Influences in Gaussian Space

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Functional Inequalities in Discrete Spaces with Applications,
Berkeley, 4 October, 2013

Based on joint work with Nathan Keller and Elchanan Mossel

Outline of the talk

- Find a natural definition of influences for functions $f(W_1, \ldots, W_n)$ where W_i are i.i.d. standard Gaussians.

- Gaussian analogues of many fundamental results of discrete harmonic analysis.
  - Kahn-Kalai-Linial (KKL) bound,
  - Threshold phenomenon for monotone events,
  - Benjamini-Kalai-Schramm (BKS) noise sensitivity theorem, etc.

- Extensions to other probability measures.

Influences of boolean functions

- $f : \{-1,1\}^n \to \{0,1\}$ with product Bernoulli measure on $\{0,1\}^n$. The influence of j-th coordinate on f

$$I_j(f) := \mathbb{P}\big(f(x) \neq f(\tau_j(x))\big),$$

  where $\tau_j(x) = (x_1, \ldots, -x_j, \ldots, x_n)$.

- Examples.
  Majority: $f(x) = \mathbf{1}_{\{\sum_{j=1}^n x_i > n/2\}}$. $I_j(f) \asymp 1/\sqrt{n}$ for all j.
  Dictator: $f(x) = x_1$. $I_1(f) = 1$ and $I_j(f) = 0$ for $j > 1$.

- Applications in phase transitions, percolation, hardness of approximation, statistical learning, social choice theory, extremal combinatorics, metric embeddings, ...

Why useful?

- Geometric/ Isoperimetric meaning: For set $A \subseteq \{-1,1\}^n$,

$$\sum_{j=1}^{n} I_j(A) = \frac{1}{2^{(n-1)}} \underbrace{\#\{\text{ edges between } A \text{ and } A^c\}}_{\text{Edge boundary of } A}.$$

- KKL's lower bound on max influence
- Connection to Russo's formula
- ...
- ...

Kahn - Kalai- Linial (KKL) theorem

- $f : \{-1,1\}^n \to \{0,1\}$

  Efron-Stein bound:

$$\mathrm{Var}(f) \le \sum_{j=1}^{n} I_j(f)$$

$$\Rightarrow \max_{1\le j\le n} I_j(f) \ge \frac{\mathrm{Var}(f)}{n}$$

- Nontrivial bound by KKL ('88):

$$\max_{1\le j\le n} I_j(f) \ge \mathrm{Var}(f).\Omega(\frac{\log n}{n})$$

  (also holds for product $\mathrm{Ber}(p)$ measure)

Russo's formula

- Let μ_p denote the $\mathrm{Ber}(p)$.
- Clearly if $A \subseteq \{-1,1\}^n$ is increasing then $p \mapsto \mu_p^{\otimes n}(A)$ is monotone increasing.
- Russo's Lemma: $A \subseteq \{-1,1\}^n$ increasing,

$$\frac{d\mu_p^{\otimes n}(A)}{dp} = \sum_{j=1}^{n} I_j^p(A).$$

An application: threshold phenomenon for monotone sets

Theorem (Threshold phenomenon, Friedgut & Kalai '96)

Let $A \subseteq \{-1,1\}^n$ *monotone transitive. Then*

$$\mu_p^{\otimes n}(A) \geq \epsilon \quad \Rightarrow \quad \mu_q^{\otimes n}(A) \geq 1-\epsilon$$

where $q = p + c\log(1/2\epsilon)(\log n)^{-1}$.

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- Erdös-Rényi random graph: Take a complete graph on n vertices. Let $N = \binom{n}{2}$. Each of N edges is present independently with probability p.
- Graph property: events that are closed under relabeling of the vertices.
- A = nontrivial monotone graph property $\subset \{0,1\}^N$.
  e.g. connected, triangle-free, hamiltonian, non-planar, ...
- Bourgain & Kalai '97: The threshold interval for monotone graph properties is $\sim (\log n)^{-2+o(1)}$.

Influences in continuous probability space

- How can we define influences for $f : (\mathbb{R}^n, \mu^{\otimes n}) \to \{0, 1\}$?
  Several existing definitions.

- BKKKL influence (Bourgain-Kahn-Kalai-Katznelson-Linial '92)

  $$\mathbb{P}\Big\{x \in \mathbb{R}^n : f \text{ is not constant on the fiber } (x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)\Big\}.$$

  - KKL bound still holds.
  - Lacks geometric meaning.

- 'Variance' influence of f (Hatami'09, Mossel-O'Donnell-Oleszkiewicz '09)

  $$\mathbb{E}\mathrm{Var}_i(f | x \setminus \{x_i\}).$$

  - Reasonable
  - No KKL type bound.

    On $[0, 1]^n$, $f(x) = \mathbf{1}_{\{\max x_i \le 1 - n^{-1}\}}$, $\quad \mathbb{E}[f] = e^{-1}$, $\quad I_j(f) \asymp n^{-1} \quad \forall j$.

Our definition : Geometric influence

The *geometric influence* of the j-th coordinate on $A \subseteq (\mathbb{R}^n, \nu^{\otimes n})$ is

$$I_j^{\mathcal{G}}(A) := \int \nu^+(A_j^x)\nu^{\otimes n}(dx) \in [0, +\infty]$$

where

$$A_j^x := \left\{y \in \mathbb{R} : (x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_n) \in A\right\}$$

and

$$\nu^+(A_j^x) := \liminf_{r \downarrow 0} \frac{\nu(A_j^x + [-r, r]) - \nu(A_j^x)}{r}.$$

is the surface measure (lower Minkowski content) of the section A_j^x.

e.g. $\nu^+([a, b]) = \phi(a) + \phi(b)$, when ν has a density ϕ.

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- $I_j^{\mathcal{G}}(A) = \int |\partial_j \mathbf{1}_A| \nu^{\otimes n}(dx)$ (L^1-norm of $\partial_j \mathbf{1}_A$).
- The definition of $I_j^{\mathcal{G}}(A)$ does not depend on the product structure of the measure $\nu^{\otimes n}$.

Lemma

Let ν be a probability measure on $\mathbb{R}$ with a 'nice' density. Let $A \subset \mathbb{R}^n$ be a monotone set. Then

$$\liminf_{r \downarrow 0} \frac{\nu^{\otimes n}(A + [-r, r]^n) - \nu^{\otimes n}(A)}{r} = \sum_{i=1}^{n} I_i^{\mathcal{G}}(A).$$

- In literature, $\liminf_{r \downarrow 0} \frac{\nu^{\otimes n}(A+[-r,r]^n) - \nu^{\otimes n}(A)}{r}$ is called boundary under uniform enlargement.
- Also true for convex sets. But not for general sets, e.g. $\mathbb{Q}^n$.

KKL-type bound for Gaussian measure

Theorem (Keller-Mossel-S-12)

Consider the product spaces $\mathbb{R}^n$ *endowed with the product Gaussian measure* $\mu^{\otimes n}$ *. Then for any Borel-measurable set* $A \subset \mathbb{R}^n$ *with* $\mu^{\otimes n}(A) = t$

$$\max_{1 \le i \le n} I_i^{\mathcal{G}}(A) \ge ct(1-t)\frac{\sqrt{\log n}}{n},$$

where $c > 0$ *is a universal constant.*

KKL-type bound for Gaussian measure

Theorem (Keller-Mossel-S-12)

Consider the product spaces $\mathbb{R}^n$ endowed with the product Boltzmann measure $\mu_\rho^{\otimes n}$. Then for any Borel-measurable set $A \subset \mathbb{R}^n$ with $\mu^{\otimes n}(A) = t$

$$\max_{1 \le i \le n} I_i^{\mathcal{G}}(A) \ge ct(1-t)\frac{(\log n)^{1-1/\rho}}{n},$$

where $c > 0$ is a universal constant.

- The Boltzmann measure μ_ρ with parameter $\rho \ge 1$ is given by

$$\mu_\rho(dx) := C_\rho e^{-|x|^\rho} dx, \quad x \in \mathbb{R}.$$

KKL-type bound for Gaussian measure

Theorem (Keller-Mossel-S-12)

Consider the product spaces $\mathbb{R}^n$ endowed with the product . Then for any Borel-measurable set $A \subset \mathbb{R}^n$ with $\mu^{\otimes n}(A) = t$

$$\max_{1 \le i \le n} I_i^{\mathcal{G}}(A) \ge ct(1-t)\frac{(\log n)^{1-1/\rho}}{n},$$

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- The Boltzmann measure μ_ρ with parameter $\rho \ge 1$ is given by

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- The proof uses isoperimetric inequality for Boltzmann measures

  Isoperimetric function: $\mathcal{I}_{\mu_\rho}(t) \ge c \min(t, 1-t)(-\log \min(t, 1-t))^{1-1/\rho}$

  $+$ the original KKL bound via h-influences.

Tightness

- Example: Semi-infinite box.

  Let $B_n := [-\infty, b_n]^n$ where b_n is chosen such that $\mu_\rho^{\otimes n}(B_n) = t < 1/2$ fixed. Then
  $$I_j^{\mathcal{G}}(B_n) \leq ct\frac{(\log n)^{1-1/\rho}}{n},$$
  for all $1 \leq j \leq n$.

Talagrand's variance bound

Theorem (Talagrand '94)

For any $f : \{-1,1\}^n \to \mathbb{R}$ *with product Bernoulli measure* $\nu^{\otimes n}$

$$\mathrm{Var}(f) \le C \sum_{j=1}^{n} \frac{\|D_j f\|_2^2}{1 + \log(\|D_j f\|_2 / \|D_j f\|_1)}$$

$D_j f(x) := f(\tau_j x) - f(x)$.

If $f = \mathbf{1}_A$ with $\nu^{\otimes n}(A) = t$, then the above inequality becomes

$$t(1-t) \le C' \sum_{j=1}^{n} \frac{I_j(A)}{-\log I_j(A)} \qquad (\|D_j f\|_2^2 = I_j(A), \|D_j f\|_1 = I_j(A)).$$

$\Rightarrow$ KKL bound.

Theorem (Gaussian analogue)

Consider $\mathbb{R}^n$ *with the product Boltzmann measure* $\mu_\rho^{\otimes n}$*. Let* $A \subset \mathbb{R}^n$*.*
If $\mu^{\otimes n}(A) = t$*, then*

$$t(1-t) \le C \sum_{j=1}^{n} \frac{I_j^{\mathcal{G}}(A)}{(-\log I_j^{\mathcal{G}}(A))^{1-1/\rho}}.$$

Talagrand's bound in Markov semigroup (Cordero-Erausquin, Ledoux)

- Let P_t be reversible Markov semigroup on $\mathcal{X}$ with generator L and μ is the invariant probability measure. The associated Dirichlet form
$$\mathcal{E}(f,g) = -\int fLgd\mu.$$
- Suppose
$$\mathcal{E}(f,f) = \sum_{i=1}^{n} \int_{\mathcal{X}} \Gamma_i(f)^2 d\mu,$$
and for some $\kappa \in \mathbb{R}$,
$$\Gamma_i(P_t(f)) \le e^{\kappa t} P_t(\Gamma_i(f)).$$
- Suppose L also satisfies log-Sobolev inequality
$$\rho \mathrm{Ent}(f^2) \le 2\mathcal{E}(f,f).$$

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Theorem (Cordero-Erausquin, Ledoux '12)

$$\mathrm{Var}_\mu(f) \le C(\rho,\kappa) \sum_{i=1}^{n} \frac{\|\Gamma_i f\|_2^2}{1 + \log(\|\Gamma_i f\|_2 / \|\Gamma_i f\|_1)},$$

where $C(\rho,\kappa) = 4e^{(1+\kappa/\rho)_+}/\rho$.

L^1 norm of $\Gamma_i(f)$ serves as a natural candidate for 'influence' of f in the i-th direction

Examples

- Bonami-Beckner semigroup. μ = uniform measure on $\{-1, 1\}^n$. $\mathcal{E}(f, f) = \frac{1}{4} \int \sum_{i=1}^{n} |D_i f|^2 d\mu$. $\kappa = 0$ and $\rho = 1$.

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- Gaussian case (Ornstein-Uhlenbeck semigroup). $\mu(dx) = \otimes_{i=1}^n e^{-V_i(x)} dx$ on $\mathbb{R}^n$. The semigroup is generated by $L = \sum_{i=1}^n \partial_i^2 - V_i'(x)\partial_i$.

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  Under the assumption $V_i'' \geq c > 0$, $\kappa = -c$ and $\rho = c$.

- Transposition walk on S_n. μ = uniform measure on S_n. T = set of transpositions.

$$\mathcal{E}(f,f) = \frac{1}{|T|}\int \sum_{s\in T} |D_s f|^2 d\mu, \quad D_s f(\sigma) = f(s\sigma) - f(\sigma).$$

  $\kappa = 0$ and $\rho \sim (n \log n)^{-1}$.
  O'Donnell-Wimmer '09, O'Donnell-Wimmer '13

Talagrand's bound for sets

- In the Gaussian set-up, when $f = \mathbf{1}_A$, the meaning of $\|\partial_i f\|_2$ is not clear.
- Cordero-Erausquin and Ledoux proved that for all $\|f\|_\infty \leq 1$,

$$\mathrm{Var}_\mu(f) \leq C \sum_{i=1}^{n} \frac{\|\partial_i f\|_1 (1 + \|\partial_i f\|_1)}{\sqrt{1 + \log^+(1/\|\partial_i f\|_1)}}$$

  which implies KKL bound for Gaussian measures.
- Useful fact: For every $\|f\|_\infty \leq 1$ and every $0 < t < 1/2$,

$$\|\partial_i P_t f\|_\infty \leq \frac{1}{\sqrt{t}} \qquad\qquad P_t = \text{ OU semigroup.}$$

  It helps to bound $\|\partial_i P_t f\|_p, p > 1$ by $\|\partial_i P_t f\|_1$.

Russo's formula and threshold

Lemma

Let $\mu_\theta = N(\theta, 1)$. *Let* $A \subseteq \mathbb{R}^n$ *be increasing.*

$$\frac{d{\mu_\theta}^{\otimes n}(A)}{d\theta} = \sum_{j=1}^{n} I_j^{\mathcal{G}}(A)$$

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Proof. $\frac{d}{d\theta}\mathbb{E}f(X_1+\theta,\ldots,X_n+\theta) = \sum_{i=1}^{n}\mathbb{E}\partial_i f(X_1+\theta,\ldots,X_n+\theta), \;\; X_i \sim N(0,1).$

Corollary

Let $\mu_\theta = N(\theta, 1)$. *Let* $A \subset \mathbb{R}^n$ *be an increasing and transitive*

$$\mu_{\theta_0}^{\otimes n}(A) > \epsilon \quad \Rightarrow \quad \mu_{\theta_1}^{\otimes n}(A) > 1 - \epsilon$$

where $\theta_1 = \theta_0 + c\log(1/2\epsilon)(\log n)^{-1/2}$.

- Threshold window is tight for $A = \{\max_i X_i \geq \mathrm{Median}(\max_i X_i)\}$.

Noise Sensitivity in discrete cubes

- Let X be uniform on $\{-1,1\}^n$ and X^η be a η-noisy copy of X.

$$X_j^\eta = \begin{cases} X_j & \text{w.p.} \quad 1-\eta \\ X_j' & \text{w.p.} \quad \eta \end{cases}, \text{ independently for each } j.$$

- For $f : \{-1,1\}^n \to \mathbb{R}$ and $\eta \in (0,1)$, the noise sensitivity of f,

$$\mathrm{VAR}(f,\eta) = \mathbb{E}[f(X)f(X^\eta)] - \mathbb{E}[f(X)]\mathbb{E}[f(X^\eta)].$$

$f_k : \{-1,1\}^{n_k} \to \mathbb{R}$ is asymptotically noise-sensitive if

$$\mathrm{VAR}(f_k,\eta) \stackrel{k\to\infty}{\to} 0 \quad \forall \eta > 0$$

$$\Big(\Leftrightarrow \qquad \sum_{0<|S|\le d} \hat{f}_k(S)^2 \stackrel{k\to\infty}{\to} 0 \quad \forall d \ge 1 \Big)$$

- Example: The event of having a L-R crossing in the box $[0,n]^2$ for the critical percolation in $\mathbb{Z}^2$.

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- Example: The event of having a L-R crossing in the box $[0,n]^2$ for the critical percolation in $\mathbb{Z}^2$.

- Benjamini, Kalai, Schramm (1999): $A_k \subseteq \{-1,1\}^{n_k}$ is asymptotically noise-sensitive if

$$\sum_j I_j(A_k)^2 \overset{k\to\infty}{\to} 0.$$

Gaussian Noise Sensitivity

- $W, W' \sim N(0, I)$ independent and $W^\rho = \sqrt{1-\rho^2}W + \rho W'$.
- For $f : \mathbb{R}^n \to \mathbb{R}$ and $\rho \in (0,1)$, the Gaussian noise sensitivity of f,

$$\mathrm{VAR}^{\mathcal{G}}(f, \rho) = \mathbb{E}[f(W)f(W^\rho)] - \mathbb{E}[f(W)]\mathbb{E}[f(W^\rho)].$$

- Gaussian BKS theorem (Keller, Mossel, S. '13): $A_k \subseteq \mathbb{R}^{n_k}$ is asymptotically Gaussian noise-sensitive if

$$\sum_j I_j^{\mathcal{G}}(A_k)^2 \overset{k\to\infty}{\to} 0.$$

Quantitive BKS

Theorem (Keller and Kindler (2013))

For $f : \{-1,1\}^n \to [0,1]$ *and* $\eta \in (0,1)$

$$\mathrm{VAR}(f,\eta) \le C\Big(\sum_j I_j(f)^2\Big)^{c\eta}.$$

Theorem (Gaussian analogue)

For $f : \mathbb{R}^n \to [0,1]$ *and* $\rho \in (0,1)$,

$$\mathrm{VAR}^{\mathcal{G}}(f,\rho) \le C\Big(\sum_j \|\partial_j f\|_1^2\Big)^{c\rho^2}.$$

Proof of Quantitive Gaussian BKS

- For simplicity take $n = 1$ and $f : \mathbb{R} \to [0,1]$ in C^1.
- Use CLT to approximate (W, W^ρ) by $(\frac{X_1+\ldots+X_m}{\sqrt{m}}, \frac{X_1^\eta+\ldots+X_m^\eta}{\sqrt{m}})$ with $\eta = 1 - \sqrt{1-\rho^2}$ and $m \to \infty$.
- Use Quantitive BKS to $f(\frac{X_1+\ldots+X_m}{\sqrt{m}}) : \{-1,1\}^m \to \mathbb{R}$.
- $\mathrm{VAR}(f(\frac{X_1+\ldots+X_m}{\sqrt{m}}), \eta) \stackrel{m\to\infty}{\to} \mathrm{VAR}^{\mathcal{G}}(f(W), \rho)$ as $m \to \infty$.
- Observation.

$$\sum_{j=1}^{m} I_j^2\Big(f\Big(\frac{X_1+\ldots+X_m}{\sqrt{m}}\Big)\Big) \to 4\|f'\|_1^2 = 4I^{\mathcal{G}}(f)^2.$$

Proof.

$$\begin{aligned}
\sqrt{m} I_1\Big(f\Big(\frac{X_1+\ldots+X_m}{\sqrt{m}}\Big)\Big) &= \sqrt{m}\,\mathbb{E}\left|f\Big(\frac{1+\ldots+X_m}{\sqrt{m}}\Big) - f\Big(\frac{-1+\ldots+X_m}{\sqrt{m}}\Big)\right| \\
&\approx \sqrt{m}\frac{2}{\sqrt{m}}\mathbb{E}\left|f'\Big(\frac{X_2+\ldots+X_m}{\sqrt{m}}\Big)\right| \\
&\approx 2\mathbb{E}|f'(W)|
\end{aligned}$$

Gaussian BKS using semigroup techniques

- Bouyrie (2013) gave a direct proof using standard semigroup tools.

- For any f on the Gaussian space and $t > 0$,

$$\mathrm{VAR}(f, \sqrt{1 - e^{-4t}}) = \mathrm{Var}(P_t f) \leq 4e^{-t} \Big(\sum_{j=1}^{n} \|\partial_j f\|_1^2 \Big)^{\frac{1-e^{-t}}{2}} \|f\|_2^{1+e^{-t}}.$$

[P_t = OU semigroup.]

Correlation of increasing sets

Theorem (Talagrand' 96)

For any pair of increasing subsets $A, B \subseteq \{-1, 1\}^n$,

$$\nu^{\otimes n}(A \cap B) - \nu^{\otimes n}(A)\nu^{\otimes n}(B) \geq c\varphi(\sum_{i=1}^{n} I_i(A)I_i(B)),$$

where $\nu^{\otimes n}$ = *product Bernoulli measure and* $\varphi(x) = x/\log(e/x)$.

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where $\mu^{\otimes n}$ = *standard Gaussian measure and* $\varphi(x) = x/\log(e/x)$.

Is there a direct semigroup proof?

Thank You !!