Strong Noise Sensitivity and Random Graphs

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Workshop on Functional Inequalities in Discrete Spaces

Joint work with Eyal Lubetzky

Outline of talk

- Noise sensitivity of Boolean functions and the Benjamini-Kalai-Schramm Theorem.

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- (Strong) noise sensitivity in the Erdős-Rényi $\mathcal{G}(n, p)$ model.
- Some sketch of arguments.

1. Boolean functions and Noise sensitivity

Basic Set up for Noise Sensitivity

- $$x := x_1, \ldots, x_n \;\; \text{i.i.d.} \begin{cases} 1 & \text{with probability } p_n \,, \\ 0 & \text{with probability } 1 - p_n \,. \end{cases}$$

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While not obvious, if this approaches 0 for one ϵ, it does so for all ϵ.

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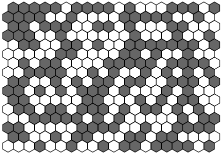
Percolation on the hexagonal lattice ($p_n = 1/2$).



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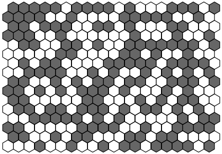

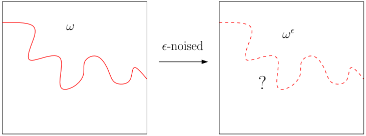

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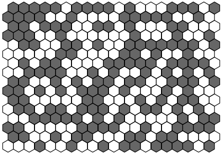

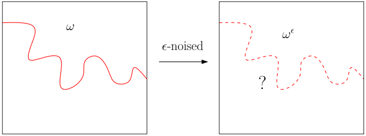

Theorem (Benjamini, Kalai & Schramm 1999)

Percolation crossings are noise sensitive.

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- Clique containment: all influences are of order $(\log n)^2/n^2$.

Influences are relevant for noise sensitivity

Theorem (Benjamini, Kalai & Schramm 1999)

If p_n is constant and

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- The proof uses Fourier analysis and hypercontractivity.

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  - The giant component of $\mathcal{G}(n, p)$ emerges at $p = 1/n$
- Don't worry that p_n sometimes refers to the p used when the number of variables is n and sometimes refers to the edge probability for $\mathcal{G}(n, p)$ which has $\binom{n}{2}$ variables.

2. Witnesses and Strong Noise sensitivity

Our functions are now always assumed to be monotone.

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A 1-witness (1-certificate or "min-term") is a minimal subset of the variables with the property that if all these bits are 1, then the function is guaranteed to be 1.

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- Example: If f is the property that a graph is connected, then a 1-witness would be a spanning tree.
- Let $\mathcal{W}_1 = \mathcal{W}_1(f)$ denote the set of 1-witnesses of some monotone Boolean function f and similarly for $\mathcal{W}_0 = \mathcal{W}_0(f)$.

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Recall:

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A sequence of monotone Boolean functions $f_n : \{0,1\}^n \to \{0,1\}$ is called **1-strongly noise sensitive ($\mathrm{StrSens}_1$)** (w.r.t. (p_n)) if for any $\epsilon > 0$,

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0-strongly noise sensitive ($\mathrm{StrSens}_0$) is defined analogously.

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- The notion of StrSens_1 is intimately connected to the structure of the set of witnesses.
- $\text{StrSens}_1 \not\Leftrightarrow \text{StrSens}_0$: $\mathcal{W}_1$ and $\mathcal{W}_0$ can differ greatly (like 3-SAT).
- Unlike NS, the defining condition of StrSens_1 can depend upon ϵ.

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- This example was of interest since the influences are of order $\frac{\log n}{n}$ which was later shown to be optimal by Kahn, Kalai and Linial.

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- However, one can show that the sequence is not StrSens_0; the effect of conditioning on a 0-witness to be 0 is much more drastic. Note also the 0-witnesses have much pairwise overlap.

3. Quantitative noise sensitivity results for the Erdős-Rényi random graph model.

Theorem

The event that there exists a cycle of length contained in $[n^{1/3}, 2n^{1/3}]$ is noise sensitive at $p_n = 1/n$ and is in fact 1-strong noise sensitive:

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More precisely:

- $\epsilon_n \gg n^{-1/3}$ *implies*

$$\max_{W \in \mathcal{W}_1} \mathbb{P}\Big[f_n(x^{\epsilon_n}) = 1 \mid x_W \equiv 1\Big] - \mathbb{P}\Big[f_n(x^{\epsilon_n}) = 1\Big] = o(1)\,.$$

- $\epsilon_n \ll n^{-1/3}$ *implies*

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- $\epsilon_n \asymp n^{-1/3}$ *implies neither noise sensitivity nor noise stability.*

Noise sensitivity and the critical window

- The quantitative result involving $n^{-1/3}$ matches the critical window for the Erdős-Rényi random graph; intuitively:
  - $\epsilon_n = O(n^{-1/3})$: noised graph stays within the critical window (correlation).
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  - $\epsilon_n \gg n^{-1/3}$: noised graph leaves the critical window (decorrelates).
- The argument of the first result is based on a second moment calculation involving the number of 1-witnesses which are satisfied for x and x^ϵ.

Minimum degree at least k is 0-strong noise sensitive

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The event that each vertex has degree at least k is noise sensitive at $p_n = \frac{\log n + (k-1) \log \log n}{n}$ and is in fact 0-strong noise sensitive:

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"Proof" and Consequences of 0-strong noise sensitivity for Minimum degree at least k

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Corollary

(1). For $\mathcal{G}(n, \frac{\log n}{n})$, the events "Connectivity" and "containing a perfect matching" are noise sensitive and even the same quantitative ($\frac{1}{\log n}$) results hold.

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Corollary

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Definition

A graph is called **strictly balanced** if its edge/vertex ratio is strictly larger than all of its subgraphs.

Noise sensitivity for containing a graph

Theorem

1. If H_n is strictly balanced with $1 \ll \ell_n \leq \left(\frac{\log n}{\log \log n}\right)^{1/2}$ edges, then (f_n) is noise sensitive, and furthermore, it is 1-strong noise sensitive.

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One takes $p_n = \lambda/n$ where $\lambda > 4$ and H to be two triangles connected by a path of length $\lfloor \frac{3}{2} \log_\lambda n \rfloor$.

Some final differences between NS and StrSens_1: Dependence on ϵ

Recall in noise sensitivity,

$$\lim_{n\to\infty} \mathbb{P}\big[f_n(x^\epsilon) = 1 | f_n(x) = 1\big] - \mathbb{P}\big[f_n(x^\epsilon) = 1\big] = 0$$

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1-strongly noise sensitive (StrSens_1) (w.r.t. (p_n)) if for any $\epsilon > 0$,

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It turns out the above quantity going to 0 can depend on the value of ϵ. One composes iterated 5-majority with tribes.

Some final differences between NS and StrSens_1: Superstability

Definition (Benjamini, Kalai, Schramm)

A sequence of Boolean functions $f_n : \{0,1\}^n \to \{0,1\}$ is called **noise stable** (w.r.t. (p_n)) if

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For all n and ϵ, $\mathbb{P}\Big[f_n(x^\epsilon) = 0 \mid x_W \equiv 1\Big] = \epsilon/2$.

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Example: Iterated 5-Majority
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Thank you very much!