Implementing Randomized Matrix Algorithms in Parallel and Distributed Environments

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Outline

Goal: *very* large-scale "vector space analytics"

Small-scale and **medium-scale**:

- Model data by graphs and matrices
- Compute eigenvectors, correlations, etc. in RAM

Very **large-scale**:

- Model data with flat tables and the relational model
- Compute with join/select and other "counting" in, e.g., Hadoop

Can we "bridge the gap" and do "vector space computations" at very large scale?

- Not obviously yes: exactly computing eigenvectors, correlations, etc. is subtle and uses lots of comminication.
- Not obviously no: lesson from random sampling algorithms is you can get ϵ-approximation of optimal with very few samples.

Why randomized matrix algorithms?

Traditional matrix algorithms (direct & iterative methods, interior point, simplex, etc.) are designed to work in RAM and their performance is measured in floating-point operations per second (FLOPS).

- **Traditional algorithms** are NOT well-suited for:
  - problems that are very large
  - distributed or parallel computation
  - when communication is a bottleneck
  - when the data must be accessed via "passes"

- **Randomized matrix algorithms** are:
  - faster: better theory
  - simpler: easier to implement
  - implicitly regularize: noise in the algorithm avoids overfitting
  - inherently parallel: exploiting modern computer architectures
  - more scalable: modern massive data sets

Traditional algorithms

- **for ℓ_2 regression**:
  - direct methods: QR, SVD, and normal equation ($O(mn^2 + n^2)$ time)
    - ★ Pros: high precision & implemented in LAPACK
    - ★ Cons: hard to take advantage of sparsity & hard to implement in parallel environments
  - iterative methods: CGLS, LSQR, etc.
    - ★ Pros: low cost per iteration, easy to implement in some parallel environments, & capable of computing approximate solutions
    - ★ Cons: hard to predict the number of iterations needed
- **for ℓ_1 regression**:
  - linear programming
  - interior-point methods (or simplex, ellipsoid? methods)
  - re-weighted least squares
  - first-order methods

Nearly all traditional algorithms for low-rank matrix problems, continuous optimization problems, etc. boil down to variants of these methods.

Over-determined/over-constrained regression

An ℓ_p *regression problem* is specified by a design matrix $A \in \mathbb{R}^{m \times n}$, a response vector $b \in \mathbb{R}^m$, and a norm $\|\cdot\|_p$:

$$\text{minimize}_{x \in \mathbb{R}^n} \quad \|Ax - b\|_p.$$

Assume $m \gg n$, i.e., many more "constraints" than "variables." Given an $\epsilon > 0$, find a $(1+\epsilon)$-approximate solution $\hat{x}$ in relative scale, i.e.,

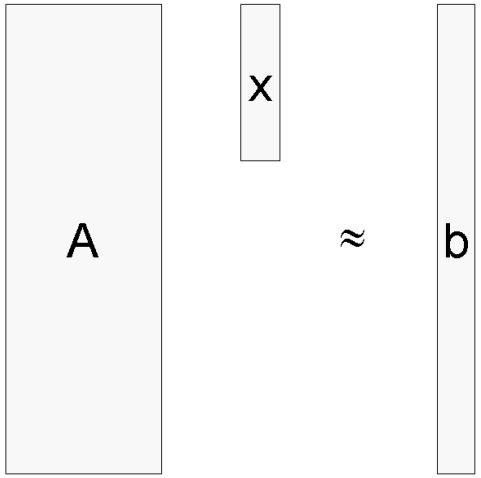
$$\|A\hat{x} - b\|_p \leq (1+\epsilon)\|Ax^* - b\|_p,$$

where x^* is a/the optimal solution.

- $p = 2$: *Least Squares Approximation:* Very widely-used, but highly non-robust to outliers.
- $p = 1$: *Least Absolute Deviations:* Improved robustness, but at the cost of increased complexity.

Large-scale environments and how they scale

- Shared memory
  - cores: $[10, 10^3]^*$
  - memory: $[100\text{GB}, 100\text{TB}]$
- Message passing
  - cores: $[200, 10^5]^\dagger$
  - memory: $[1\text{TB}, 1000\text{TB}]$
  - CUDA cores: $[5 \times 10^4, 3 \times 10^6]^\ddagger$
  - GPU memory: $[500\text{GB}, 20\text{TB}]$
- MapReduce
  - cores: $[40, 10^5]^\S$
  - memory: $[240\text{GB}, 100\text{TB}]$
  - storage: $[100\text{TB}, 100\text{PB}]^\P$
- Distributed computing
  - cores: $[-, 3 \times 10^5]^\|$.



*http://www.sgi.com/pdfs/4358.pdf

†http://www.top500.org/list/2011/11/100

‡http://i.top500.org/site/50310

§http://www.cloudera.com/blog/2010/04/pushing-the-limits-of-distributed-processing/

¶http://hortonworks.com/blog/an-introduction-to-hdfs-federation/

‖http://fah-web.stanford.edu/cgi-bin/main.py?qtype=osstats

Outline

Two important notions: leverage and condition

(Mahoney, "Randomized Algorithms for Matrices and Data," FnTML, 2011.)

- **Statistical leverage.** (Think: eigenvectors. Important for low-precision.)
  - The *statistical leverage scores* of A (assume $m \gg n$) are the diagonal elements of the projection matrix onto the column span of A.
  - They equal the ℓ_2-norm-squared of any orthogonal basis spanning A.
  - They measure:
    - ★ how well-correlated the singular vectors are with the canonical basis
    - ★ which constraints have largest "influence" on the LS fit
    - ★ a notion of "coherence" or "outlierness"
  - Computing them exactly is as hard as solving the LS problem.

- **Condition number.** (Think: eigenvalues. Important for high-precision.)
  - The *ℓ_2-norm condition number* of A is $\kappa(A) = \sigma_{\max}(A)/\sigma^{+}_{\min}(A)$.
  - $\kappa(A)$ bounds the number of iterations; for ill-conditioned problems (e.g., $\kappa(A) \approx 10^6 \gg 1$), the convergence speed is very slow.
  - Computing $\kappa(A)$ is generally as hard as solving the LS problem.

These are for the ℓ_2-norm. Generalizations exist for the ℓ_1-norm.

Meta-algorithm for ℓ_2-norm regression

(Drineas, Mahoney, etc., 2006, 2008, etc., starting with SODA 2006; Mahoney FnTML, 2011.)

1: Using the ℓ_2 statistical leverage scores of A, construct an importance sampling distribution $\{p_i\}_{i=1}^m$.

2: Randomly sample a small number of constraints according to $\{p_i\}_{i=1}^m$ to construct a subproblem.

3: Solve the ℓ_2-regression problem on the subproblem.

A naïve version of this meta-algorithm gives a $1+\epsilon$ relative-error approximation in roughly $O(mn^2/\epsilon)$ time (DMM 2006, 2008). (Ugh.)

Meta-algorithm for ℓ_2-norm regression, cont.

(Drineas, Mahoney, etc., 2006, 2008, etc., starting with SODA 2006; Mahoney FnTML, 2011.[‡‡])

We can make this meta-algorithm "fast" in RAM:[**]

- This meta-algorithm runs in $O(mn \log n/\epsilon)$ time in RAM if:
  - we perform a Hadamard-based random projection and sample uniformly in a randomly rotated basis, or
  - we quickly computing approximations to the statistical leverage scores and using those as an importance sampling distribution.

We can make this meta-algorithm "high precision" in RAM:[††]

- This meta-algorithm runs in $O(mn \log n \log(1/\epsilon))$ time in RAM if:
  - we use the random projection/sampling basis to construct a preconditioner and couple with a traditional iterative method.

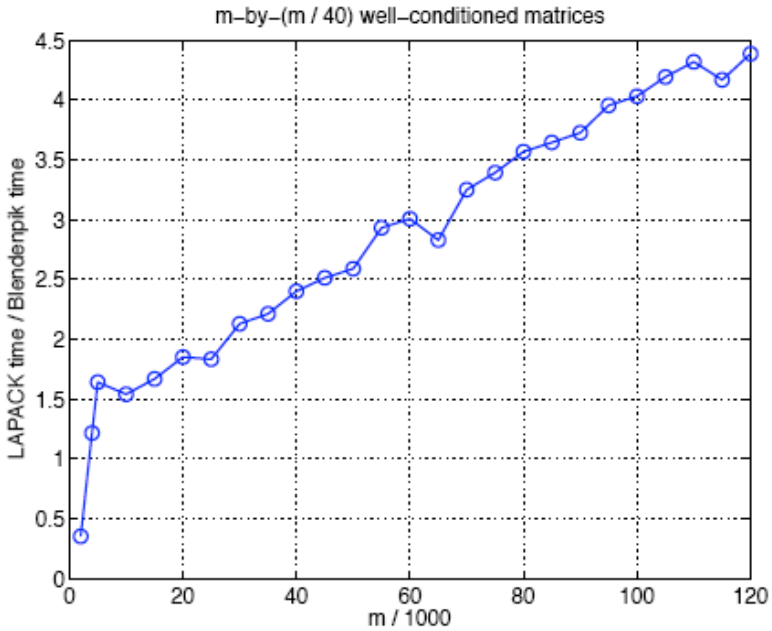
[**] (Sarlós 2006; Drineas, Mahoney, Muthu, Sarlós 2010; Drineas, Magdon-Ismail, Mahoney, Woodruff 2011.)

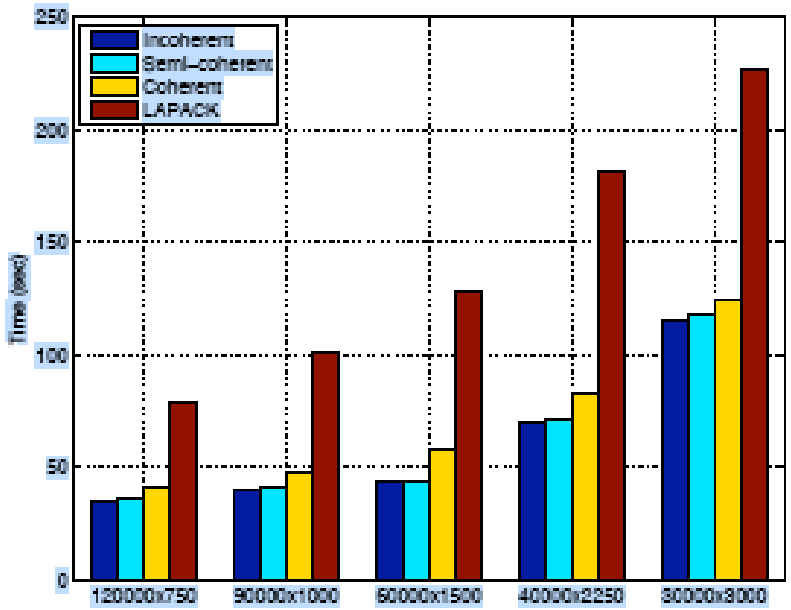
[††] (Rokhlin & Tygert 2008; Avron, Maymounkov, & Toledo 2010; Meng, Saunders, & Mahoney 2011.)

[‡‡] (Mahoney, "Randomized Algorithms for Matrices and Data," FnTML, 2011.)

Randomized regression in RAM: Implementations

Avron, Maymounkov, and Toledo, SISC, 32, 1217–1236, 2010.



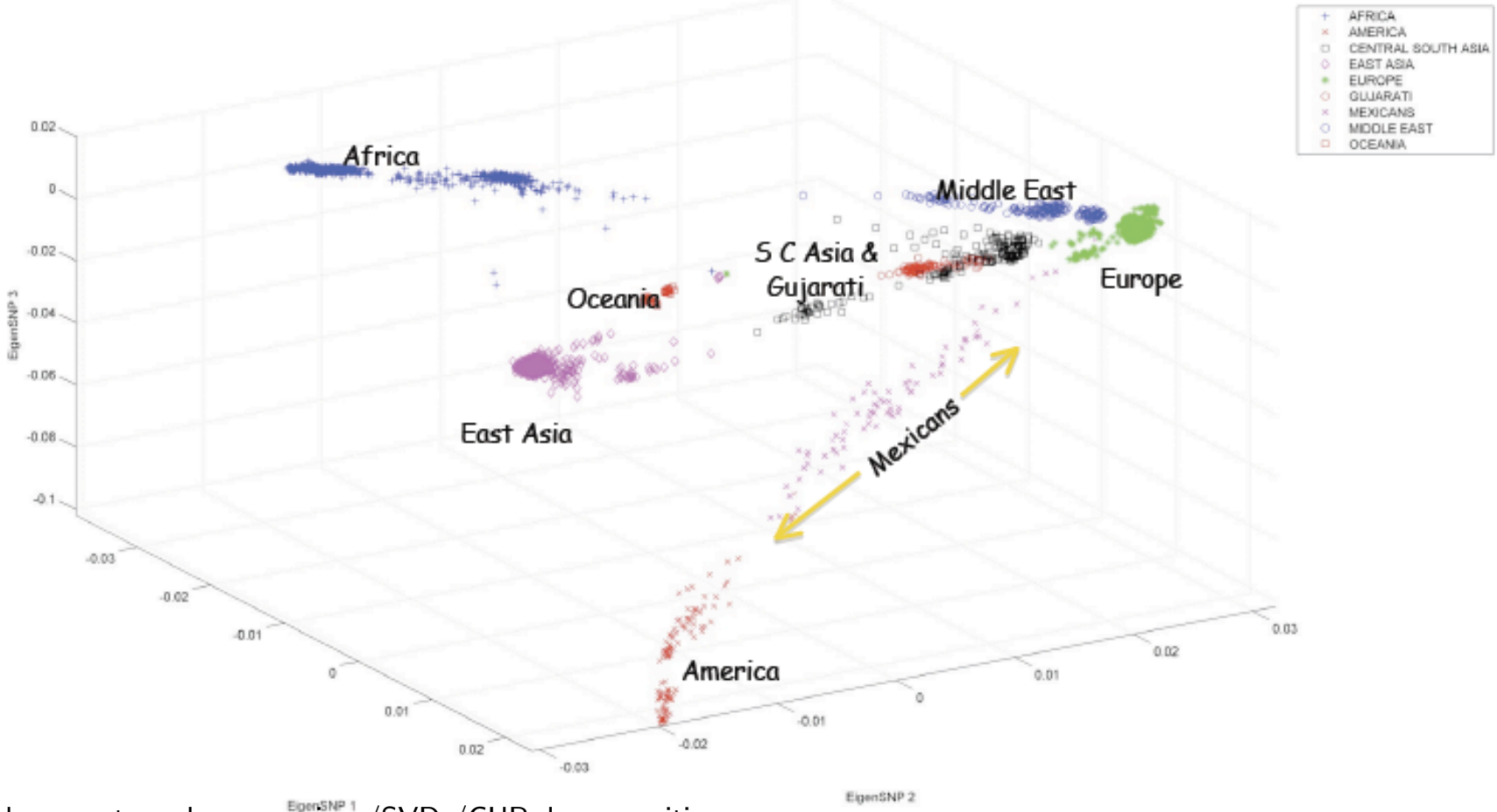


Conclusions:

- *Randomized algorithms "beats Lapack's direct dense least-squares solver by a large margin on essentially any dense tall matrix."*
- *These results "suggest that random projection algorithms should be incorporated into future versions of Lapack."*

Randomized regression in RAM: Human Genetics

Paschou et al., PLoS Gen '07; Paschou et al., J Med Gen '10; Drineas et al., PLoS ONE '10; Javed et al., Ann Hum Gen '11.



Computing large rectangular regressions/SVDs/CUR decompositions:

- In commodity hardware (e.g., a 4GB RAM, dual-core laptop), using MatLab 7.0 (R14), the computation of the SVD of the dense $2,240 \times 447,143$ matrix A takes about 20 minutes.
- Computing this SVD is not a one-liner—we can not load the whole matrix in RAM (runs out-of-memory in MatLab).
- Instead, compute the SVD of AAT.
- In a similar experiment, compute 1,200 SVDs on matrices of dimensions (approx.) $1,200 \times 450,000$ (roughly, a full leave-one-out cross-validation experiment).

A retrospective

Randomized matrix algorithms:

- BIG success story in high precision scientific computing applications and *large-scale* statistical data analysis!
- Can they *really* be implemented in parallel and distributed environments for *LARGE-scale* statistical data analysis?

Outline

Algorithm LSRN (for strongly over-determined systems)

(Meng, Saunders, and Mahoney 2011)

1: Choose an oversampling factor $\gamma > 1$, e.g., $\gamma = 2$. Set $s = \lceil \gamma n \rceil$.
2: Generate $G = \text{randn}(s, m)$, a Gaussian matrix.
3: Compute $\tilde{A} = GA$.
4: Compute $\tilde{A}$'s economy-sized SVD: $\tilde{U}\tilde{\Sigma}\tilde{V}^T$.
5: Let $N = \tilde{V}\tilde{\Sigma}^{-1}$.
6: Iteratively compute the min-length solution $\hat{y}$ to

$$\text{minimize}_{y \in \mathbb{R}^r} \quad \|ANy - b\|_2.$$

7: Return $\hat{x} = N\hat{y}$.

Why we choose Gaussian random projection

(Meng, Saunders, and Mahoney 2011)

Gaussian random projection

- has the best theoretical result on conditioning,
- can be generated super fast,
- uses level 3 BLAS on dense matrices,
- speeds up automatically on sparse matrices and fast operators,
- still works (with an extra "allreduce" operation) when A is partitioned along its bigger dimension.

So, although it is "slow" (compared with "fast" Hadamard-based projections i.t.o. FLOPS), it allows for better communication properties.

Theoretical properties of LSRN

(Meng, Saunders, and Mahoney 2011)

- In exact arithmetic, $\hat{x} = x^*$ almost surely.
- The distribution of the spectrum of AN is the same as that of the pseudoinverse of a Gaussian matrix of size $s \times r$.
- $\kappa(AN)$ is independent of all the entries of A and hence $\kappa(A)$.
- For any $\alpha \in (0, 1 - \sqrt{r/s})$, we have

$$\mathcal{P}\left(\kappa(AN) \leq \frac{1+\alpha+\sqrt{r/s}}{1-\alpha-\sqrt{r/s}}\right) \geq 1 - 2e^{-\alpha^2 s/2},$$

  where r is the rank of A.

So, if we choose $s = 2n \geq 2r$, we have $\kappa(AN) < 6$ w.h.p., and hence we only need around 100 iterations to reach machine precision.

Implementation of LSRN

(Meng, Saunders, and Mahoney 2011)

- Shared memory (C++ with MATLAB interface)
  - Multi-threaded ziggurat random number generator (Marsaglia and Tsang 2000), generating 10^9 numbers in less than 2 seconds using 12 CPU cores.
  - A naïve implementation of multi-threaded dense-sparse matrix multiplications.

- Message passing (Python)
  - Single-threaded BLAS for matrix-matrix and matrix-vector products.
  - Multi-threaded BLAS/LAPACK for SVD.
  - Using the Chebyshev semi-iterative method (Golub and Varga 1961) instead of LSQR.

Solving real-world problems

matrix	m	n	nnz	rank	cond	DGELSD	$A\backslash b$	Blendenpik	LSRN
landmark	71952	2704	1.15e6	2671	1.0e8	29.54	0.6498^*	-	17.55
rail4284	4284	1.1e6	1.1e7	full	400.0	> 3600	1.203^*	OOM	136.0
tnimg_1	951	1e6	2.1e7	925	-	630.6	1067^*	-	36.02
tnimg_2	1000	2e6	4.2e7	981	-	1291	$> 3600^*$	-	72.05
tnimg_3	1018	3e6	6.3e7	1016	-	2084	$> 3600^*$	-	111.1
tnimg_4	1019	4e6	8.4e7	1018	-	2945	$> 3600^*$	-	147.1
tnimg_5	1023	5e6	1.1e8	full	-	> 3600	$> 3600^*$	OOM	188.5

Table: Real-world problems and corresponding running times. DGELSD doesn't take advantage of sparsity. Though MATLAB's backslash may not give the min-length solutions to rank-deficient or under-determined problems, we still report its running times. Blendenpik either doesn't apply to rank-deficient problems or runs out of memory (OOM). LSRN's running time is mainly determined by the problem size and the sparsity.

Iterating with LSQR

(Paige and Saunders 1982)

Code snippet (Python):

```
u     = A.matvec(v) - alpha*u
beta  = sqrt(comm.allreduce(np.dot(u,u)))
...
v     = comm.allreduce(A.rmatvec(u)) - beta*v
```

Cost per iteration:

- two matrix-vector multiplications
- two cluster-wide synchronizations

Iterating with Chebyshev semi-iterative (CS) method

(Golub and Varga 1961)

The strong concentration results on $\sigma^{\max}(AN)$ and $\sigma^{\min}(AN)$ enable use of the CS method, which requires an accurate bound on the extreme singular values to work efficiently.

Code snippet (Python):

```
v   = comm.allreduce(A.rmatvec(r)) - beta*v
x  += alpha*v
r  -= alpha*A.matvec(v)
```

Cost per iteration:

- two matrix-vector multiplications
- one cluster-wide synchronization

LSQR vs. CS on an Amazon EC2 cluster

(Meng, Saunders, and Mahoney 2011)

solver	N_{nodes}	$N_{\text{processes}}$	m	n	nnz	N_{iter}	T_{iter}	T_{total}
LSRN w/ CS	2	4	1024	4e6	8.4e7	106	34.03	170.4
LSRN w/ LSQR						84	41.14	178.6
LSRN w/ CS	5	10	1024	1e7	2.1e8	106	50.37	193.3
LSRN w/ LSQR						84	68.72	211.6
LSRN w/ CS	10	20	1024	2e7	4.2e8	106	73.73	220.9
LSRN w/ LSQR						84	102.3	249.0
LSRN w/ CS	20	40	1024	4e7	8.4e8	106	102.5	255.6
LSRN w/ LSQR						84	137.2	290.2

Table: Test problems on an Amazon EC2 cluster and corresponding running times in seconds. Though the CS method takes more iterations, it actually runs faster than LSQR by making only one cluster-wide synchronization per iteration.

Outline

"Everything generalizes" from ℓ_2 regression to ℓ_1 regression

(But "everything generalizes messily" since ℓ_1 is "worse" that ℓ_2.)

- A matrix $U \in \mathbb{R}^{m \times n}$ is $(\alpha, \beta, p = 1)$-*conditioned* if $|U|_1 \leq \alpha$ and $\|x\|_\infty \leq \beta \|Ux\|_1, \ \forall x$; and ℓ_1-*well-conditioned basis* if $\alpha, \beta = \text{poly}(n)$.

- Define the ℓ_1 *leverage scores* of an $m \times n$ matrix A, with $m > n$, as the ℓ_1-norms-squared of the rows of an ℓ_1-well-conditioned basis of A.

- Define the ℓ_1-*norm condition number* of A, denoted by $\kappa_1(A)$, as:

$$\kappa_1(A) = \frac{\sigma_1^{\max}(A)}{\sigma_1^{\min}(A)} = \frac{\max_{\|x\|_2=1} \|Ax\|_1}{\min_{\|x\|_2=1} \|Ax\|_1}.$$

  This implies: $\sigma_1^{\min}(A)\|x\|_2 \leq \|Ax\|_1 \leq \sigma_1^{\max}(A)\|x\|_2, \quad \forall x \in \mathbb{R}^n$.

Meta-algorithm for ℓ_1-norm regression

1: Using an ℓ_1-well-conditioned basis for A, construct an importance sampling distribution $\{p_i\}_{i=1}^m$ from the ℓ_1-leverage scores.
2: Randomly sample a small number of constraints according to $\{p_i\}_{i=1}^m$ to construct a subproblem.
3: Solve the ℓ_1-regression problem on the subproblem.

A naïve version of this meta-algorithm gives a $1+\epsilon$ relative-error approximation in roughly $O(mn^5/\epsilon^2)$ time (DDHKM 2009). (Ugh.)

But, as with ℓ_2 regression:

- We can make this algorithm run much faster in RAM by
  - approximating the ℓ_1-leverage scores quickly, or
  - performing an "ℓ_1 projection" to uniformize them approximately.
- We can make this algorithm work at higher precision in RAM at large-scale by coupling with an iterative algorithm.

Conditioning: finding an ℓ_1 well-conditioned basis

Recall, given an $n \times d$ matrix A and $p \in [1, \infty]$, we want to find a low-distortion embedding $\Pi \in \mathbb{R}^{s \times n}$ s.t. $s = \mathcal{O}(\text{poly}(d))$ and

$$1/\,\mathcal{O}(\text{poly}(d)) \cdot \|Ax\|_p \leq \|\Pi Ax\|_p \leq \mathcal{O}(\text{poly}(d)) \cdot \|Ax\|_p, \quad \forall x \in \mathbb{R}^d.$$

There are two main ways:

Lemma (Conditioning via QR on low-distortion embedding)

Given a low-distortion embedding matrix Π of $\mathcal{A}_p$, let R be the "R" matrix from the QR decomposition of ΠA. Then, AR^{-1} is ℓ_p-well-conditioned.

Lemma (Conditioning via ellipsoidal rounding)

Given an $n \times d$ matrix A and $p \in [1, \infty]$, it takes at most $\mathcal{O}(nd^3 \log n)$ time to find a matrix $R \in \mathbb{R}^{d \times d}$ such that $\kappa_p(AR^{-1}) \leq 2d$.

Making ℓ_1 regression work to low and high precision

Finding a good basis (to get a *low-precision* solution):

name	running time	κ	type	passes (for soln)
SCT[SW11]	$\mathcal{O}(mn^2 \log n)$	$\mathcal{O}(n^{5/2} \log^{3/2} m)$	QR	2
FCT [CDMMMW13]	$\mathcal{O}(mn \log n)$	$\mathcal{O}(n^{7/2} \log^{5/2} m)$	QR	2
Ellipsoid rounding [Cla05]	$\mathcal{O}(mn^5 \log m))$	$n^{3/2}(n+1)^{1/2}$	ER	n^4
Fast ER [CDMMMW13]	$\mathcal{O}(mn^3 \log m))$	$2n^2$	ER	n^2
SPC1 [MM13]	$\mathcal{O}(\text{nnz}(A) \cdot \log m)$	$\mathcal{O}(n^{\frac{13}{2}} \log^{\frac{11}{2}} n)$	QR	2
SPC2 [MM13]	$\mathcal{O}(\text{nnz}(A) \cdot \log m) + \texttt{ER\_small}$	$6n^2$	QR+ER	3
SPC3 [YMM13]	$\mathcal{O}(\text{nnz}(A) \cdot \log m) + \texttt{QR\_small}$	$\mathcal{O}(n^{\frac{19}{4}} \log^{\frac{11}{4}} n)$	QR+QR	3

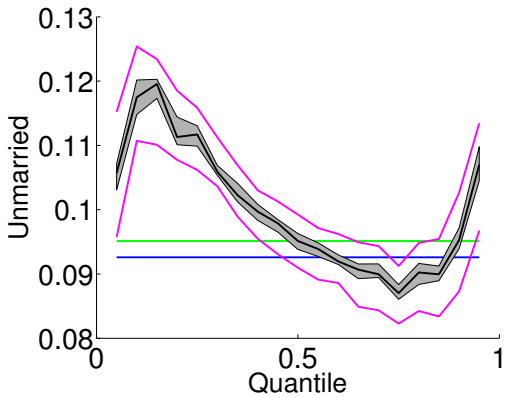
Iteratively solving (to get a *medium- to high-precision* solution):

	passes	extra work per pass
subgradient (Clarkson 2005)	$\mathcal{O}(n^4/\epsilon^2)$	—
gradient (Nesterov 2009)	$\mathcal{O}(m^{1/2}/\epsilon)$	—
ellipsoid (Nemirovski and Yudin 1972)	$\mathcal{O}(n^2 \log(\kappa_1/\epsilon))$	—
inscribed ellipsoids (Tarasov, Khachiyan, and Erlikh 1988)	$\mathcal{O}(n \log(\kappa_1/\epsilon))$	$\mathcal{O}(n^{7/2} \log n)$

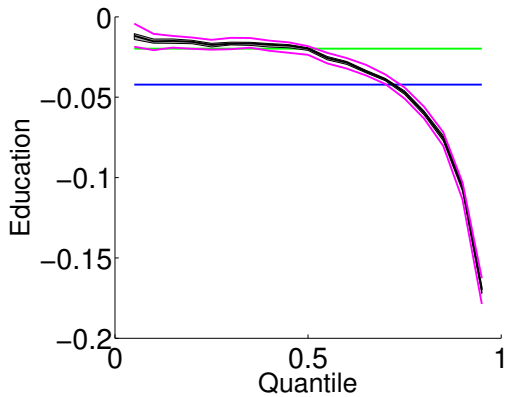
Prior work and evaluations

Evaluate on real and simulated data:

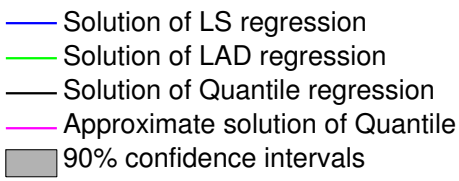
- Simulated data, size ca. $10^9 \times 10^2$, designed to have "bad" nonuniformities.
- Real US Census data, size ca. $10^7 \times 10$, or "stacked" to size ca. $10^{10} \times 10$.



(a) Unmarried



(b) Education



(c) Legend

State of the art (due to Portnoy-Koenker, 1997]):

- Standard solver for ℓ_1 regression is interior-point method `ipm`, applicable for $10^6 \times 50$-sized problems.
- Best previous sampling algorithm for ℓ_1 regression, `prqfn`, uses an interior-point method on a smaller randomly-constructed subproblem.

A **MapReduce** implementation

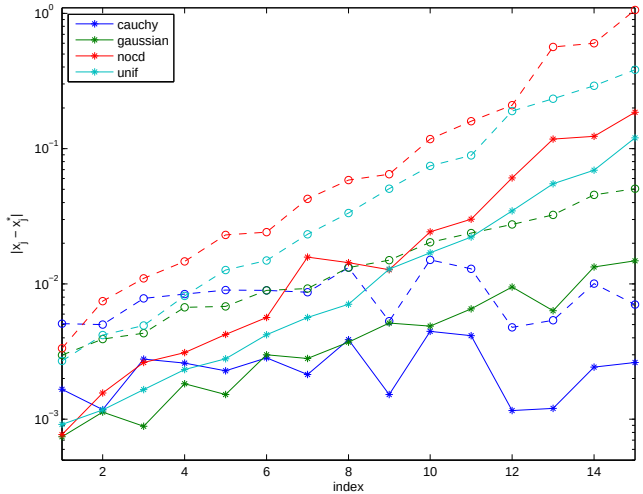
- Inputs: $A \in \mathbb{R}^{m \times n}$ and κ_1 such that

$$\|x\|_2 \leq \|Ax\|_1 \leq \kappa_1 \|x\|_2, \quad \forall x,$$

  $c \in \mathbb{R}^n$, sample size s, and number of subsampled solutions n_x.
- **Mapper:**
  1. For each row a_i of A, let $p_i = \min\{s\|a_i\|_1/(\kappa_1 n^{1/2}), 1\}$.
  2. For $k = 1, \ldots, n_x$, emit $(k, a_i/p_i)$ with probability p_i.
- **Reducer:**
  1. Collect row vectors associated with key k and assemble A_k.
  2. Compute $\hat{x}_k = \arg\min_{c^T x = 1} \|A_k x\|_1$ using interior-point methods.
  3. Return $\hat{x}_k$.

Note that multiple subsampled solutions can be computed in a single pass.

Evaluation on large-scale ℓ_1 regression problem



First (solid) and the third (dashed) quartiles of entry-wise absolute errors (on synthetic data that has "bad" nonuniformities).

	$\|x - x^*\|_1 / \|x^*\|_1$	$\|x - x^*\|_2 / \|x^*\|_2$	$\|x - x^*\|_\infty / \|x^*\|_\infty$
CT (Cauchy)	[0.008, 0.0115]	[0.00895, 0.0146]	[0.0113, 0.0211]
GT (Gaussian)	[0.0126, 0.0168]	[0.0152, 0.0232]	[0.0184, 0.0366]
NOCD	[0.0823, 22.1]	[0.126, 70.8]	[0.193, 134]
UNIF	[0.0572, 0.0951]	[0.089, 0.166]	[0.129, 0.254]

First and the third quartiles of relative errors in 1-, 2-, and ∞-norms on a data set of size $10^{10} \times 15$. CT (and FCT) clearly performs the best. GT is worse but follows closely. NOCD and UNIF are much worse. (Similar results for size $10^{10} \times 100$ if SPC2 is used.)

The Method of Inscribed Ellipsoids (MIE)

MIE works similarly to the bisection method, but in a higher dimension.

Why do we choose MIE?

- Least number of iterations
- Initialization using all the subsampled solutions
- Multiple queries per iteration

At each iteration, we need to compute (1) a function value and (2) a gradient/subgradient.

- For each subsampled solution, we have a hemisphere that contains the optimal solution.
- We use all these solution hemispheres to construct initial search region.

Constructing the initial search region

Given any feasible $\hat{x}$, let $\hat{f} = \|A\hat{x}\|_1$ and $\hat{g} = A^T \text{sign}(A\hat{x})$. we have

$$\|x^* - \hat{x}\|_2 \leq \|A(x^* - \hat{x})\|_1 \leq \|Ax^*\|_1 + \|A\hat{x}\|_1 \leq 2\hat{f},$$

and, by convexity,

$$\|Ax^*\|_1 \geq \|A\hat{x}\|_1 + \hat{g}^T(x^* - \hat{x}),$$

which implies $\hat{g}^T x^* \leq \hat{g}^T \hat{x}$.

Hence, for each subsampled solution, we have a hemisphere that contains the optimal solution.

We use all these hemispheres to construct the initial search region $\mathcal{S}_0$.

Computing multiple f and g in a single pass

On MapReduce, the IO cost may dominate the computational cost, which requires algorithms that could do more computation in a single pass.

- Single query:

$$f(x) = \|Ax\|_1, \quad g(x) = A^T \text{sign}(Ax).$$

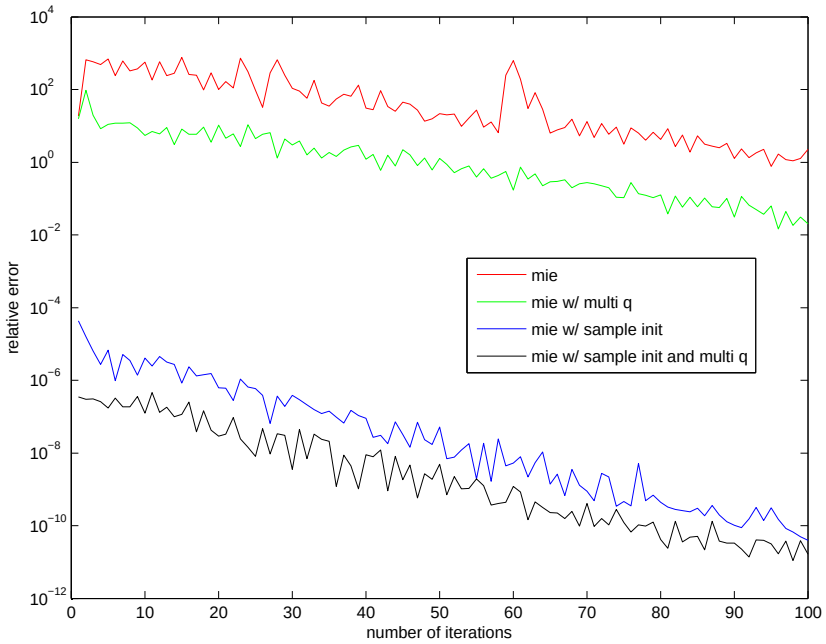
- Multiple queries:

$$F(X) = \text{sum}(|AX|, 0), \quad G(X) = A^T \text{sign}(AX).$$

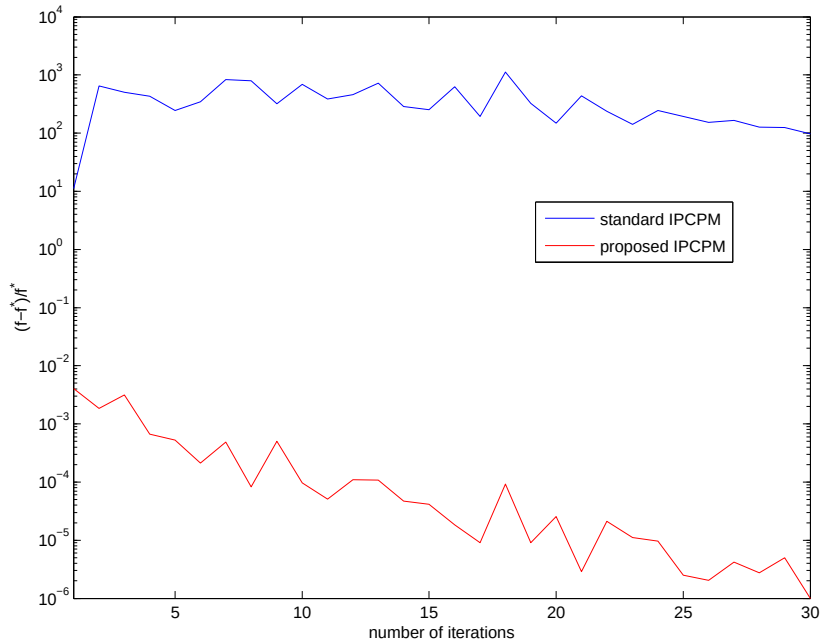
An example on a 10-node Hadoop cluster:

- $A : 10^8 \times 50$, 118.7GB.
- A single query: 282 seconds.
- 100 queries in a single pass: 328 seconds.

MIE with sampling initialization and multiple queries



(d) size: $10^6 \times 20$



(e) size: $5.24e9 \times 15$

Comparing different MIE methods on large/LARGE ℓ_1 regression problem.

Conclusion

- Randomized regression in parallel & distributed environments: different design principles for high-precision versus low-precision
  - Least Squares Approximation
  - Least Absolute Deviations
  - Extensions to Quantile Regression, Kernel-based Learning, Etc.

- Algorithms require more computation than traditional matrix algorithms, but they have better communication profiles.
  - On MPI: Chebyshev semi-iterative method vs. LSQR.
  - On MapReduce: Method of inscribed ellipsoids with multiple queries.
  - Look beyond FLOPS in parallel and distributed environments.