Spectrahedra and directional derivatives of determinants

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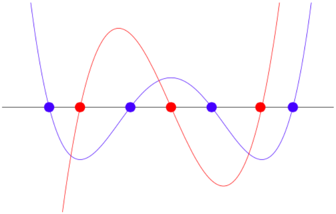
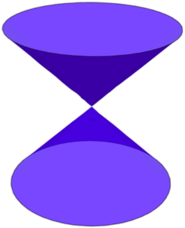
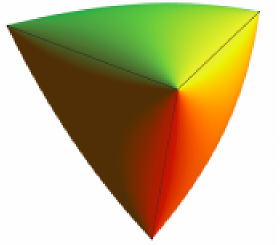
Hyperbolic polynomials





Spectrahedra



Directional derivatives

$$E_{n-1}(X) = \frac{d}{dt} \det(X + tI)\big|_{t=0}$$

Hyperbolic polynomials

A polynomial p homogeneous of degree d in n variables is hyperbolic with respect to $e \in \mathbb{R}^n$ if

- $p(e) \neq 0$
- for all $x \in \mathbb{R}^n$, all roots of $t \mapsto p(x - te)$ are real

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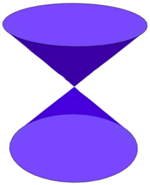
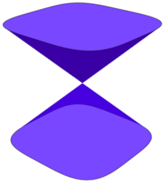


$$p(x, y, z) = -x^2 - y^2 + z^2$$

hyperbolic w.r.t. $e = (0, 0, 1)$



$$p(x, y, z) = -x^4 - y^4 + z^4$$

not hyperbolic

Hyperbolicity cones

If p is hyperbolic w.r.t. $e \in \mathbb{R}^n$ define hyperbolicity cone as

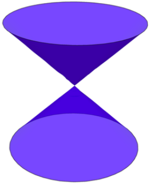
$$\Lambda_+(p, e) = \{x \in \mathbb{R}^n : \text{all roots of } t \mapsto p(x - te) \text{ non-negative}\}$$

Theorem (Gårding 1959)
If p is hyperbolic w.r.t. e then $\Lambda_+(p, e)$ is convex.

Example

$$p(x, y, z) = -x^2 - y^2 + z^2$$

- hyperbolic w.r.t. $e = (0, 0, 1)$
- Hyperbolicity cone is second-order/Lorentz/ice-cream cone



Key examples

p has definite determinantal representation

$$p(x) = \det\left(\sum_{i=1}^{n} A_i x_i\right)$$

- $A_1, \ldots, A_n$ are $d \times d$ symmetric matrices
- $\sum_{i=1}^{n} A_i e_i \succ 0$

Hyperbolicity cone is spectrahedron

$$\Lambda_+(p, e) = \left\{ x \in \mathbb{R}^n \ : \ \sum_{i=1}^{n} A_i x_i \succeq 0 \right\}$$

Examples:

- Polyhedral cone: $p(x) = \prod_i (a_i^T x)$ with e in interior
- Positive semidefinite cone $p(X) = \det(X)$ with e pos def.

Hyperbolic programming

$$\text{minimize}_x \langle c, x \rangle \text{ subject to } \begin{cases} Ax = b \\ x \in \Lambda_+(p, e). \end{cases}$$

Theorem (Güler 1997)

$-\log_e(p)$ is a self-concordant barrier for $\Lambda_+(p, e)$

Special cases

- Linear programming
- Second-order cone programming
- Semidefinite programming

Is hyperbolic programming more general than semidefinite programming?

Derivative relaxations/Renegar derivatives

If p is hyperbolic w.r.t. e then directional derivative

$$D_e p(x) = \frac{d}{dt} p(x + te)\Big|_{t=0} \quad \text{is hyperbolic w.r.t. } e$$

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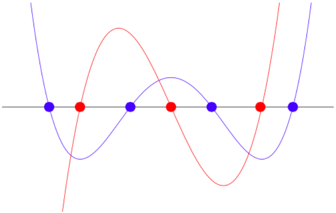

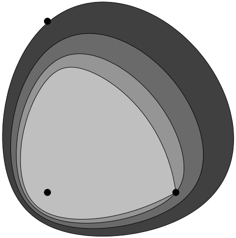

Geometrically: the derivative relaxation is bigger!

$$\Lambda_+(p, e) \subseteq \Lambda_+(D_e p, e).$$

Examples: elementary symmetric polynomials

If $e_n(x) = x_1 x_2 \cdots x_n$ then

$$\begin{aligned} D_{\mathbf{1}_n} e_n(x) &= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} x_1 \cdots x_n \\ &= \text{elementary sym. poly. of degree } n-1 \text{ in } n \text{ variables} \\ &= e_{n-1}(x) \end{aligned}$$

Repeatedly differentiate in

- same direction $\longrightarrow$ all elementary sym. poly.
- different directions $\longrightarrow$ (essentially) permanent

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$$D_{I_n} \det(X) = \text{sum of } (n-1) \times (n-1) \text{ principal minors of } X$$
$$= E_{n-1}(X) = e_{n-1}(\lambda(X))$$

Repeat to get all elementary sym. poly. in eigenvalues

Lax conjecture

Lax Conjecture: Every hyperbolic polynomial in 3 variables has definite determinantal representation.

Helton-Vinnikov Theorem: the Lax Conjecture is true

Generalized Lax conjecture

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Algebraic version:

If p is hyperbolic w.r.t. e then there exists q such that

- qp has a definite determinantal representation
- hyp. cone of q $\supseteq$ hyp. cone of p.

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definite determinantal rep. $\Longrightarrow$ cone spectrahedral
$\Longrightarrow$ cone projected spectrahedral

Lax-type problems for derivatives

Lax conjecture for derivatives
If $\Lambda_+(p, e)$ is a spectrahedron then

$$\Lambda_+(D_e p, e) \text{ is a spectrahedron.}$$

Would imply hyperbolicity cones are spectrahedra for

- permanents, mixed discriminants
- elementary symmetric polynomials (in eigenvalues)

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Theorem (S. 2017)
If p has a definite determinantal representation then
$$\Lambda_+(D_e p, e) \text{ is a spectrahedron.}$$

Spectrahedral descriptions

Hyperbolicity cones known to be spectrahedra

- Sanyal (2013): $\Lambda_+(e_{n-1}, \mathbf{1}_n)$ of size $n-1$
- Brändén (2014): $\Lambda_+(e_k, \mathbf{1}_n)$ of size $O(n^{k-1})$
- Amini (2016):
  hyp. cones assoc. with multivariate matching polynomials
- Kummer (2016):
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(Projected) spectrahedral descriptions

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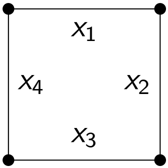
Hyperbolicity cones known to be projected spectrahedra

- Zinchenko (2008): $\Lambda_+(e_{n-1}, \mathbf{1}_n)$
- Parrilo, S. (2015): $\Lambda_+(E_k, I_n)$ of size $O(n^2 \min\{k, n-k\})$
- Netzer, Sanyal (2015): Smooth hyperbolicity cones

Example: Sanyal's representation



Spanning tree polynomial:

$$x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4$$

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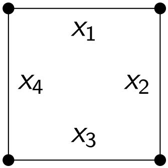

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Definite determinantal representation:

Let $\tilde{L}_{C_n}(x)$ be edge-weighted reduced Laplacian of n-cycle

$$\begin{aligned}\det(\tilde{L}_{C_n}(x)) &= n\,(\text{spanning tree polynomial of } C_n) \\ &= n\, e_{n-1}(x)\end{aligned}$$

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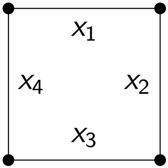

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Spectrahedral representation

$$\Lambda_+(e_{n-1}, \mathbf{1}_n) = \{x \in \mathbb{R}^n \ : V^T \operatorname{diag}(x) V \succeq 0\}$$

where columns of V are a basis for $\mathbf{1}_n^\perp$ = cycle space$^\perp$

Main result

Theorem (S. 2017)
$\Lambda_+(E_{n-1}, I_n)$ has a spectrahedral rep. of size $\binom{n+1}{2} - 1$.
If $B_1, B_2, \ldots, B_{\binom{n+1}{2}-1}$ is a basis for $n \times n$ symmetric matrices with trace zero and $[\mathcal{B}(X)]_{ij} = \mathrm{tr}(B_i X B_j)$ then

$$\Lambda_+(E_{n-1}, I_n) = \{X \in \mathcal{S}^n \ : \ \mathcal{B}(X) \succeq 0\}$$

Corollaries

- If p has a definite determinantal representation then derivative relaxation is a spectrahedron.
- Spectrahedral rep. of $\Lambda_+(e_{n-2}, \mathbf{1}_n)$ of size $\binom{n}{2} - 1$.

Sketch of proof: "geometric"

Sanyal's representation of $\Lambda_+(e_{n-1}, \mathbf{1}_n)$

$$\Lambda_+(e_{n-1}, \mathbf{1}_n) = \{x \in \mathbb{R}^n \ : \ y^T \operatorname{diag}(x) y \geq 0 \ \text{ for all } y \in \mathbf{1}_n^\perp\}$$

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New representation of $\Lambda_+(E_{n-1}, I_n)$

$$\Lambda_+(E_{n-1}, I_n) = \{X \in \mathcal{S}^n \quad \operatorname{tr}(YXY) \geq 0 \ \text{ for all } \ Y \in I_n^\perp\}$$

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Establish this by showing

$$\Lambda_+(e_{n-1}, \mathbf{1}_n) = \{x \in \mathbb{R}^n \ : \ \operatorname{tr}(Y \operatorname{diag}(x) Y) \geq 0 \ \text{ for all } \ Y \in I_n^\perp\}$$

(diagonal of symmetric matrix is majorized by its eigenvalues)

Sketch of proof: algebraic

Polynomial identity

$$\overbrace{c\prod_{i<j}(\lambda_i(X)+\lambda_j(X))}^{q(X)}\,e_{n-1}(\lambda(X)) = \det(\mathcal{B}(X))$$

(constant $c > 0$ depends on choice of basis in definition of $\mathcal{B}$)

Sketch of proof: algebraic

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Consequence:

$$\Lambda_+(q, I) \cap \Lambda_+(E_{n-1}, I_n) = \{X : \mathcal{B}(X) \succeq 0\}.$$

Sketch of proof: algebraic

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Separate argument:

$$\Lambda_+(q, I) \supseteq \Lambda_+(E_{n-1}, I_n)$$

(Use description of $\Lambda_+(p, e)$ from Kummer et al. 2015)

Some open questions

- Are $\Lambda_+(E_k, I_n)$ spectrahedra for $k = 3, 4, \ldots, n-2$?
- Lower bounds on size of spectrahedral representations?
  (Quadratic cones: Kummer (2016))

Spectral spectrahedra

Let C be a permutation invariant spectrahedron. Is

$$\lambda^{-1}[C] = \{X : \lambda(X) \in C\}$$

a spectrahedron?

Special case of Lax conjecture since $\lambda^{-1}[C]$ a hyp. cone
(Bauschke, Güler, Lewis, Sendov 2001)

Summary

- What is the relationship between hyperbolic and semidefinite programming?
- Are hyperbolicity cones (projected) spectrahedra?
- Main result: showed explicit family of hyperbolicity cones that are spectrahedra

Preprint:

- ‘A spectrahedral representation of the first derivative relaxation of the positive semidefinite cone’ `https://arxiv.org/abs/1707.09150`

THANK YOU!