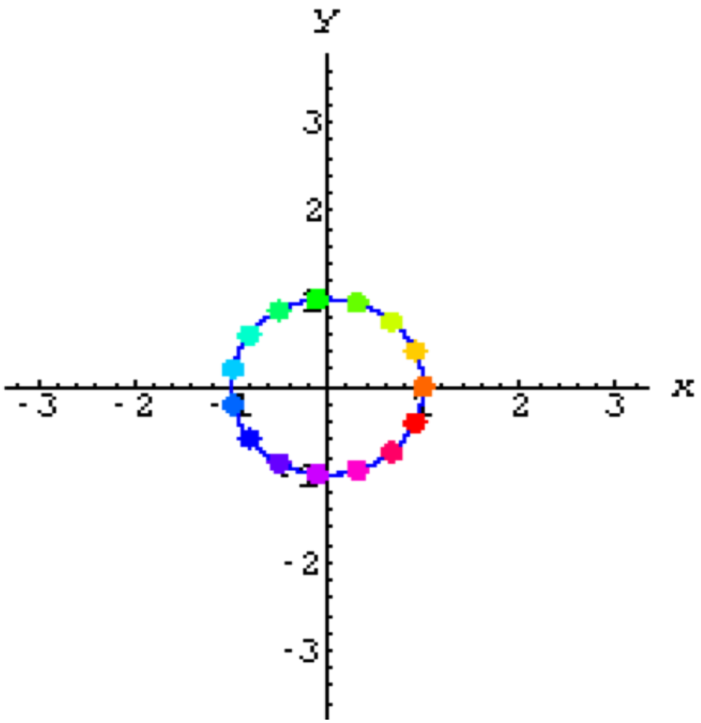


Gif by Todd Will

Every unit vector x
is mapped to $x \cdot f(x)$

The result is the Ellipsoid

$$X_f = \{x \in \mathbb{R}^d \mid f(x) \le 1\}$$
$$= \{x \in \mathbb{R}^d \mid ||DV^T x|| \le 1\}$$

where $P = UDV^T$ is the SVD of A,
and we have an exact "coreset"

$$||Px|| = ||UDV^T x|| = ||DV^T x||$$

From Sensitivity Lens

$$\frac{k(p,x)}{f(x)} = \frac{|px|^2}{||Px||^2} = \left|\frac{px}{||Px||}\right|^2 = \left|\frac{uDV^Tx}{||UDV^Tx||}\right|^2$$

$$= \left|\frac{uDV^Tx}{||DV^Tx||}\right|^2 = \left|u \cdot \frac{DV^Tx}{||DV^Tx||}\right|^2 \leq ||u||^2$$

$$\sum_{i=1}^{n} ||u_i||^2 = ||U||_F^2 = d$$

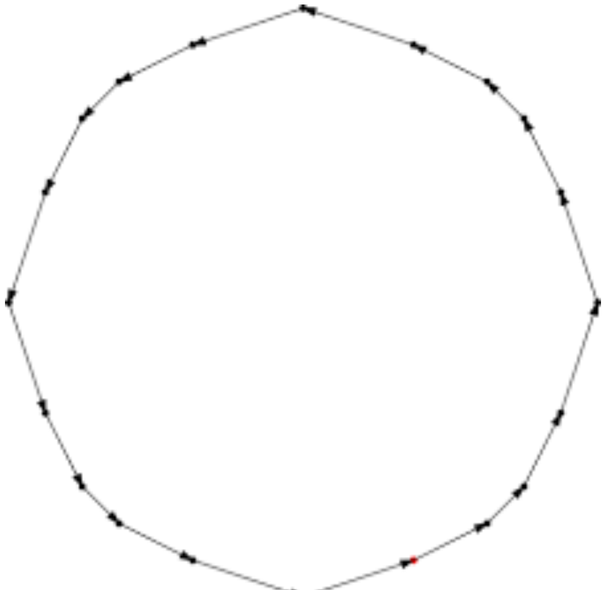
The general case

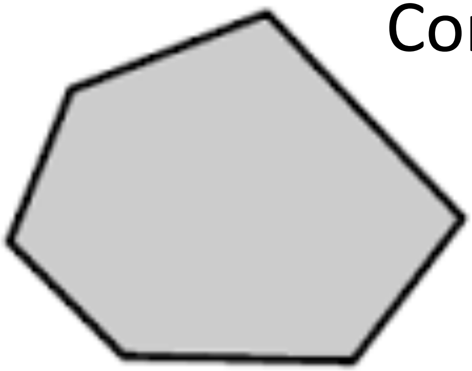
- Example: $k(p,x) = |px|$

$$f(x) = ||Px||_1$$

- Every unit vector x is mapped to $x \cdot f(x)$
- The result is a convex shape

$$X_f = \{ x \in \mathbb{R}^d \mid f(x) \leq 1 \}$$
$$= \{x \in \mathbb{R}^d \mid ||Ax||_1 \leq 1\}$$



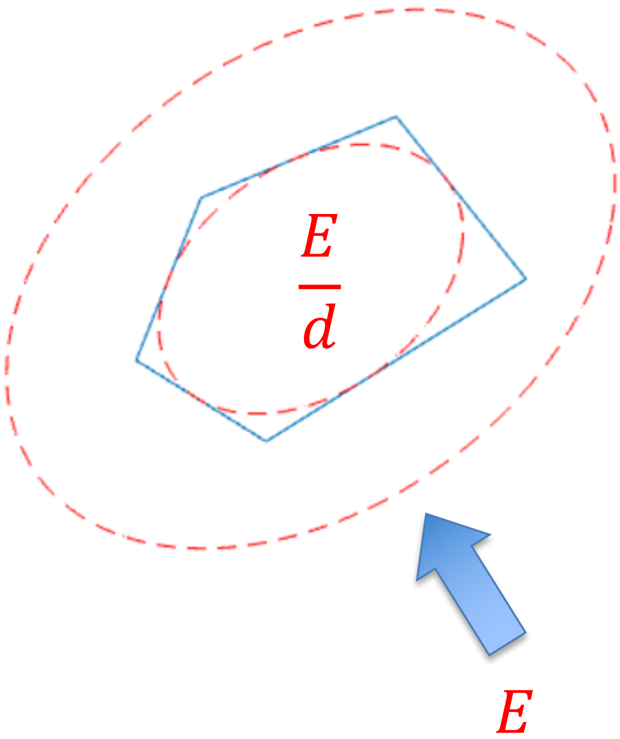


Complexity > $n^d > n$

Theorem (John's Ellipsoid)

- Every convex body contains an ellipsoid $\frac{E}{d}$ such that E contains it.
- For a $E \in \mathbb{R}^{d\times d}$ and every $x \in \mathbb{R}^{d}$:

$$f(x) \sim \left|\left|Ex\right|\right| = ||DV^T x||$$



- We define $P = UDV^T$ as the *f*-SVD of *P*
- Cons: (i) only d-approximation

  (ii) not subset of input point set P

From Sensitivity Lens

$$\frac{k(p,x)}{f(x)} = \frac{|px|}{||Px||_1} = \frac{|px|}{||UDV^T x||_1} \approx \frac{|uDV^T x|}{||DV^T x||_2} \leq ||u||_1$$

$$\sum_{i=1}^{n} ||u\_i||_1 = ?$$

Sensitivity for convex optimization

- We want to minimize/answer

$$f(x) \sim \sum_{p \in P} k(p, x)$$

- $k(p, x) \sim g(|px|)$

- $a \cdot k(p, x) \sim k(p, a \cdot x)$
- Otherwise, we use level sets for X_f

Main Theorem [F., Tukan]

The sensitivity of a point $p \in P$ is at most

$$\max_x \frac{k(p,x)}{f(x)} \leq \sum_{i=1}^{d} k(p, E^{-1} e_i)$$

and the total sensitivity (~size of coreset):

$$\sum_{p \in P} s(p) \in d^{O(1)}$$

Proof Sketch - sensitivity

$$\frac{k(p,x)}{f(x)} \sim \frac{k(p,x)}{||Ex||} \sim k\left(p, \frac{x}{||Ex||}\right) = k(uE, E^{-1}y)$$

$$y = \frac{Ex}{||Ex||}$$

$$\sim g(|uy|) \leq g(|u|_2) \leq g(|u|_1)$$

$$= g\left(\sum_{i=1}^{d} |ue_i|\right) \sim \sum_{i=1}^{d} g(|ue_i|)$$

$$\sim \sum_{i=1}^{d} k(uE, E^{-1}e_i) = \sum_{i=1}^{d} k(p, E^{-1}e_i)$$

Proof Sketch – total sensitivity

$$\sum_{p\in P}\sum_{i=1}^{d} k(p, E^{-1}e_i) = \sum_{i=1}^{d}\sum_{p\in P} k(p, E^{-1}e_i)$$

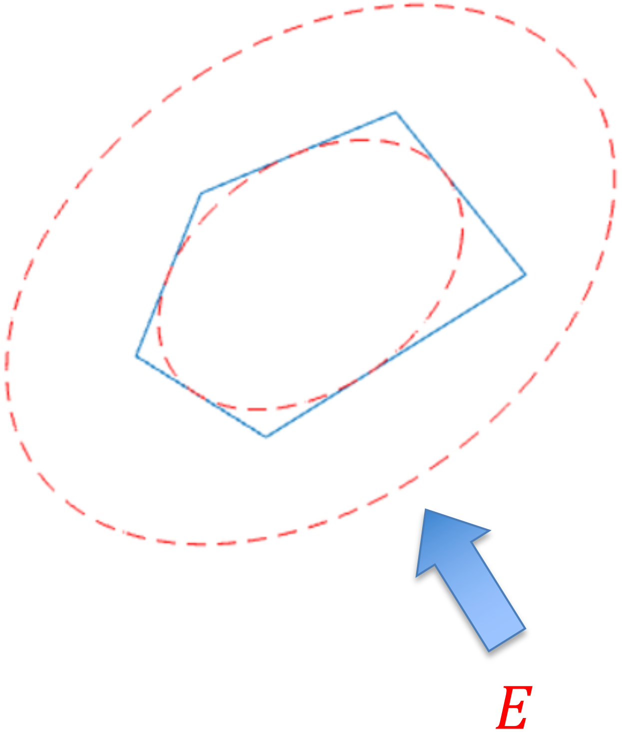
$$= \sum_{i=1}^{d} f(E^{-1}e_i) \sim \sum_{i=1}^{d} \left|\left|E \cdot E^{-1}e_i\right|\right| \sim$$

$$\sum_{i=1}^{d} \left|\left|e_i\right|\right| = d$$

How do we compute the ellipsoid E?

$$X_f = \{ x \in \mathbb{R}^d \mid f(x) \le 1 \}$$

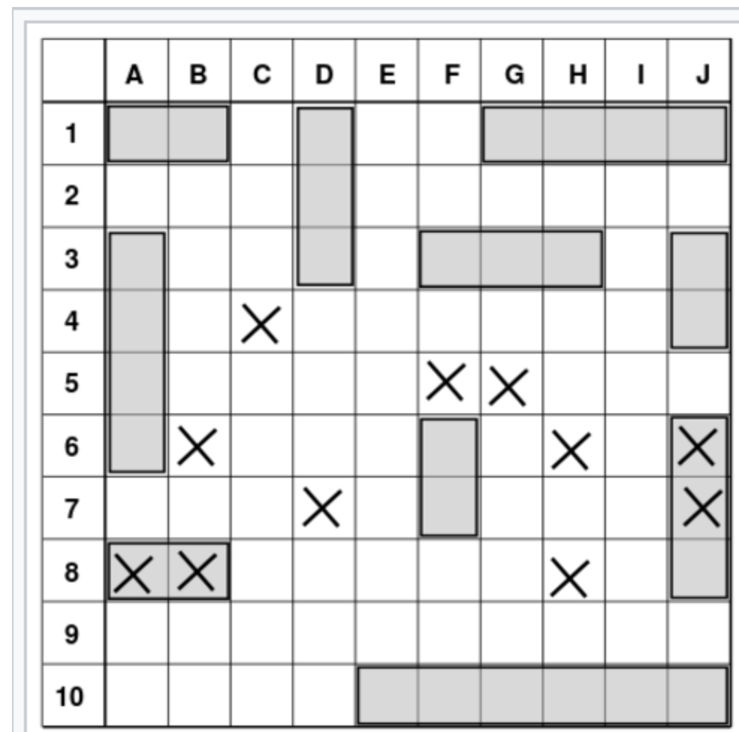
$$f(x) \sim \left|| Ex |\right| = ||DV^T x||$$



Only using oracle membership.

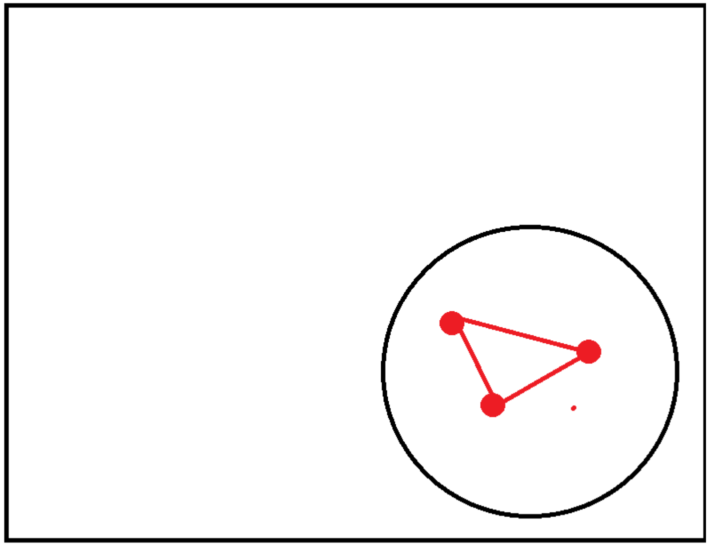
Path Planning in the Dark

- In control space we know start & destination configurations
- Can only ask boolean queries regarding feasible positions
- As in Battleships (game)

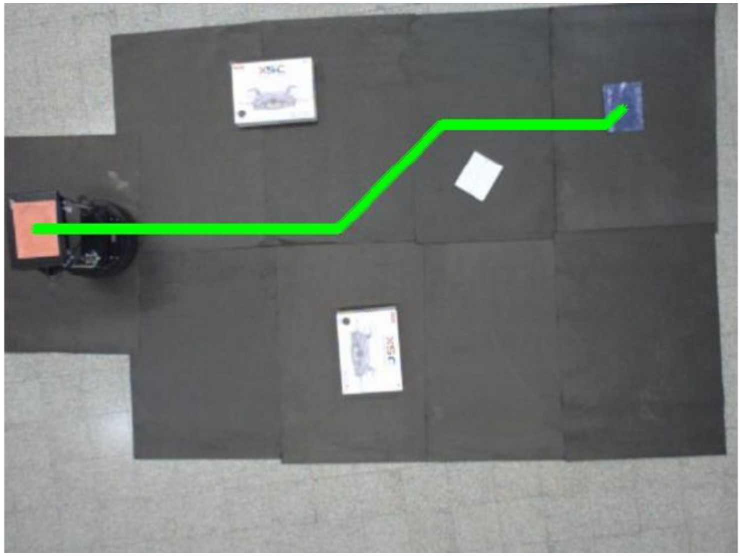


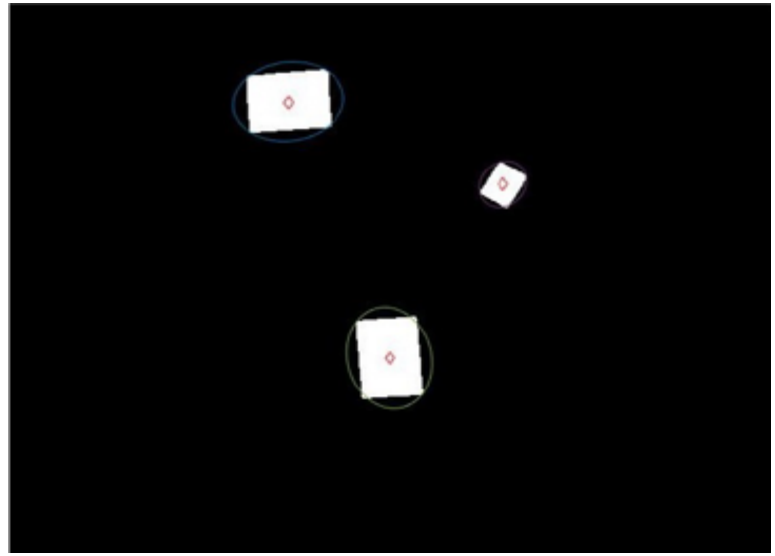
Path Planning in the Dark

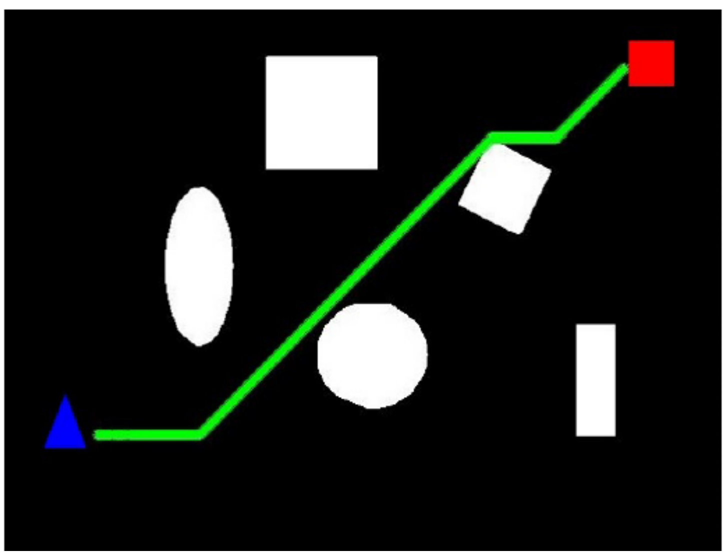
- We want minimum number of queries for maximum approximation error
- Existing algorithms have no guarantee for optimality
- Approximation by convex polygons

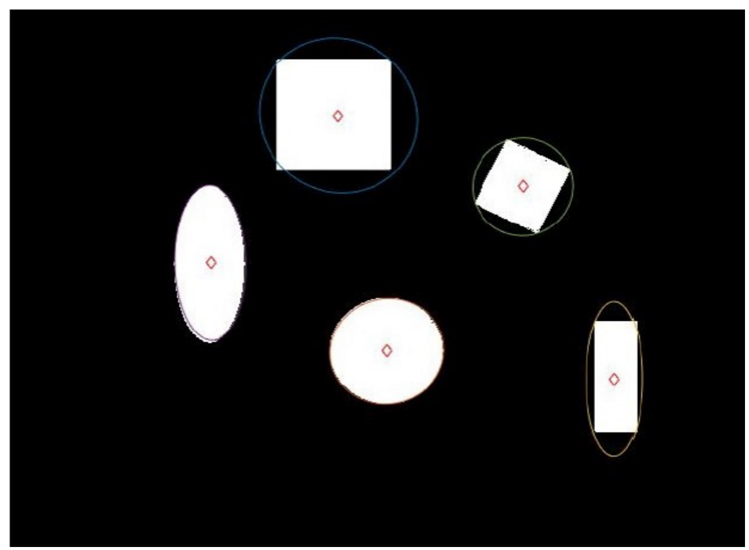


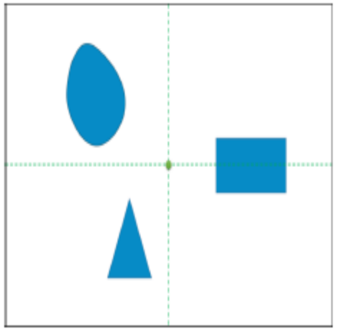
Path Planning



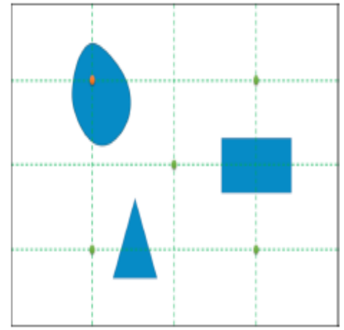




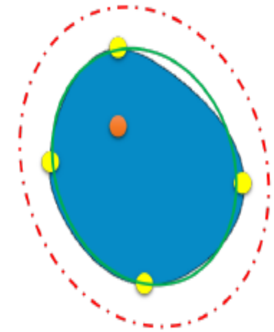




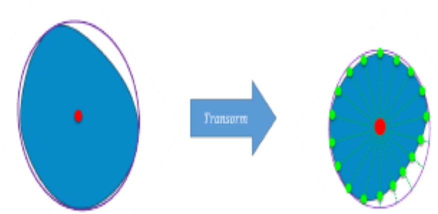
(a) Epsilon grid sampling; First iteration



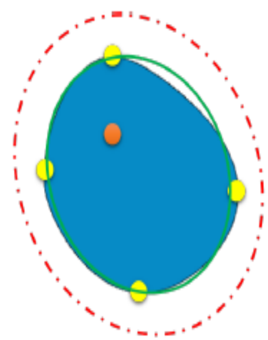
(b) Epsilon grid sampling; Second iteration



(c) d^{2d} approximation to John Ellipsoid

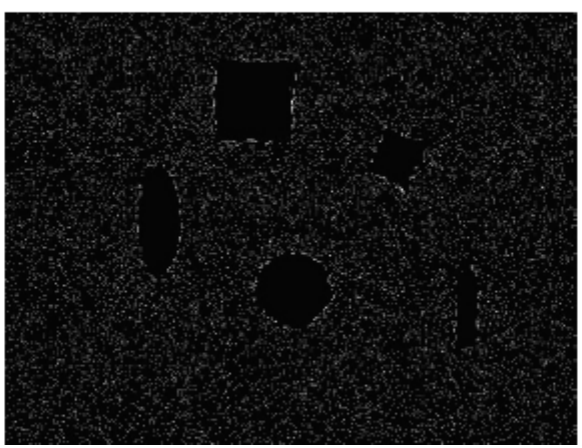


(d) Applying "Epsilon Star" on the transform space

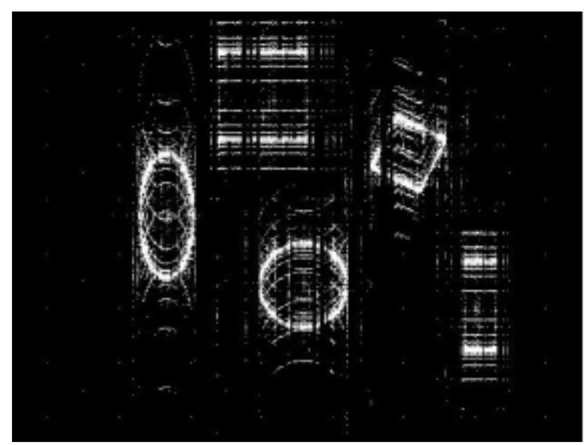


(e) $1 + \epsilon$ approximation to the real convex bodies

RRT



Our Algorithm



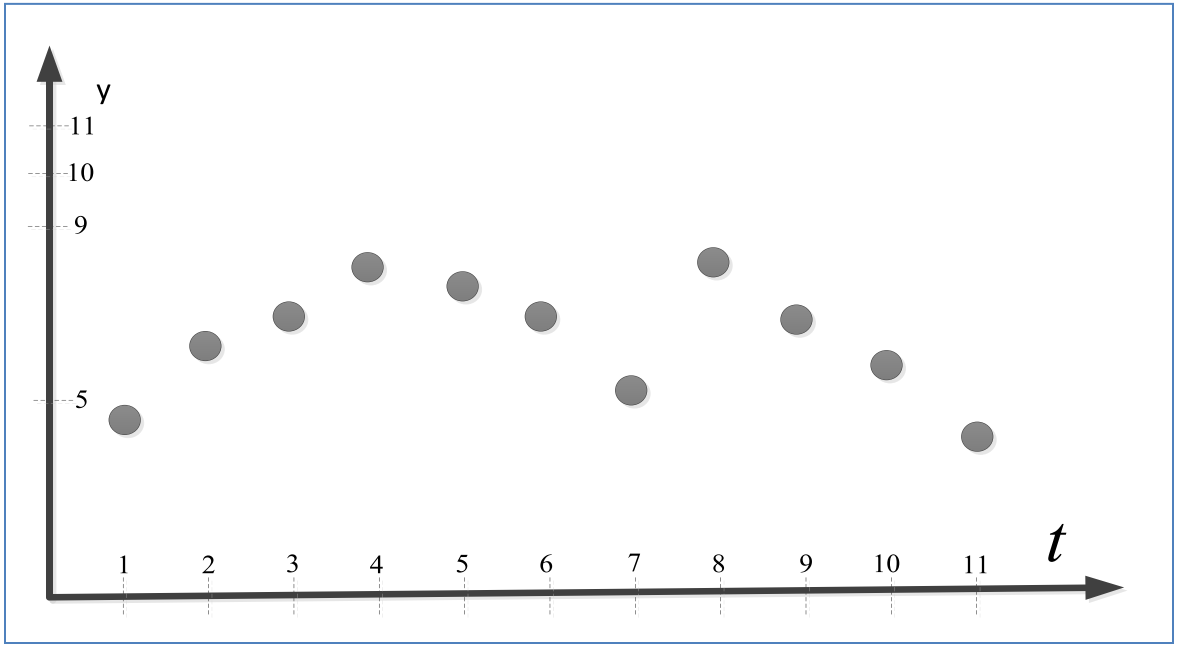
Open Problems

- More Coresets
  - Deep learning, Decision trees, Sparse data
  - Robotics: Optimal 3D Navigation and Mapping
- Private Coresets, [STOC'11, with Fiat et al.]
- Homomorphic Encryption Coresets
  [with A. Akavia, H. Shaul]
- Generic software library for robotics & big data
  - Coresets on Demand on the cloud
- Sensor Fusion (GPS+Video+Audio+Text+..)

FaceLocatiors.mp4

k – Segment Queries

Input: *d*-dimensional signal *P* over time



Coreset for k-means

[Feldman, Sohler, Monemizadeh, SoCG'07]

Coreset for k-means can be computed by choosing points from the distribution:

$$\text{sensitivity}(p) = \frac{dist(p,q^*)}{\sum_{p'} dist(p',q^*)} + \frac{1}{n_p}$$

q^* = k-means of P

n_p = number of points in the cluster of p

$|C| = \frac{k \cdot d}{\epsilon^2}$

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[SODA'13, Feldman, Schmidt, ..]

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[SODA'13, Feldman, Schmidt, ..]

The chicken-and-egg problem

1. We need approximation to compute the coreset
2. We compute coreset to get a fast approximation to a problem

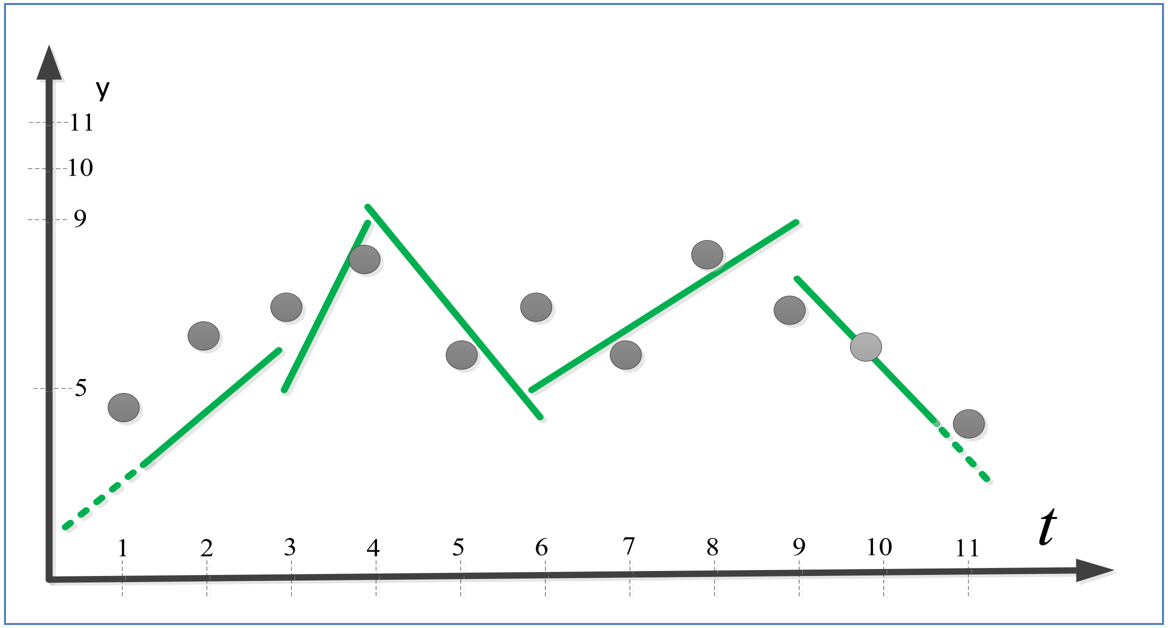
Lee-ways:

I. Bi-criteria approximation
II. Heuristics
III. polynomial time reduced to linear time by the merge-reduce tree

k – Segment Queries

Input: *d*-dimensional signal *P* over time

Query: *k* segments over time



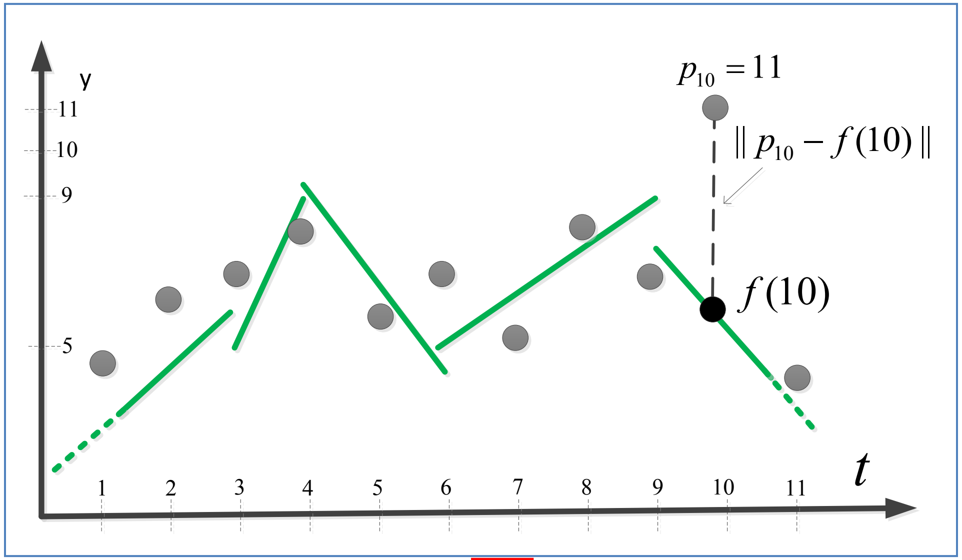
k-Piecewise linear function *f* over *t*

k − Segment Queries

Input: d-dimensional signal P over time

Query: k segments over time

Output: Sum of squared distances from P

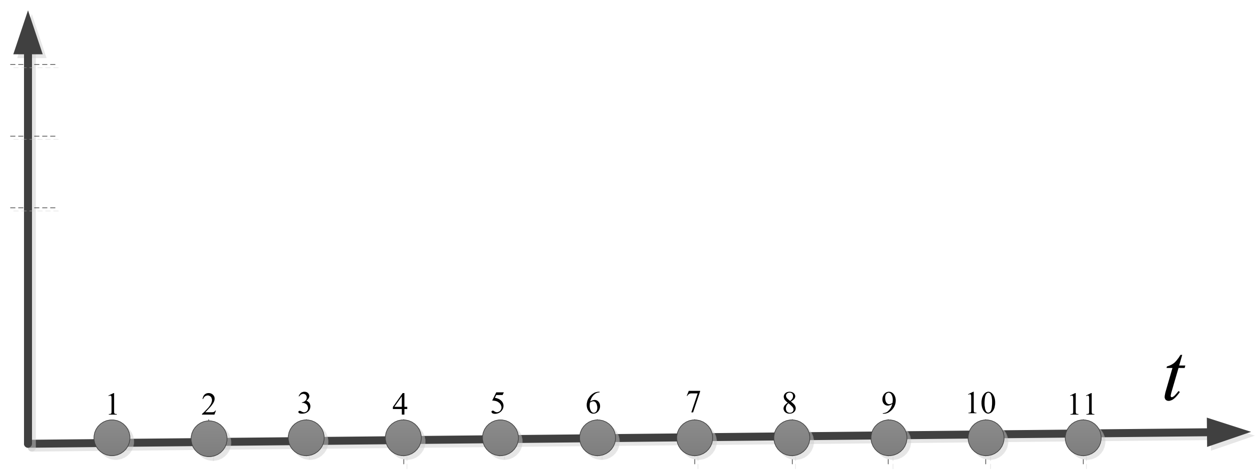


$$\mathrm{cost}(P, f) := \sum_t \| f(t) - p_t \|^2$$

Observation:

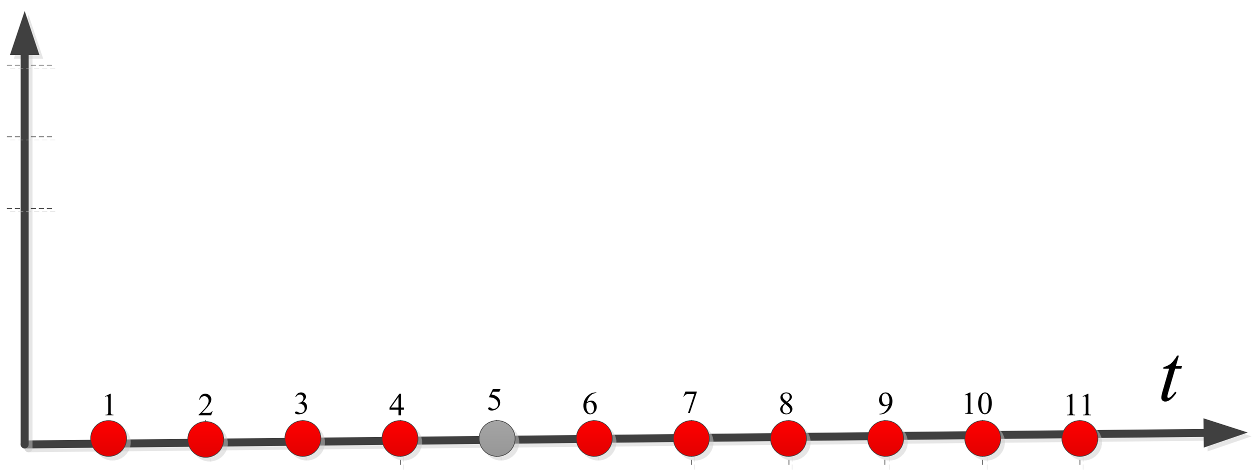
No small coreset $C \subset P$ exists for k-segment queries

Input P: *n* points on the *x*-axis



Input P: *n* points on the *x*-axis

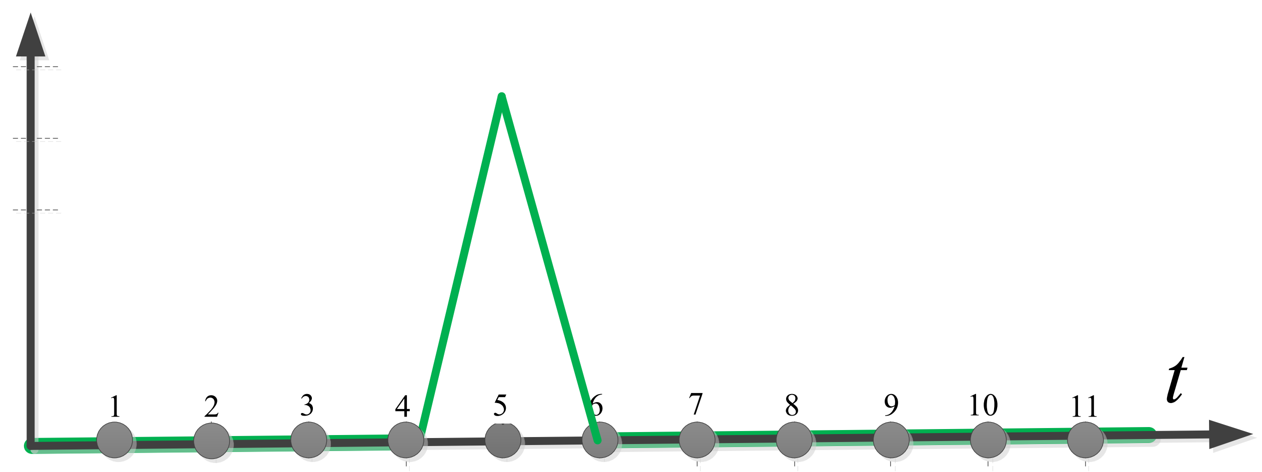
Coreset *C*: all points except one



Input P: *n* points on the *x*-axis

Coreset *C*: all points except one

Query *f*: covers all except this one



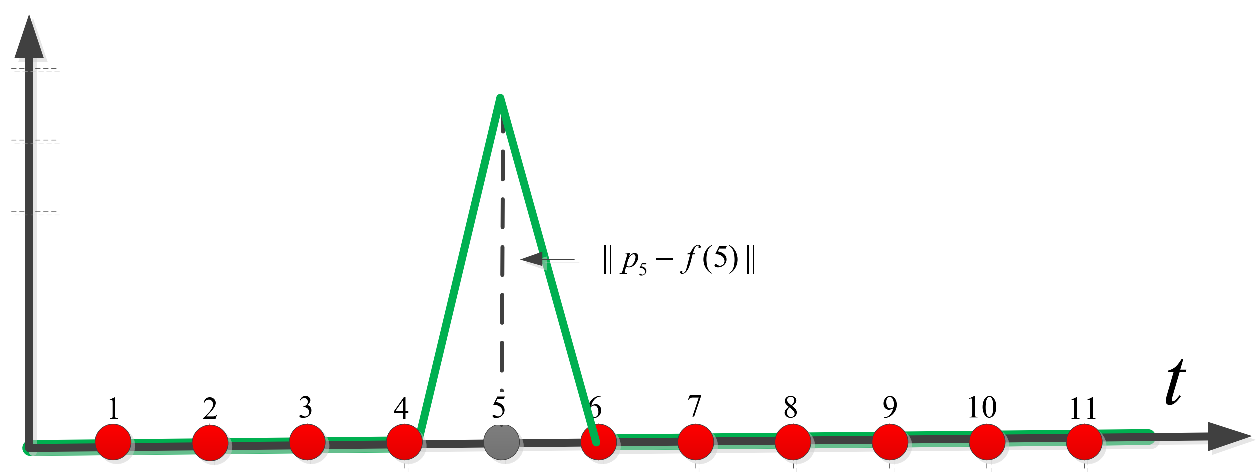
Input P: n points on the x-axis

Coreset C: all points except one

Query f: covers all except this one

$\text{Cost}(P, f) > 0$

$\text{Cost}(C, f) = 0$



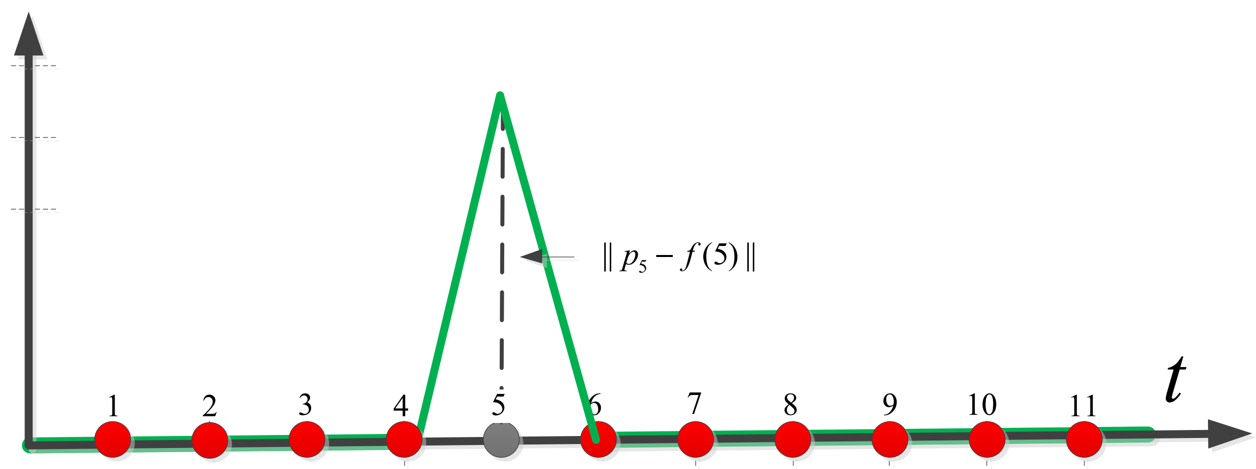
Input P: n points on the x-axis

Coreset C: all points except one

Query f: covers all except this one

$$\frac{\text{Cost}(P, f) > 0}{\text{Cost}(C, f) = 0}$$

Unbounded factor approximation



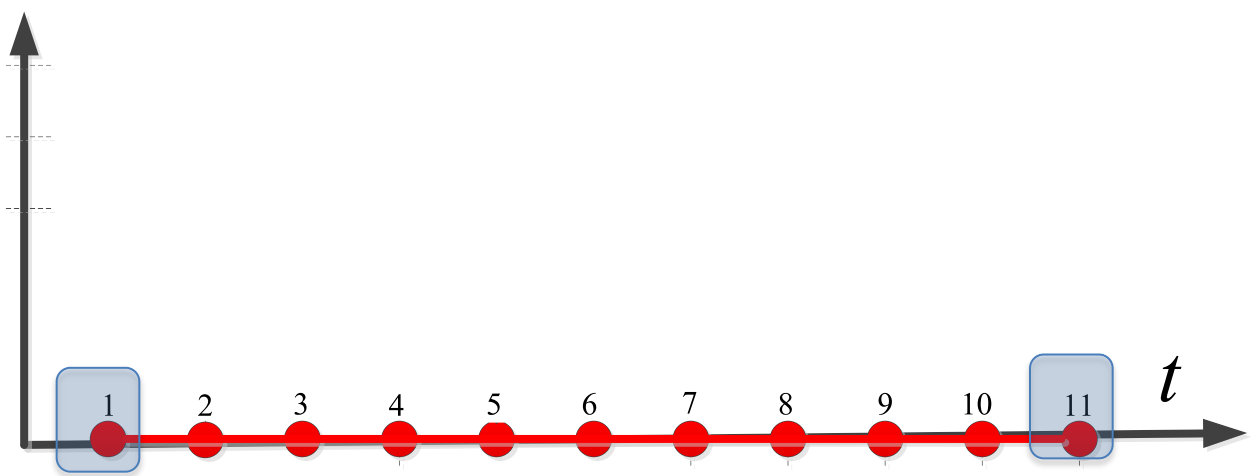
For every point p:

Sensitivity(p) = $\max_{q \in Q} \frac{dist(p,q)}{\sum_{p'} dist(p',q)} = 1$

Total sensitivities: n

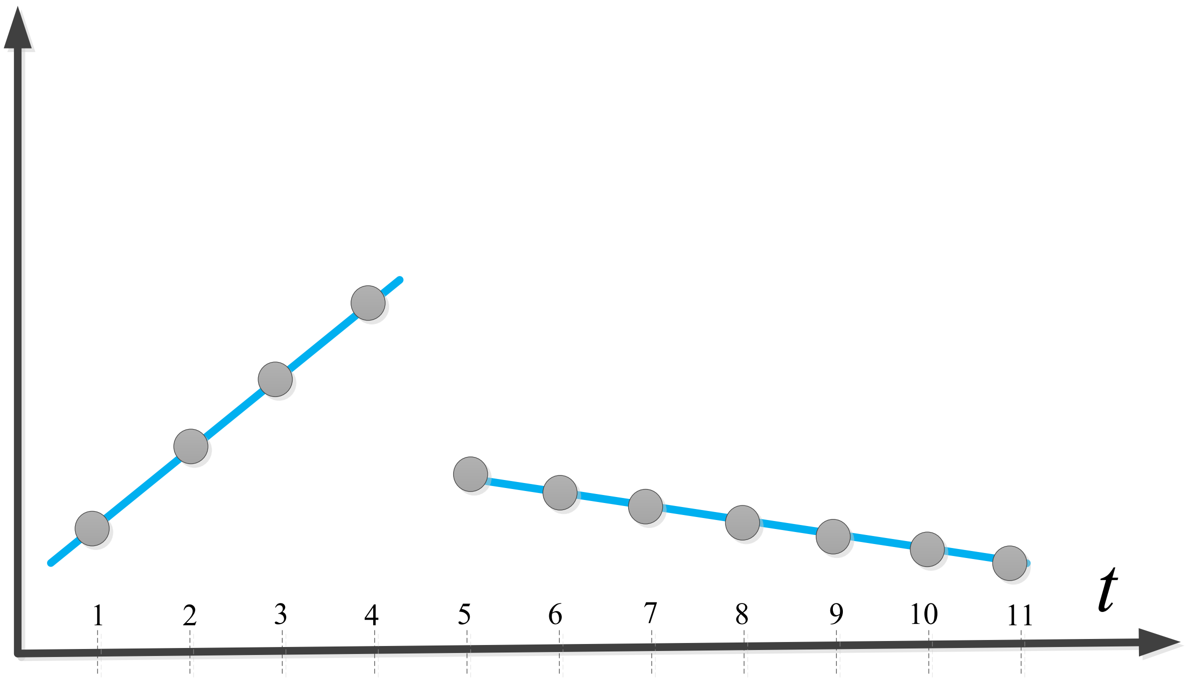
Observation:

Points on a segment can be stored by the two indexes of their end-points



Observation:

Points on a segment can be stored by the two indexes of their end-points and the slope of the segment



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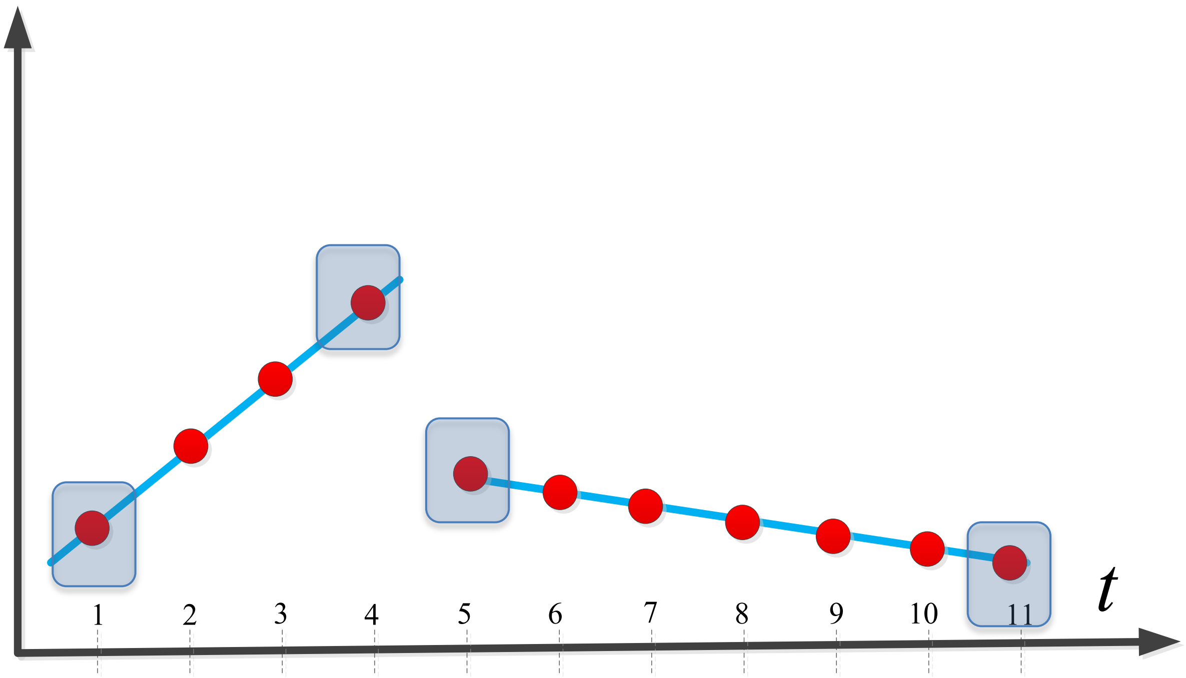
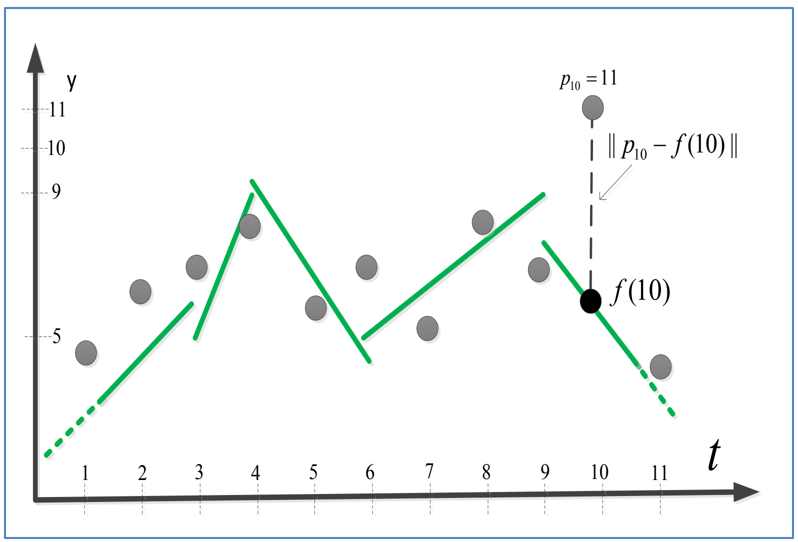


Definition: Coreset

A weighted set $C \cancel{\subset} P$ such that
for every k-segment f :

$$\mathrm{cost}(P, f) \sim \mathrm{cost}_w(C, f)$$



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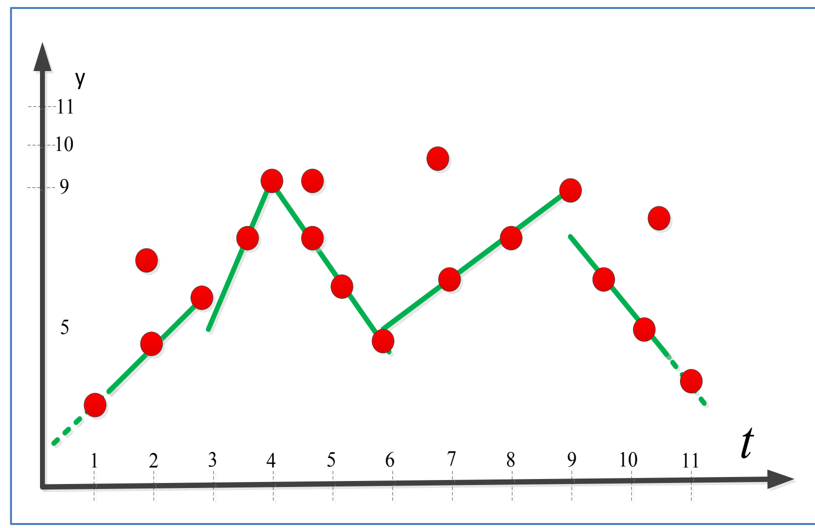

$$\sum_{t} \|f(t) - pt\|$$

$$\sum_{p_t \in C} w(p_t) \cdot \|f(t) - pt\|$$

Surprising Applications

1. (1-epsilon) approximations:
   Heuristics work better on coresets

2. Running constant factor on epsilon-coresets helps

3. Coreset for one problem is good for a lot of unrelated problems

4. Coreset for O(1) points

Implementation

- The worst case and sloppy (constant) analysis is not so relevant

- In Thoery:
  a random sample of size $1/\epsilon$ yields $(1+\epsilon)$ approximation with probability at least $1-\delta$.
  In Practice:
  Sample s points, output the approximation ϵ and its distribution

- Never implement the algorithm as explained in the paper.

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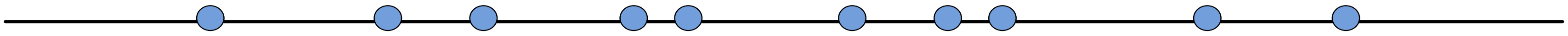
$|C| = \cancel{\frac{k \cdot d}{\epsilon^2}} \frac{k \cdot \left(\frac{k}{\varepsilon}\right)}{\epsilon^2}$ [SODA'13, Feldman, Schmidt, ..]

Coreset for Enclosing Balls $P \subseteq \mathbb{R}$



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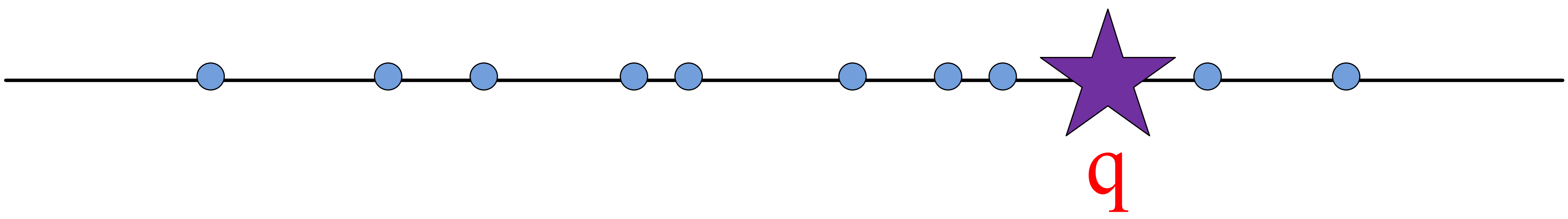

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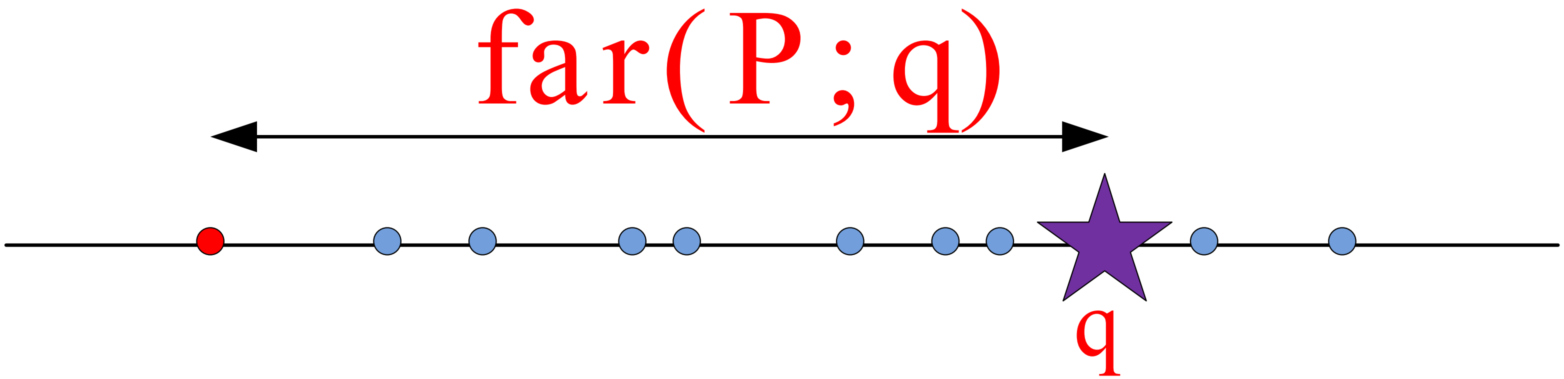
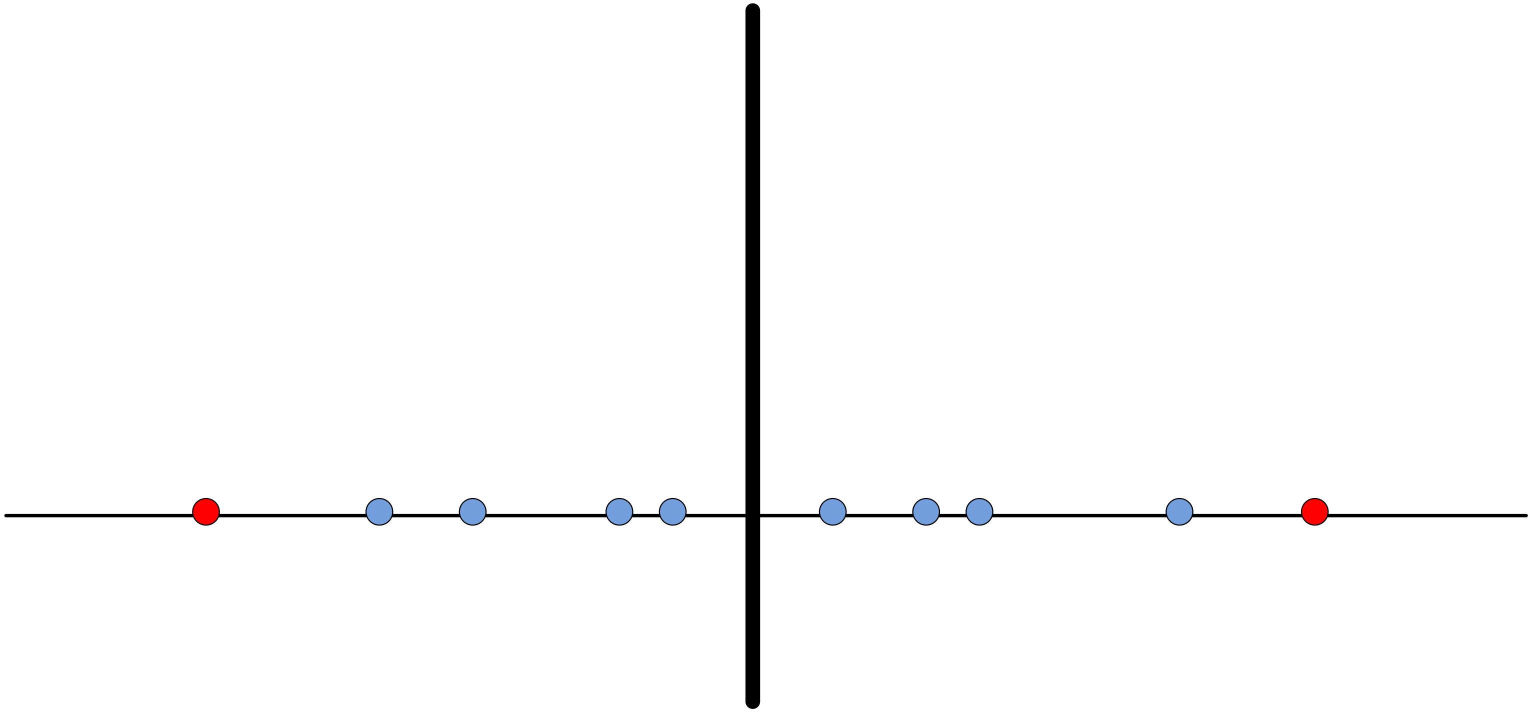


Coreset for Enclosing Balls $P \subseteq \mathbb{R}$

The fathest point from every query $q \in \mathbb{R}$

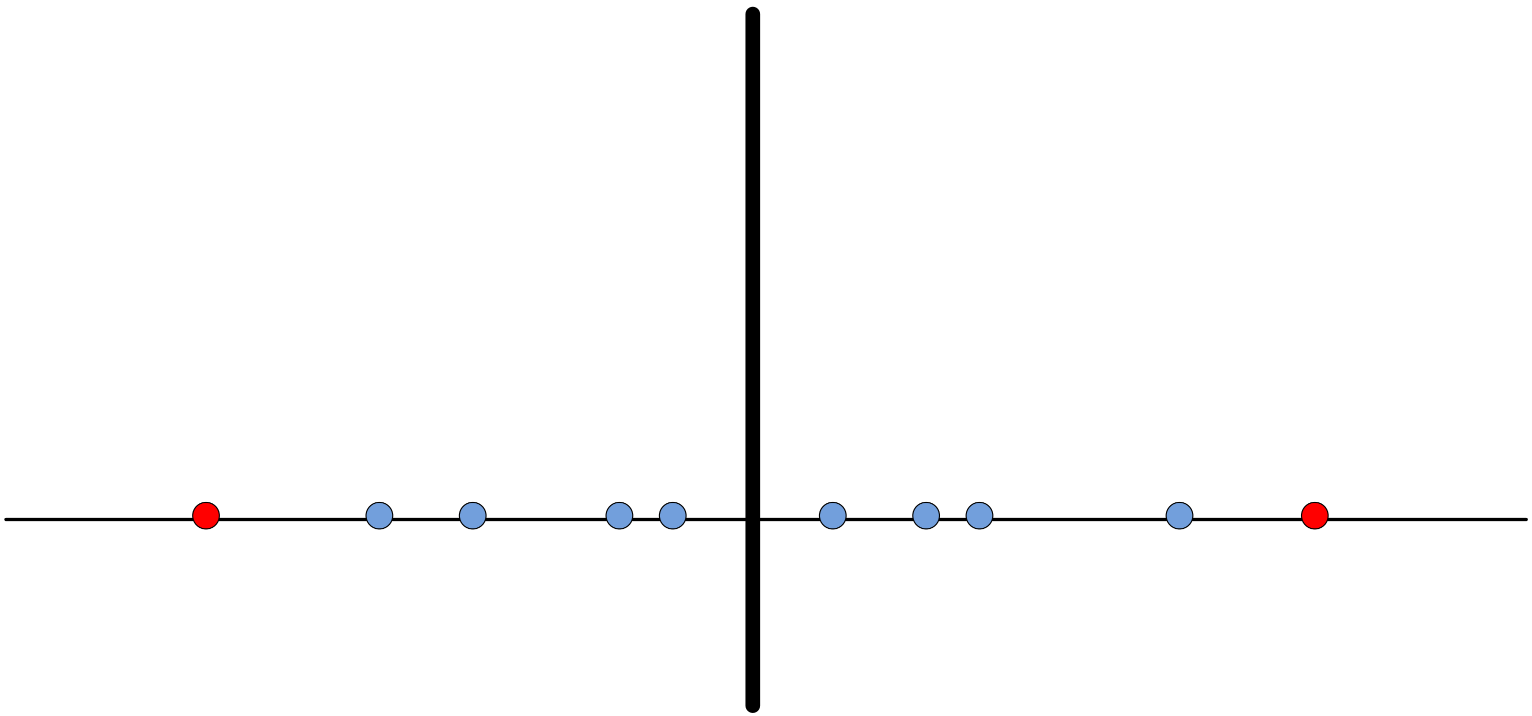
is a red point



Coreset for Enclosing Balls $P \subseteq \mathbb{R}$

The fathest point from every query $q \in R^d$

is a red point



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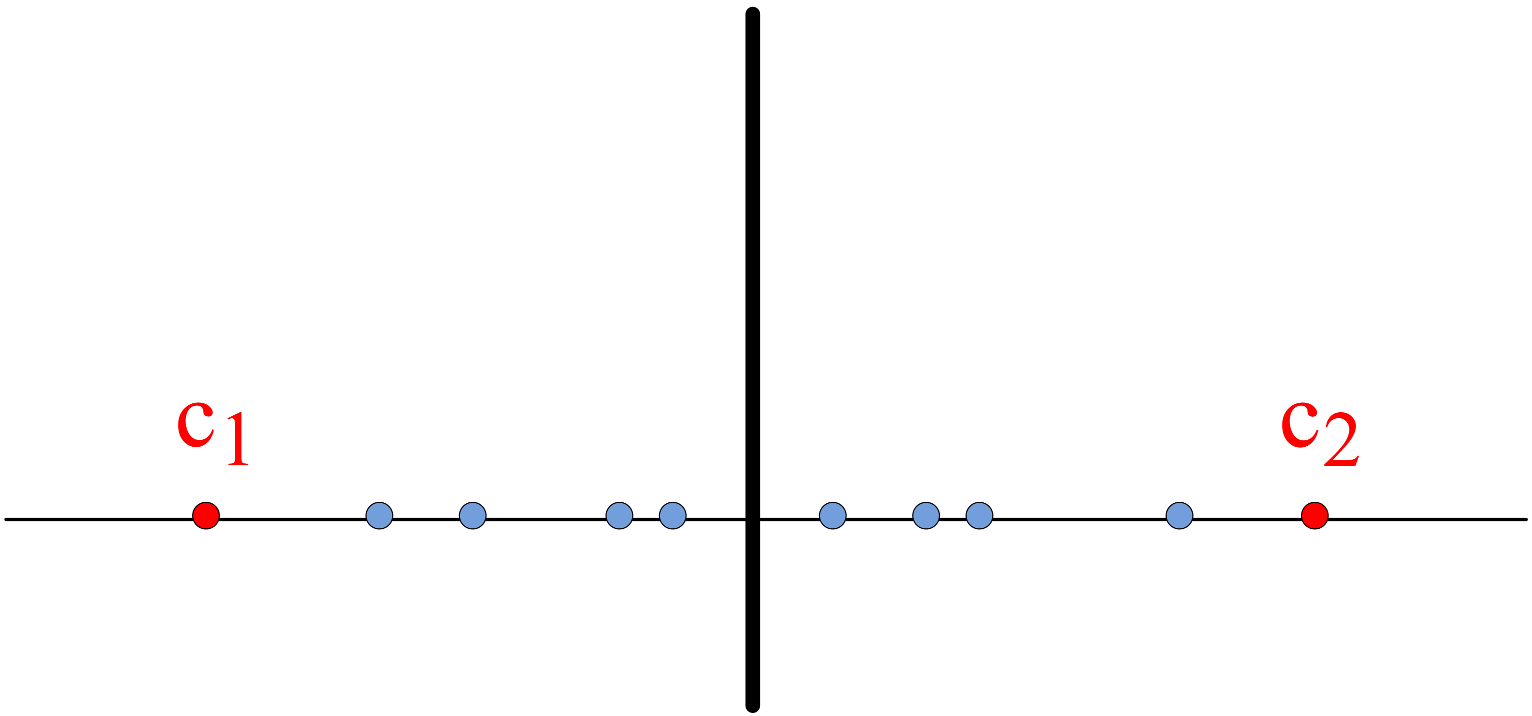

$C := f c_1 ; c_2 g$ is a coreset for P

Coreset Techniques

Graph Theory
Sparsifiers
Batson, Speilman, Srivastava, …

Computational Geometry
Coresets
Har-Peled, Agarwal, Sohler, Chen

Matrix Approximation
Volume Sampling
Clarkson, Mahoney, Drineas …

Combinatorial Geometry
ε-nets, ε-approximations
Haussler, Welzl, Alon, Matousek, Sharir,…

Statistics
Importance Sampling
Srinivasan, Ripley, , …

PAC-Learning
ε-sample
Vapnik, Chervonenkis, Valiant, … .

Compressed Sensing
Sketches
Property Testing

ICRA’14 (With Rus, Paul and Newman)