Graph Decompositions and Unique Games

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Two Results

Approximating unique games using low diameter graph decomposition,

joint work with Vedat Levi Alev

Graph clustering using effective resistance

joint work with Vedat Levi Alev, Nima Anari, Shayan Oveis Gharan

Motivation

To approximate Unique Games for graphs with many small eigenvalues.

For graphs with few small eigenvalues, it is known how to approximate Unique Games efficiently using the eigenspace enumeration technique by [Kolla].

We don't know how to solve the problem in such generality.

Well known graph class with many small Laplacian eigenvalues is the class of K_r-minor free graphs and bounded genus graphs [Kelner, Lee, Price, Teng].

Min-Uncut

Let's start with the simplest interesting case of Unique Games.

The max-cut problem:

Given an undirected graph G=(V,E), find a partition of V into two sets V_1 and V_2 and maximize the number of edges between them.

The min-uncut problem:

Given an undirected graph G=(V,E), find a partition of V into two sets V_1 and V_2 and minimize the number of edges in the same sets.

Min-Uncut

Planar graphs: Exact [Hadlock]

General graphs: $\sqrt{\log n}$ [Charikar, Makarychev, Makarychev]

K_r minor free graphs: O(r)

Graphs with genus g: O(log g)

Linear Programming Relaxation

Find the minimum number of edges to remove so that the remaining graph is bipartite.

Polynomial Time Solvability

This cycle LP has exponentially many constraints, but it can be solved in polynomial time using the ellipsoid method, using a shortest path algorithm as a separation oracle.

Remark: Many algorithms for unique games use these “labelled extended graphs”, but we cannot because it destroys K_r-minor freeness.

Low Diameter Graph Decomposition

Powerful tool in designing approximation algorithms for K_r-minor free graphs (e.g. [Bansal, Feige, Krauthgamer, Makarychev, Nagarajan, Naor, Schwartz]).

[Abraham, Gavoille, Gupta, Neiman, Talwar]

Given any weighted graph G=(V,E,w) and $\Delta>0$, there exists a distribution of partition $V=(V_1, \ldots, V_k)$ such that

1. Each edge is cut with probability at most $O(r)\, w_e / \Delta$.
2. The diameter of each part is at most Δ, i.e.

Rounding

Want: Cut into bipartite components.

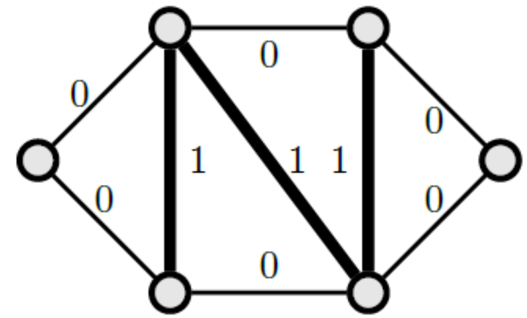
Know: Cut into low diameter components.

Hope: low diameter components => bipartite components.

- Set $w_e = x_e$
- The odd cycle constraint says that each cycle is of length at least one.
- If we could show that there exists a pair of vertices u,v in C such that dist(u,v) > 1/4, then we can run the low diameter graph decomposition with Δ = 1/4 and cut all odd cycles.
- Then the expected cost is O(r LP).

Rounding

But the hope is not always true.



We observe that edges with large fractional value is the only obstruction.

Lemma. If x_e <= 1/2 for all e, then for every odd cycle there exists a pair of vertices u,v with dist(u,v) > 1/4.

Then we first pick all edges with x_e >= 1/2, and then run decomposition.

Proof by Picture

Lemma. If x_e <= 1/2 for all e, then for every odd cycle there exists a pair of vertices u,v with dist(u,v) > 1/4.

Unique Games Conjecture

An important special case of unique games is 2LIN(R).

We are given linear equations of the form $x_i + x_j \equiv c_{ij} \pmod{R}$.

[Khot] For every ϵ, there exists R such that for 2LIN(R), it is NP-hard to distinguish the following two cases:

- YES: there is an assignment that satisfies $1-\epsilon$ fraction of constraints.
- NO: no assignments satisfies more than ϵ fraction of constraints.

Linear Unique Games

We can generalize the cycle cutting algorithm by defining “inconsistent cycles”, and get the same results as in min-uncut:

- K_r minor free graphs: O(r)
- Graphs with genus g: O(log g)

The approximation ratio is independent of R.

General Unique Games

We first use the low diameter graph decomposition result in [Abraham, Gavoille, Gupta, Neiman, Talwar], and then find a shortest path tree in each component, and do the propagation rounding in [Gupta, Talwar].

- K_r minor free graphs: $O(r\sqrt{\epsilon})$
- Graphs with genus g: $O(\sqrt{\log(g)}\,\epsilon)$

Graph Decomposition

To design an algorithm for the unique games conjecture, we can afford to delete say 1% of the edges.

We know that Unique Games are "easy" on expander graphs [Arora, Khot, Kolla, Steurer, Tulsiani, Vishnoi], and on graphs with bounded diameter [Gupta, Talwar].

Suppose we can delete 1% of edges such that each remaining component is of constant expansion or constant diameter, then done.

The subexponential time algorithm by [Arora,Barak,Steurer] is to decompose the graph into graphs with few small Laplacian eigenvalues.

Graph Decomposition

But there are graphs (e.g. hypercube) such that if we delete constant fraction of edges, we can only ensure that each component

- is of conductance $\Omega(1 / \log n)$.
- is of diameter $O(\log n)$.

This log(n) loss is too bad for the unique games problem.

Question: Is there a "relaxed property" which is close to high conductance and low diameter so that we can prove a stronger graph decomposition result (without a log(n) factor loss)?

Effective Resistance

We think about the weighted graph as an electrical network.

Each edge is of conductance w(e), or of resistance 1/w(e).

The effective resistance between two vertices u and v is defined as the potential difference between u and v when one unit of current is injected to u and extracted from v.

Equivalently, it is the minimum energy of a unit flow from u to v, where the energy of a flow is defined as $\sum_{e\in E} r_e f_e^2$.

Effective Resistance as a Relaxed Property

It is easy to see that Reff(u,v) <= dist(u,v).

It is also known that Reff(u,v) = O(1) in a constant expansion graph.

So it can be seen as a relaxed property of low diameter and high expansion.

And it is intuitively a good relaxation, as we usually think of Reff(u,v) being small as saying that there are many disjoint short paths between u and v.

Applications of Effective Resistance

Traditionally, effective resistance is closely related to some parameters of random walks, such as hitting time [Chandra, Raghavan, Ruzzo, Smolensky, Tiwari], cover time [Matthews], and the probability of an edge in a random spanning tree [Kirchhoff].

Recently, it is used in spectral sparsification [Spielman, Srivastava], computing maximum flow [Christiano, Kelner, Madry, Spielamn, Teng], finding thin tree [Anari, Oveis Gharan], and generating random spanning trees [Kelner, Madry] [Madry, Straszak, Tarnawski].

It may also be a natural distance measure to do graph clustering.

Graph Decomposition using Reff

Theorem. For every weighted graph G=(V,E,w) and large enough constant $\Delta>0$, there exists a partition $V=(V_1, \ldots, V_k)$ such that

1. The resistance diameter is small, i.e.

$$\max_{u,v\in V_i} Reff_{G[V_i]}(u,v) \leq \Delta n / w(E).$$

2. Few edges between clusters, i.e.

$$w(E - \cup E(V_i)) \leq w(E) / \Delta^{1/3}$$

d-Regular Graphs

Corollary. For every unweighted d-regular graph G=(V,E) and large enough constant $\Delta>0$, there exists a partition V=(V_1, ..., V_k) such that

1. Each component is of resistance diameter $O(\Delta/d)$.
2. Only $O(|E|/\Delta^{1/3})$ edges are deleted.

Note that any d-regular graph has resistance diameter $\Omega(1/d)$, and d-regular expander has resistance diameter $O(1/d)$. So, although we cannot decompose a graph into expander, we can decompose a graph into graphs with essentially optimal electrical property.

Technical Result

Theorem. If $w(\delta(S)) \geq c\, vol(S)^{1/2+\epsilon}$

for every some $\epsilon > 0$ and for all S separating u and v, then

$$Reff(u,v) \leq \left(1/\deg(u)^{2\epsilon} + 1/\deg(v)^{2\epsilon}\right) 1/c^2 \epsilon .$$

The main point is that the expansion requirement for large sets is very mild (i.e. sublinear).

Proof Ideas

The technical result is to look at the potential vectors, and argue that if each level set is of mild expansion, then the potential will drop slowly. The argument is a little similar to some proofs of Cheeger's inequality.

The graph decomposition result is to apply the technical result recursively, and since the conductance is sublinear, it is a geometric decreasing sequence (e.g. we don't lose a log(n) factor).

Open Questions

1. Can the decomposition be computed in near linear time?

2. Are there some problems that can be solved effectively in graphs with bounded resistance diameter?

Thank you.