A lower bound on the positive semidefinite rank of convex bodies

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Semidefinite programming lifts

- C convex body. A semidefinite lift of C is a representation:

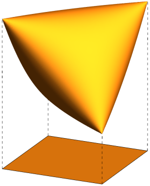
$$C = \pi(S)$$

where π linear map and S spectrahedron ($A_0, \ldots, A_n \in \mathbf{S}^m$):

$$S = \{x \in \mathbb{R}^n : A_0 + x_1 A_1 + \ldots + x_n A_n \succeq 0\}$$

Size of lift $= m$

- $\text{rank}_{\text{psd}}(C) =$ size of smallest SDP lift of C



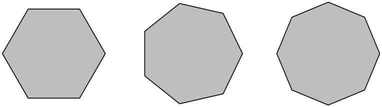
Example:

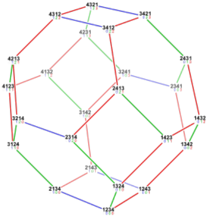
$$[-1,1]^2 = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \exists u \in \mathbb{R} \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & u \\ x_2 & u & 1 \end{bmatrix} \succeq 0 \right\}$$

Positive semidefinite rank

- Constructing SDP lifts: sum-of-squares method

- Lower bounds: Psd rank of some basic convex sets unknown (regular polygons, permutahedron, ...)





Wikipedia, "Permutohedron"

A lower bound for LP lifts

For a polytope P, let $\text{rank}_{LP}(P)$ be the size of its smallest LP lift.

Theorem (Goemans)

If P is a polytope then $\text{rank}_{LP}(P) \geq \log_2(\#\textit{vertices}(P))$.

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Proof.

Assume $P = \pi(Q)$ and Q has m facets.

- If x is a vertex of P then $\pi^{-1}(\{x\})$ is a face of Q
- Any face of Q is an intersection of facets (Q is a polytope)

Thus $\#\text{vertices}(P) \leq \#\text{faces}(Q) \leq 2^m$, i.e., $m \geq \log_2(\#\text{vertices}(P))$. □

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Bound can be tight, e.g., regular N-gon, or permutahedron

SDP lifts: a bound using quantifier elimination

Assume $C = \pi(S)$ where $(A_0, \ldots, A_n \in \mathbf{S}^m)$:

$$S = \{x \in \mathbb{R}^n : A_0 + x_1 A_1 + \ldots + x_n A_n \succeq 0\}$$

- Write explicit polynomial inequalities that describe S
- Quantifier elimination $\to$ polynomial equalities/inequalities that describe C $\to$ bound on the degree of the boundary of C.

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This approach gives (see [Gouveia, Parrilo, Thomas])

$$\text{rank}_{\text{psd}}(C) \geq \Omega\left(\sqrt{\frac{\log d}{n \log \log d}}\right)$$

where d is degree of boundary of C. **Problems:**

- Constants hard to make explicit (most likely very large)
- Tight?

Main results

If $C \subset \mathbb{R}^n$ is a convex body, the *polar* of C is

$$C^o = \{c \in \mathbb{R}^n : \langle c, x \rangle \leq 1 \; \forall x \in C\}.$$

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Theorem (Fawzi-Safey El Din)

There exist convex bodies C such that $\mathrm{rank}_{\mathrm{psd}}(C) \leq \sqrt{20 \log d}$ where the degree d of the algebraic boundary of C^o can be made arbitrary large.

Preliminaries: KKT conditions

Consider a semidefinite program

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\begin{array}{ll}
\text{maximise} & c^T x \\
\text{subject to} & A(x) := A_0 + x_1 A_1 + \cdots + x_n A_n \succeq 0 \quad (\textbf{linear matrix inequality})
\end{array}
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KKT conditions (assuming certain regularity conditions) A point x is optimal if, and only if, there exists $Z \in \mathbf{S}^m$ (Lagrange multiplier) such that

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\begin{cases}
A(x) \succeq 0, Z \succeq 0 \quad \text{(primal and dual feasibility)} \\
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Remove inequalities to get polynomial system:

$$\begin{cases} A(x)Z = 0 \quad \text{(complementary slackness)} \\ \langle A_i, Z \rangle + c_i = 0 \ (i = 1, \ldots, n) \end{cases}$$

What are the solutions of this polynomial system?

KKT system

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- [NRS] and [vBR] computed the *exact* number of solutions. More precisely, they computed number of solutions in each irreducible component of (KKT)!

[NRS] Nie-Ranestad-Sturmfels: *The algebraic degree of semidefinite programming*

[vBR] von Bothmer-Ranestad: *A general formula for the algebraic degree in SDP*

Proof of lower bound

Let C be a convex body and assume $C = \pi(S)$ where S spectrahedron.

- We exhibit a system of polynomial equations that vanishes on the boundary of ∂C^o. In fact, this system is nothing but the KKT equations. Indeed:

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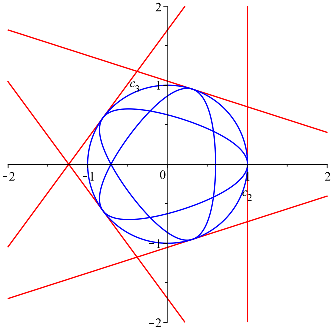
- Projecting on c we get a variety that vanishes on ∂C^o. Bézout bound tells us this variety has degree $\leq 2^{m^2}$.

Illustration of proof

$$A(x,y,s,t) = \begin{bmatrix} 1+s & t & x+s & y-t \\ t & 1-s & -y-t & x-s \\ x+s & -y-t & 1+x & -y \\ y-t & x-s & -y & 1-x \end{bmatrix}$$

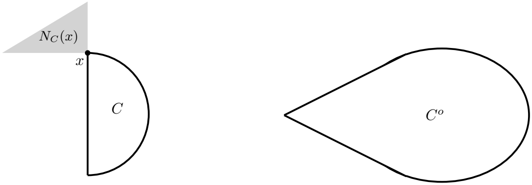
Can show that $C = \pi_{x,y}(S)$ is regular pentagon in $\mathbb{R}^2$. Variety obtained from the KKT equations:



- Red = algebraic boundary of C^o
- Blue = spurious components

Application: vertices of spectrahedra and their shadows

A *vertex* of a convex body is a point where normal cone is *full-dimensional*.



Any convex body has at most countably many vertices (see e.g., [Schneider]).

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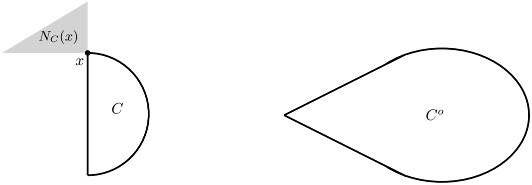

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Theorem (Fawzi-Safey El Din)

If C has a SDP representation of size m then C has at most 2^{m^2} vertices.

Proof.

Each vertex of C contributes a linear factor in the boundary of C^o. □

Tightness of bound

Theorem (Fawzi-Safey El Din)

There exist convex bodies C such that $\text{rank}_{\text{psd}}(C) \leq \sqrt{20 \log d}$ *where the degree d of the algebraic boundary of C^o can be made arbitrary large.*

Main idea

- The convex bodies C are "*random spectrahedra*" of appropriate dimension.
- For these spectrahedra, we can use the exact formulas for the degree of the KKT equations computed in:
  - Nie-Ranestad-Sturmfels: *The algebraic degree of semidefinite programming*
  - von Bothmer-Ranestad: *A general formula for the algebraic degree in SDP*

Proof of tightness

- Nie-Ranestad-Sturmfels: If C is a *generic* spectrahedron defined by $A(x) := A_0 + x_1 A_1 + \ldots + x_n A_n \in \mathbf{S}^m$ then:

$$\partial_a C^o \subseteq \bigcup_{r \in \text{Pataki range}} \mathcal{V}_r$$

where each $\mathcal{V}_r$ is irreducible and has degree $\delta(n, m, r)$.

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- Complete proof by showing (using elementary calculations) that

$$\delta(n(m), m, r(m)) \geq 2^{m^2/20}$$

  for choice

$$n = n(m) \sim m^2/4 \text{ and } r = r(m) \sim m/2.$$

Note: we observed numerically that $\delta(n(m), m, r) \geq 2^{\Omega(m^2)}$ for *all* r. Proving this would allow to prove result without using Amelunxen-Bürgisser.

Open questions

- **Polytopes:** Can we improve lower bound to $\log d$ if we assume C to be a polytope? In particular: what is the positive semidefinite rank of regular polygons in the plane?
- **Vertices:** Is the bound of 2^{m^2} on the number of vertices tight? Studying random spectrahedra as in Amelunxen-Bürgisser can be useful here...
- **Explicit:** Find *explicit* family of convex bodies that match the bound of $\sqrt{\log d}$.
- **Algebraic degree:** More systematic analysis of $\delta(n, m, r)$. Seems to have interesting properties (log-concavity, etc.) + connection with intrinsic volumes of positive semidefinite cone (cf. Amelunxen-Bürgisser).

For more, see paper on `arXiv:1705.06996`.

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