LP Rounding for Poise

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https://arxiv.org/abs/1612.01492 "Plane Gossip"

Minimum Poise Trees

- Poise of a tree = diameter of T + max degree of T
- Known [R'94]: Given an n-node graph with a spanning tree of poise P, poly-time algo to find one of poise $O(P \log n)$
- This talk: Given an n-node graph with an LP relaxation value for poise LP_P, poly-time algo to find one of poise $O(LP_P \log n)$

- Open: Given undirected graph, find a spanning tree of poise $O(\mathrm{P})$

Outline

- Approximating poise
  - Matching Based Augmentation [R, LATIN'06]
- LP Rounding
  - T-path packing
  - Dependent Flow Rounding
- Application
  - Approximating multicommodity multicast in planar graphs

Matching Based Augmentation

- Iterative construction heuristic: Subgraph added at each iteration identified by examining the optimal solution (typically a matching variant)
- Each iteration’s cost related to that of optimal (Performance ratio is of the order of the number of iterations)

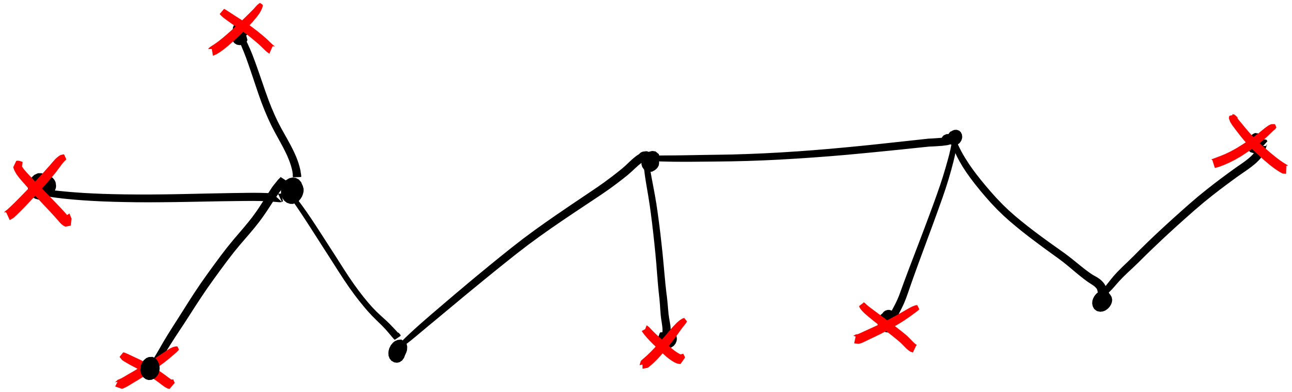
Example: $\log_2 n$ approximation for Minimum Spanning Tree

- Begin with nodes in singleton clusters
- While more than two clusters
  - Compute a minimum-cost matching M between clusters
  - Add M to the solution
  - Merge clusters connected by matching edges

Why is cost(M) ≤ cost(MST)?

A Tree-Pairing Lemma

- Given an even number of nodes of a tree, there is a pairing of these nodes such that the tree-paths between the pairs are edge-disjoint



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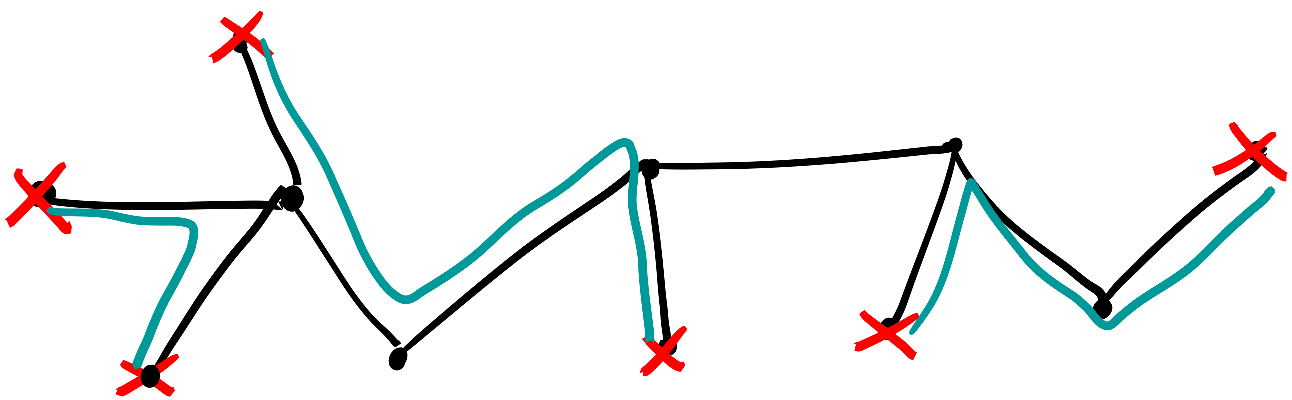


Pairing minimizing the total path length has this property

Proposed Application

- Identify representatives in each connected component of the current solution
- Use lemma to pair them up in a hypothetical optimal solution
- Infer the resulting matching problem that needs to be solved to augment the current solution (to halve the number of components)

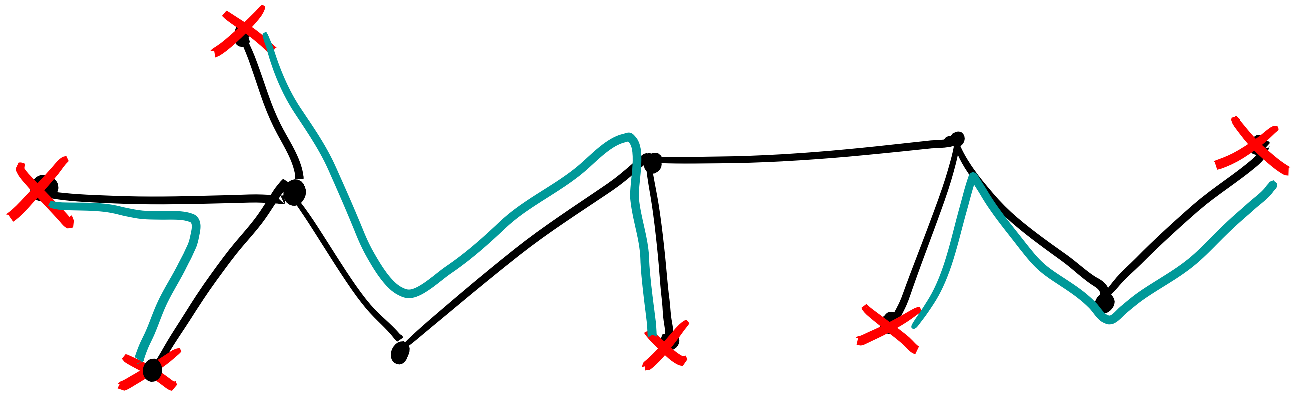
Let’s try MBA on Poise

Simpler formulation:

Given G, find spanning tree T Min Max-degree(T)

s.t. T-path-length between any pair of nodes $\leq$ D

What are the requirements of the matching subproblem?



Diameter-bounded min-degree trees

Min Max-deg(T) s.t. max path length $\leq$ L

Matching subproblem for MBA:

Match representatives using paths to

- Minimize node congestion due to paths
- All matching paths are of length at most L

Algorithm Sketch

- Start with empty subgraph, all nodes are reps
- Iterate until connected
  - Set up a length-L bounded min-degree matching problem on current reps, solve and add to the solution
  - Pick a rep for each component

Additional Complication

Diameter is not additive in its effect on objective like degree

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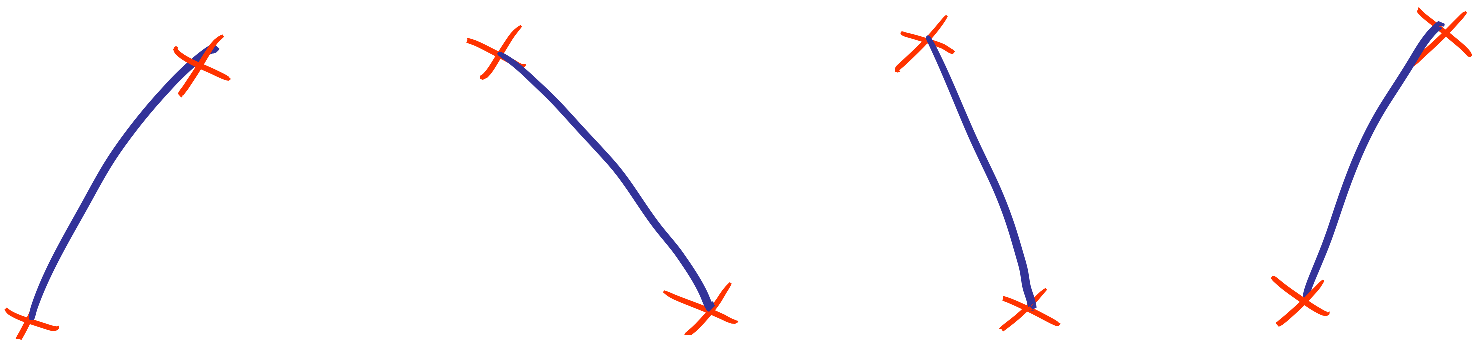

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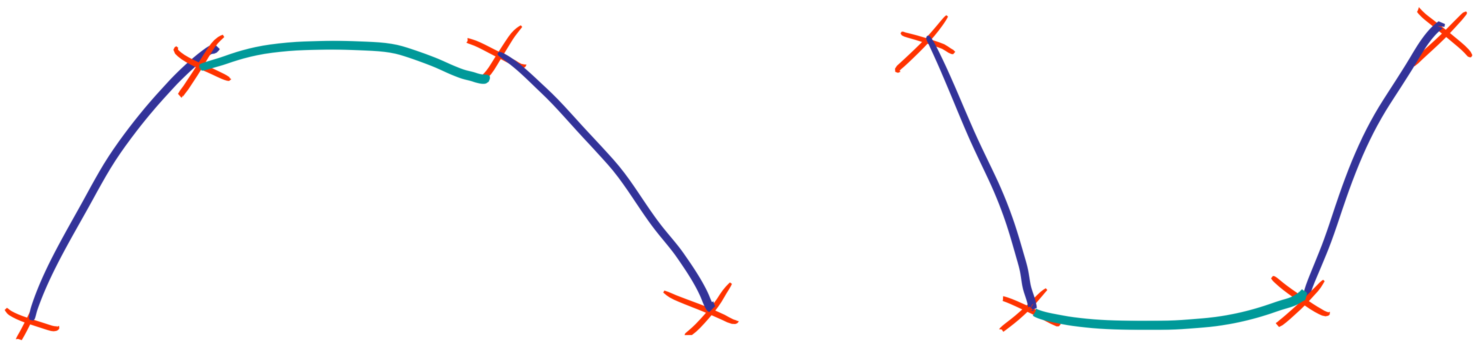

… and can grow despite each subgraph being bounded in diameter

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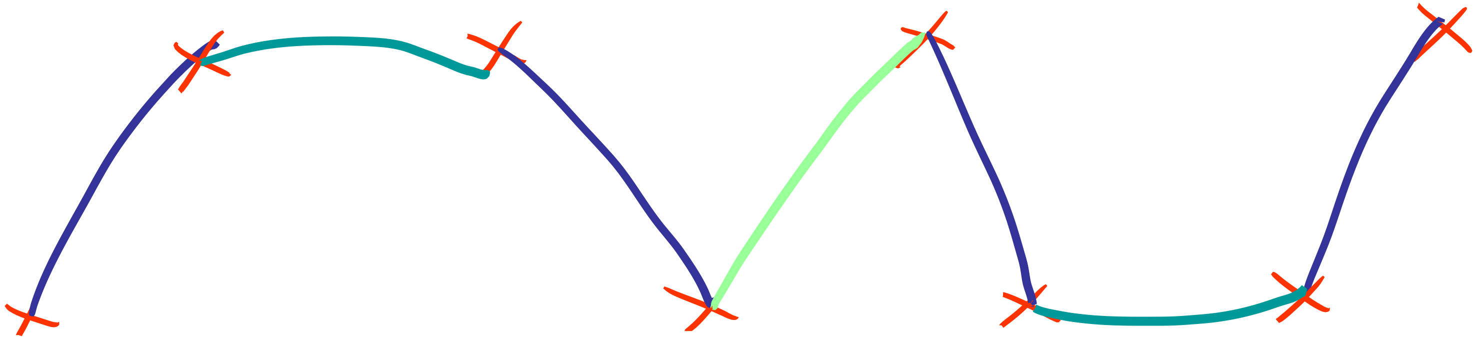

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Simple fix

Promote one rep from each pair to bound diameter by number of iterations

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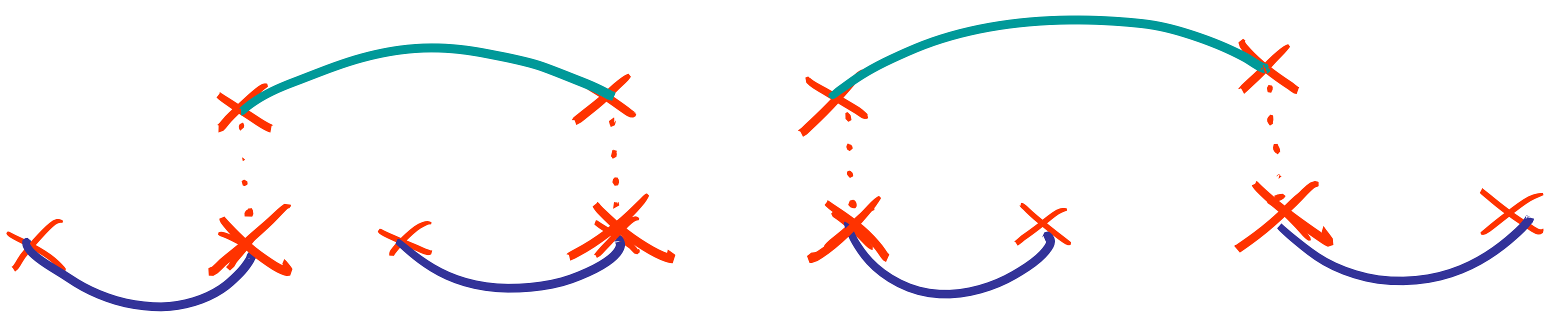

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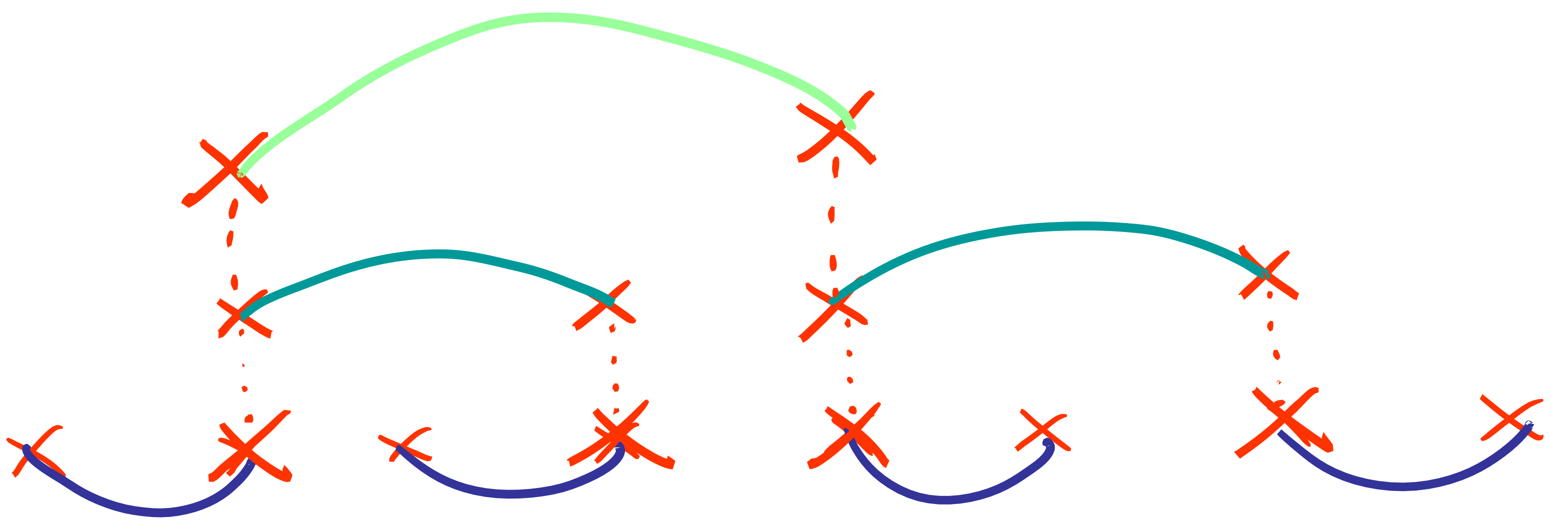

Algorithm

- Start with empty subgraph, all nodes are reps
- Iterate until connected
  - Set up a length-L bounded min-degree matching problem on current reps, solve and add to the solution
  - Retain one rep from each matched pair
- Choose a minimum-diameter tree of the final subgraph

Diameter-bounded min-degree trees

Min Max-deg(T) s.t. max path length $\leq$ L

- MBA-technique using the tree pairing lemma leads to a minimum node-congestion bounded-length path matching problem (Can be approximated well)
- Promoting one of two representatives in a cluster ensures bounded diameter growth

Theorem: MBA algorithm gives a spanning tree within $O(\log n)$ of the minimum poise of the optimal tree

Diameter-bounded min-degree trees

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- MBA-technique using the tree pairing lemma leads to a minimum node-congestion bounded-length path matching problem (Can be approximated well)
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Goal: MBA algorithm that gives a spanning tree within $O(\log n)$ of an LP for minimum poise of optimal tree

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LP for Steiner Poise L

$$
\begin{array}{lll}
\text{minimize} & L = L_1 + L_2 & (POISE-L) \\
\text{subject to} & \sum_{e\in\delta(v)} x(e) \le L_1 & \forall v \in V \\
 & \sum_{P\in\mathcal{P}(t,r)} y_t(P) = 1 & \forall t \in R \\
 & \sum_{P\in\mathcal{P}(t,r)} \ell(P) y_t(P) \le L_2 & \forall t \in R \\
 & \sum_{P\in\mathcal{P}(t,r):e\in P} y_t(P) \le x(e) & \forall e \in E, t \in R \\
 & x(e) \in \{0,1\} \text{ for } e \in E, & \\
 & y_t(P) \in \{0,1\} \text{ for } t \in R, P \in \mathcal{P}_L(t,r). &
\end{array}
$$

x = Choice of edges in low poise subgraph

y = Choice of path from terminals to root

Use MBA strategy for poise

- Merge clusters containing terminals to reduce by constant fraction in each phase
  - Compute matchings between cluster centers
  - Path lengths used in matching and node degrees both at most target L
  - Promote one of two centers as new cluster center to ensure bounded diameter for final cluster

How to connect reps using a paths of small length and node congestion extracted from the LP?

T-path packing

Given undirected Eulerian graph, terminals T, a path between two distinct terminals is a T-path

Theorem [Lovasz 1976, Cherkassy 1976]: If connectivity from t in T to any other terminal is C(t), can find a packing of T-paths of cardinality

$$\sum_{\{t \in T\}} \frac{C(t)}{2}$$

(Best possible)

Use a Filtered LP solution

- Convert LP solution x to POISE LP to a multigraph with inter-terminal connectivity M
- $C(t) \geq M$ for each terminal t, so apply Theorem
- However paths may be much longer than L
- Filter away paths longer than 4L and argue a constant fraction still retained (Why?)
- Scale by 1/M and convert to a flow packing
- Use flow packing for dependent rounding

Rounding Flow Paths

Given fractional flow paths of length at most 4L causing expected node congestion at most 2L from each of the selected centers, round into one integral flow path per center.

Use [KarpLRTVV, Algorithmica'87] which bounds additive violation of congestion by the column density of the packing system

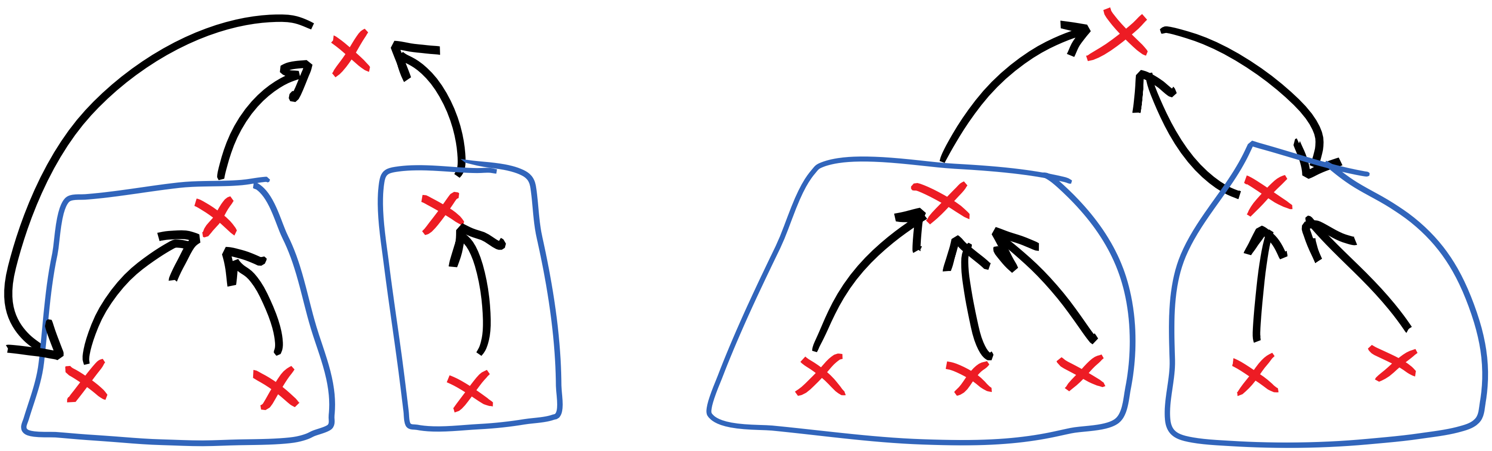
$$\sum_{\{p \in P(r,t)\}} y(p) = 1 \; \forall \, t$$

$$\sum_{t} \sum_{\{v \in p : p \in P(r,t)\}} y(p) \; \leq 2L_c \;\; \forall \, v$$

By filtering, the column density is at most $4L_d$

Details of MBA

- Solution to rounded LP is one path out of a constant fraction of cluster centers
- Build auxiliary graph and retain collection of stars
- Elect star center as cluster center to retain low diameter over phases



New LP Rounding via T-paths

LP rounding of natural formulation for minimum poise Steiner trees

$$O(\mathrm{P}^{*}\log k)$$

k = number of source-sink pairs

P^{*} = LP optimum for minimum poise

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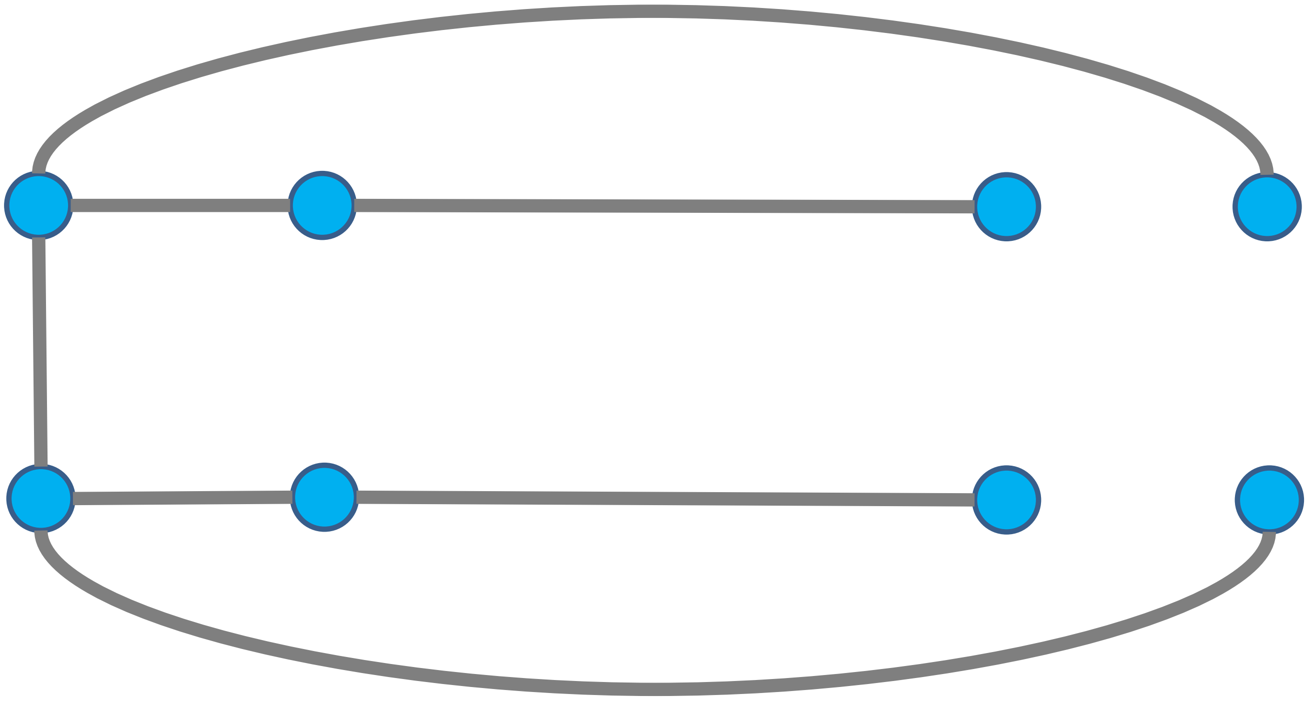
Broadcast



Minimum Telephone Multicast Time Problem

Given:

- A graph $G(V, E)$
- A source node r and a set of terminal nodes R

Inform the terminals of the message of r.

How?

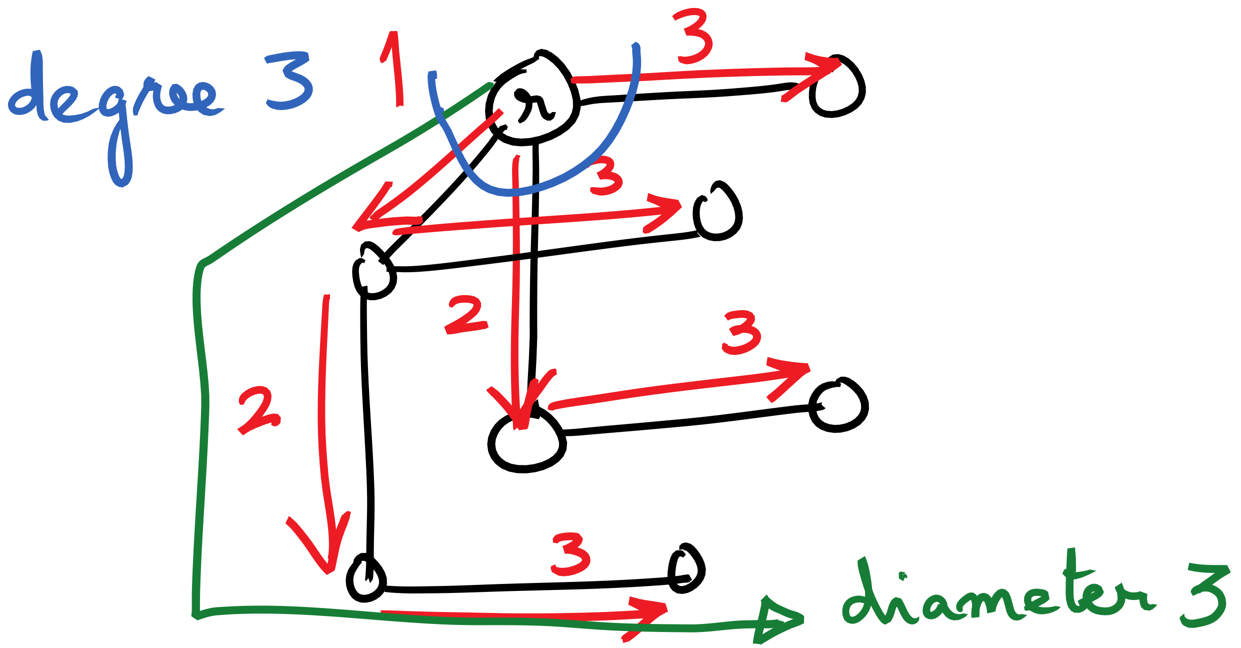
- Disjoint pairs of adjacent vertices exchange information in rounds

Goal:

- Use the minimum number of rounds to inform R

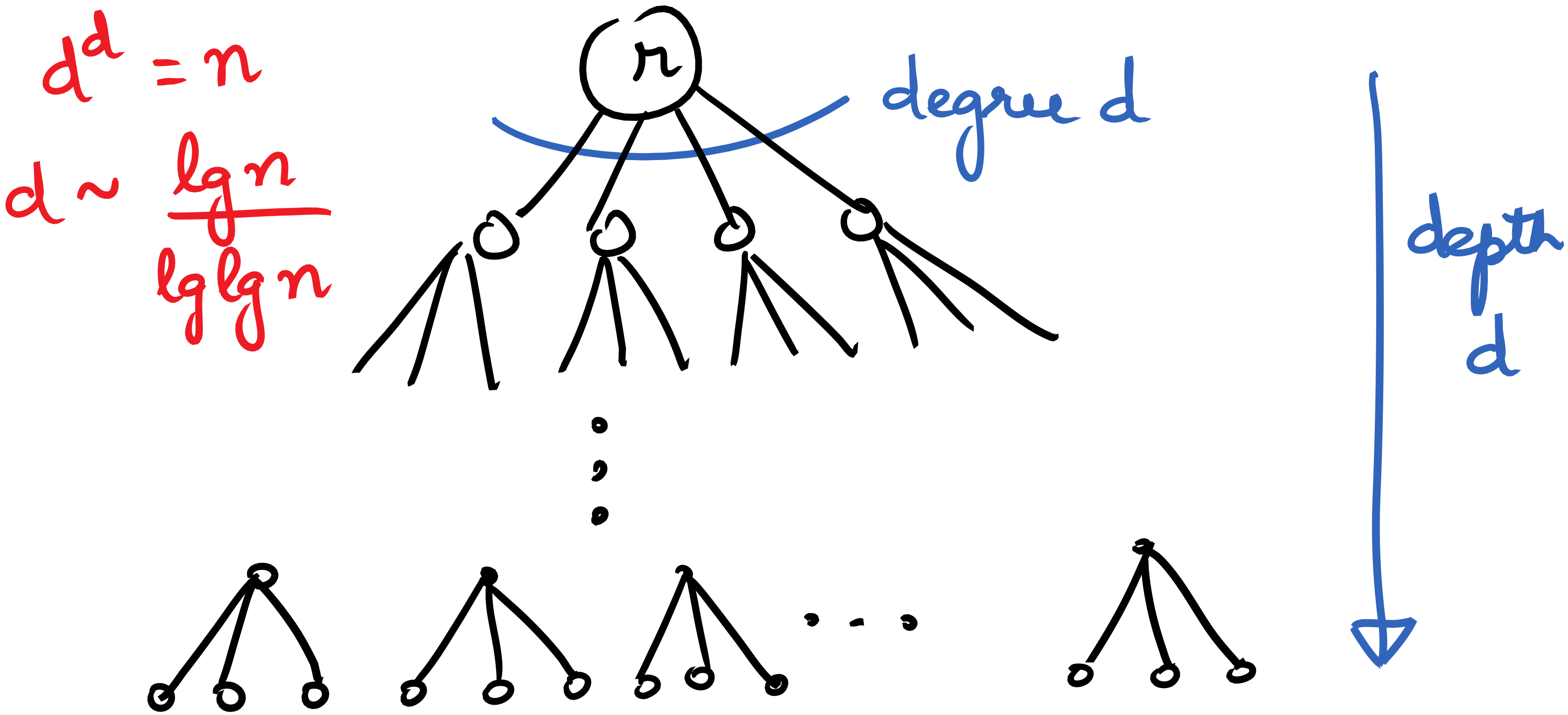
A new spanning tree objective

- Use critical arcs used in broadcast to define a r-arborescence
- Diameter and max out-degree are both lower bounds on broadcast time

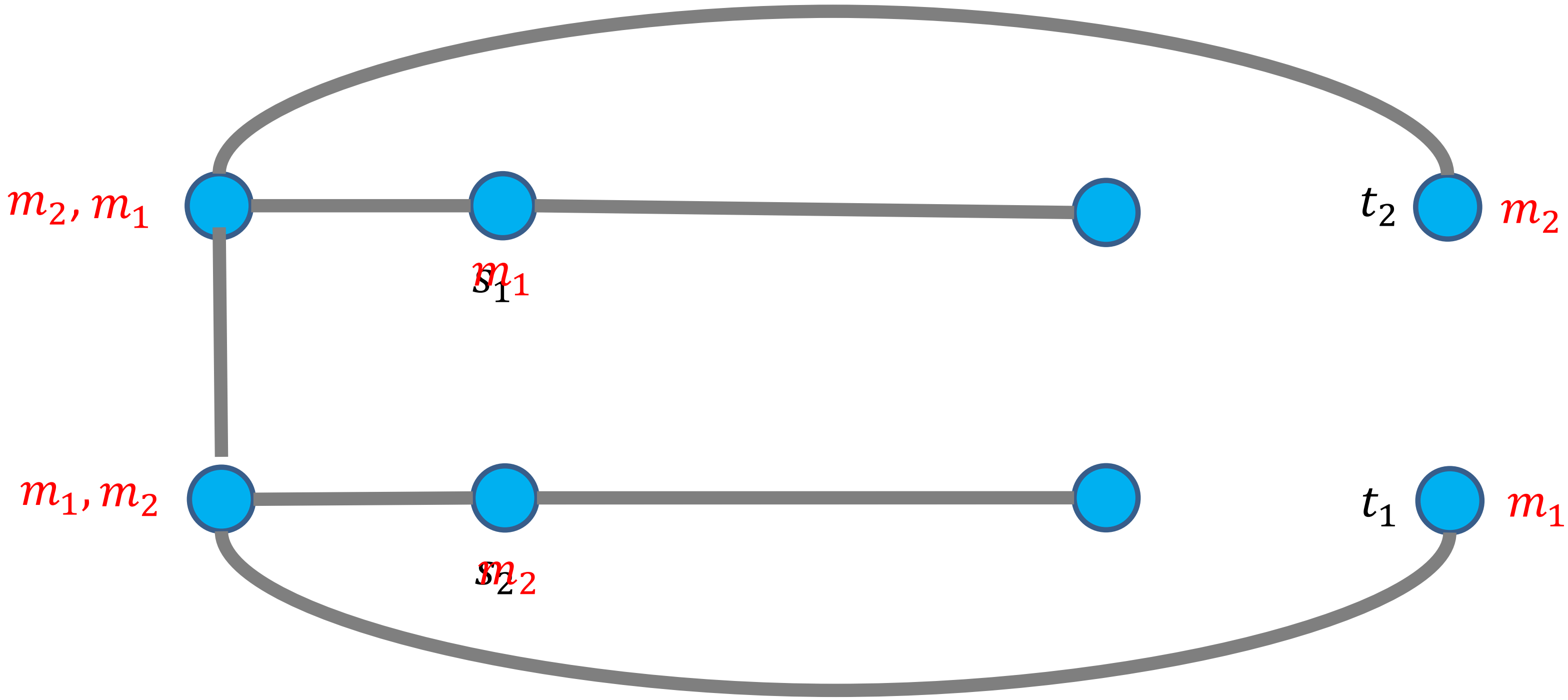


Broadcasting with Min Poise Trees

Lemma [Ravi'94]: Given a tree of poise P, can find a telephone broadcast scheme from any root within time $O\left(P\frac{\log n}{\log\log n}\right)$



Multi-commodity Multicast



Related Work

- Broadcast: $\frac{\log^2 n}{\log \log n}$ -approximation [Ravi, FOCS94]
- Improvements to $\log n$ [Guha, BarNoy, Naor, Schieber, STOC98] and $\frac{\log n}{\log \log n}$ [Elkin, Kortsarz, SODA03]
- Lower bound of $3 - \epsilon$ for undirected multicast [Elkin, Kortsarz, STOC02]
- Multicommodity multicast: $O(2^{\log \log k \cdot \sqrt{\log k}})$-approximation [Nikzad, Ravi ICALP14] (k = number of source-sink pairs)

New Result: Planar Telephone Multicast

Planar graphs

Multi-commodity Multicast

Poly-log approximation ratio

$$\mathrm{O}(OPT\ \log^3\ k\ \log n)$$

k = number of source-sink pairs

n = number of nodes

Crucially uses poise approximation from LP

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Open Problems

- O(1) Integrality gap for Steiner poise, gen Steiner poise?
- Constant-factor approximation algorithms for broadcast
- Better relation between poise and multicast time
- Improved (poly-log) approximation for multicommodity multicast in general graphs