On hyperboundedness and spectrum of Markov operators

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Plan of the talk

1. Høegh-Krohn and Simon's conjecture
2. Higher order Cheeger inequalities in the finite setting
3. An approximation procedure
4. General higher order Cheeger's inequalities
5. Quantitative links between hyperboundedness and spectrum

Reversible Markov operators

On a probability space $(S, \mathcal{S}, \mu)$, a self-adjoint operator $M : \mathbb{L}^2(\mu) \to \mathbb{L}^2(\mu)$ is said to be Markovian if μ-almost surely,

$$\forall\ f \in \mathbb{L}^2(\mu), \qquad \begin{array}{rcl} f \geqslant 0 & \Rightarrow & M[f] \geqslant 0 \\ M[\mathbb{1}] & = & \mathbb{1} \end{array}$$

So M admits a spectral decomposition: there exists a projection-valued measure $(E_l)_{l\in[-1,1]}$ such that

$$M = \int_{-1}^{1} l \, dE_l$$

and M is said to be ergodic if

$$\forall\ f \in \mathbb{L}^2(\mu), \qquad Mf = f \quad \Rightarrow \quad f \in \mathrm{Vect}(\mathbb{1})$$

namely if $(E_1 - E_{1-})[\mathbb{L}^2(\mu)] = \mathrm{Vect}(\mathbb{1})$.

Hyperboundedness

Stronger requirement: M has a spectral gap if there exists $\lambda > 0$ such that $(E_1 - E_{1-\lambda})[\mathbb{L}^2(\mu)] = \mathrm{Vect}(\mathbb{1})$. Spectral gap: the supremum of such λ.

The Markov operator M is hyperbounded if there exists $p > 2$ such that

$$\|M\|_{\mathbb{L}^2(\mu)\to\mathbb{L}^p(\mu)} \quad < \quad +\infty$$

Theorem

A self-adjoint ergodic and hyperbounded Markovian operator admits a spectral gap.

This result was conjectured by Høegh-Krohn and Simon [1972], in the context of Markovian semi-groups.

A reversible Markovian semi-group $(P_t)_{t\geqslant 0}$: a continuous family of self-adjoint Markovian operators on $\mathbb{L}^2(\mu)$ satisfying $P_0 = \mathrm{Id}$ and

$$\forall\ t, s \geqslant 0, \qquad P_t P_s \;=\; P_{t+s}$$

Ergodic if for any $f \in \mathbb{L}^2(\mu)$, $P_t[f] \to \mu[f]\mathbb{1}$ in $\mathbb{L}^2(\mu)$. Admits a spectral gap: this convergence is uniform over the unit ball of $\mathbb{L}^2(\mu)$. It corresponds to a spectral gap of the associated generator L.
The semi-group $(P_t)_{t\geqslant 0}$ is said to be hyperbounded, if there exists a time $T \geqslant 0$ such that the Markov operator P_T is hyperbounded. It follows from the previous result, applied to $M = P_T$, that a reversible ergodic hyperbounded Markov semi-group admits a spectral gap.

Hypercontractivity

The semi-group $(P_t)_{t\geqslant 0}$ is said to be hypercontractive, if there exist $p > 2$ and a time $T \geqslant 0$ such that the norm of P_T from $\mathbb{L}^2(\mu)$ to $\mathbb{L}^p(\mu)$ is 1.

The Høegh-Krohn and Simon's conjecture is easy to solve for a hypercontractive Markov semi-group, since not only its generator admits a spectral gap, but it also satisfies a logarithmic Sobolev inequality, property in fact equivalent to hypercontractivity.

It is known that hyperboundedness is itself equivalent to a non-tight logarithmic Sobolev inequality and that the existence of a spectral gap enables to tight such an inequality.

Thus the previous theorem shows that for semi-groups, hyperboundedness is equivalent of hypercontractivity.

Finite Cheeger inequality

Finite setting: S is a finite set whose points are given a positive probability by μ. We start with a Markovian generator L: a matrix $(L(x,y))_{x,y\in S}$ whose off-diagonal entries are non-negative and whose lines sum up to zero. We assume that the operator L is symmetric in $\mathbb{L}^2(\mu)$. It is well-known to be non-positive definite, let

$$0 \;=\; \lambda_1 \;\leqslant\; \lambda_2 \;\leqslant\; \cdots \;\leqslant\; \lambda_N$$

be the spectrum of $-L$.

Cheeger's inequality relates λ_2 to an isoperimetric quantity. The conductance associated to any subset $A\subset S$ with $A \neq \emptyset$:

$$j(A) \;\coloneqq\; \frac{\mu(\mathbb{1}_A L(\mathbb{1}_{A^c}))}{\mu(A)}$$

The connectivity constant of L is defined by

$$\iota_2 \;\coloneqq\; \min_{A\neq\emptyset,S} \max\{j(A), j(A^c)\} \;=\; \min_{A\,:\,0<\mu(A)\leqslant 1/2} j(A)$$

With $|L| \coloneqq \max_{x\in S} |L(x,x)|$, the Cheeger's inequality states that

$$\frac{\iota_2^2}{8\,|L|} \;\leqslant\; \lambda_2 \;\leqslant\; \iota_2$$

Connectivity spectrum

What for the other eigenvalues $\lambda_3, ..., \lambda_N$? Introduce the connectivity spectrum $(\iota_n)_{n\in[\![N]\!]}$. For $n \in \mathbb{N}$, let $\mathcal{D}_n$ be the set of n-tuples $(A_1, ..., A_n)$ of disjoint and non-empty subsets of S and

$$\iota_n \quad := \quad \min_{(A_1,\ldots,A_n)\in\mathcal{D}_n} \max_{k\in[\![n]\!]} j(A_k)$$

Clearly $\iota_1 = 0 = \lambda_1$ and for $n = 2$ the two definitions of ι_2 coincide. We conjectured that there exists a mapping $c : \mathbb{N} \to \mathbb{R}_+^*$ such that for all (S, μ, L) as above,

$$c(n)\frac{\iota_n^2}{|L|} \;\leqslant\; \lambda_n \;\leqslant\; \iota_n \qquad (1)$$

The second inequality is immediate, it amounts to consider the vector space generated by the indicator functions of n disjoints subsets in the variational characterization of λ_n through Rayleigh quotients.

Higher order Cheeger inequalities

The first inequality was recently obtained by Lee, Gharan and Trevisan [2011], in the context of weighted finite graphs (S, E, ω): E is a set of undirected edges (which may contain loops) weighted through $\omega : E \to \mathbb{R}_+$. It can be transcripted for finite Markovian generator, taking

$$\forall\ e \in E, \quad \omega(e) \quad := \quad \begin{cases} \mu(x)L(x,y) & \text{, if } e = \{x,y\} \text{ with } x \neq y \\ |L| + L(x,x) & \text{, if } e = \{x,x\} \end{cases}$$

It follows

Theorem (Lee, Gharan and Trevisan)

There exists a universal constant $\eta > 0$ such that (1) is satisfied with

$$\forall\ n \in \mathbb{N}, \qquad c(n) \quad := \quad \frac{\eta}{n^8}$$

A hyperboundedness estimate

Hyperboundedness does not look very pertinent in the finite setting, let us nevertheless derive from the above theorem a quantitative bound for this property. On the finite set S, consider a Markov kernel M symmetric in $\mathbb{L}^2(\mu)$ and denote its spectrum by

$$1 \;=\; \theta_1 \;\geqslant\; \theta_2 \;\geqslant\; \cdots \;\geqslant\; \theta_N \;\geqslant\; -1$$

Proposition

Assume that for some $n \in [\![N]\!]$, $N = \mathrm{card}(S)$, we have $\theta_n \geqslant 1 - c(n)/4$. Then we have that for any $p > 2$,

$$\|M\|^p_{\mathbb{L}^2(\mu)\to\mathbb{L}^p(\mu)} \;\geqslant\; \frac{(1-2\delta_n)^p}{2} n^{\frac{p}{2}-1}$$

where $\delta_n := \sqrt{(1-\theta_n)/c(n)} \leqslant 1/2$.

Proof (1)

Introduce the Markovian generator $L = M - \mathrm{Id}$, the eigenvalues of $-L$ are $\lambda_m = 1 - \theta_m$, $m \in [\![N]\!]$. Next consider $n \in [\![N]\!]$ as in the proposition. According to Theorem [LGT], we have

$$\iota_n \leqslant \sqrt{|L|\lambda_n / c(n)} \leqslant \sqrt{\lambda_n / c(n)} = \delta_n$$

so we can find $(A_1, \ldots, A_n) \in \mathcal{D}_n$ satisfying,

$$\forall\ k \in [\![n]\!], \qquad \delta_n \mu(A_k) \geqslant \mu[\mathbb{1}_{A_k} L[\mathbb{1}_{A_k^c}]]$$

Taking into account that

$$L[\mathbb{1}_{A_k^c}] = \mathbb{1}_{A_k} - M[\mathbb{1}_{A_k}]$$

we deduce that for any $k \in [\![n]\!]$,

$$(1 - \delta_n)\mu(A_k) \leqslant \mu[\mathbb{1}_{A_k} M[\mathbb{1}_{A_k}]]$$

Proof (2)

For any $k \in [\![n]\!]$, consider the set

$$B_k \quad := \quad \{x \in A_k \,:\, M[\mathbb{1}_{A_k}](x) \geqslant 1 - 2\delta_n\}$$

We compute that

$$\begin{aligned} \mu[\mathbb{1}_{A_k} M[\mathbb{1}_{A_k}]] &= \mu[\mathbb{1}_{B_k} M[\mathbb{1}_{A_k}]] + \mu[\mathbb{1}_{A_k \setminus B_k} M[\mathbb{1}_{A_k}]] \\ &\leqslant \mu(B_k) + (1 - 2\delta_n)(\mu(A_k) - \mu(B_k)) \end{aligned}$$

It follows from the two last bounds that

$$\frac{1}{2}\mu(A_k) \;\leqslant\; \mu(B_k) \;\leqslant\; \mu(A_k)$$

Since the sets A_1, ..., A_n are disjoint, there exists $k \in [\![n]\!]$ such that $\mu(A_k) \leqslant 1/n$. Consider $f = \mathbb{1}_{A_k}$, it appears that

$$\mu[f^2] \quad = \quad \mu(A_k)$$

Proof (3)

and since by assumption $1-2\delta_n \geqslant 0$, we get by definition of B_k,

$$\begin{aligned} \mu[|M[f]|^p] &\geqslant (1-2\delta_n)^p \mu(B_k) \\ &\geqslant \frac{(1-2\delta_n)^p}{2}\mu(A_k) \end{aligned}$$

In particular, we obtain that

$$\begin{aligned} \|M\|^p_{\mathbb{L}^2(\mu)\to\mathbb{L}^p(\mu)} &\geqslant \frac{\mu[|M[f]|^p]}{\mu[f^2]^{\frac{p}{2}}} \\ &\geqslant \frac{(1-2\delta_n)^p}{2\mu(A_k)^{\frac{p}{2}-1}} \\ &\geqslant \frac{(1-2\delta_n)^p}{2} n^{\frac{p}{2}-1} \end{aligned}$$

as announced.

■

We come back to the Høegh-Krohn and Simon's conjecture framework and approximate it by finite sets.
We need a preliminary consequence of the spectral Theorem:

Lemma

The ergodic self-adjoint Markov operator M has no spectral gap if and only if for any $\lambda > 0$, $(E_1 - E_{1-\lambda})[\mathbb{L}^2(\mu)]$ is of infinite dimension.

Indeed, the mapping $[0, 2] \ni \lambda \mapsto \dim((E_1 - E_{1-\lambda})[\mathbb{L}^2(\mu)])$ is non-decreasing. So if for some $\lambda \in (0, 2]$, we have that $\dim((E_1 - E_{1-\lambda})[\mathbb{L}^2(\mu)]) < +\infty$, it appears that the $\mathbb{Z}_+$-valued mapping $[0, \lambda] \ni l \mapsto \dim(E_1 - E_{1-l})$ has a finite number of jumps, which correspond to eigenvalues.

A contradictory argument (1)

Assume that the ergodic self-adjoint Markov operator M has no spectral gap. For any $n \in \mathbb{N}$, let $0 < \epsilon_n < 1 \wedge (c(n)/32)$ be given, where $c(n)$ is defined in Theorem [LGT]. By the above lemma, we can find $f_1, \ldots, f_n \in \mathbb{L}^2(\mu)$, which are normalized, mutually orthogonal and so that

$$\mu[f_i M f_j] \quad \begin{cases} = & 0 & , \text{if } i \neq j \\ \geqslant & (1-\epsilon_n) & , \text{if } i = j \end{cases}$$

To come back to the finite case, consider a non-decreasing family $(\mathcal{S}_N)_{N\in\mathbb{N}}$ of finite sub-σ-algebras of $\mathcal{S}$ such that

$$\bigvee_{N\in\mathbb{N}} \mathcal{S}_N \quad = \quad \sigma(f_1, \ldots, f_n)$$

A contradictory argument (2)

Fixing $N \in \mathbb{N}$, we consider μ_N the restriction of μ to $\mathcal{S}_N$, I_N the natural injection of $\mathbb{L}^2(\mu_N)$ into $\mathbb{L}^2(\mu)$ and $\mathbb{E}_N$ the conditional expectation (projection operator) with respect to $\mathcal{S}_N$. Define furthermore $M_N := \mathbb{E}_N M I_N$, which is a reversible Markov kernel on $(S_N, \mathcal{S}_N, \mu_N)$, where S_N is the finite set of atoms of $\mathcal{S}_N$.
It follows from Jensen's inequality, we have for any $p > 2$

$$\|M_N\|^p_{\mathbb{L}^2(\mu_N)\to\mathbb{L}^p(\mu_N)} \quad \leqslant \quad \|M\|^p_{\mathbb{L}^2(\mu)\to\mathbb{L}^p(\mu)}$$

Furthermore by the martingale convergence theorem, for any $f \in \mathbb{L}^2(\sigma(f_1, \ldots, f_n), \mu)$, we have in $\mathbb{L}^2(\mu)$,

$$\lim_{N\to\infty} \mathbb{E}_N[f] \quad = \quad f$$

A contradictory argument (3)

It can be deduced from the choice of $f_1, \dots, f_n$ that for N sufficiently large, M_N has n eigenvalues above $1 - 2\epsilon_n$. We can then apply our finite hyperboundedness estimate with $\delta_n := \sqrt{2\epsilon_n / c(n)} \leqslant 1/4$, to get

$$\begin{aligned}
\|M_N\|^p_{\mathbb{L}^2(\mu_N)\to\mathbb{L}^p(\mu_N)} &\geqslant \frac{(1-2\delta_n)^p}{2} n^{\frac{p}{2}-1} \\
&\geqslant 2^{-p-1} n^{\frac{p}{2}-1}
\end{aligned}$$

It follows that

$$\|M\|^p_{\mathbb{L}^2(\mu)\to\mathbb{L}^p(\mu)} \geqslant \|M_N\|^p_{\mathbb{L}^2(\mu_N)\to\mathbb{L}^p(\mu_N)} \geqslant 2^{-p-1} n^{\frac{p}{2}-1}$$

and since this is true for all $n \in \mathbb{N}$, M cannot be hyperbounded.

Extension of definitions

The previous approximation procedure also leads to the extension of Cheeger's inequalities to the general setting of self-adjoint Markov operators in $\mathbb{L}^2(\mu)$. For any $n \in \mathbb{N}$, define

$$\lambda_n(M) \quad := \quad \inf_{H\,:\,\dim(H)=n} \max\left\{ \frac{\mu[f(\mathrm{Id}-M)[f]]}{\mu[f^2]} \; : \; f \in H\backslash\{0\} \right\}$$

(in the general framework, these quantities are no longer necessarily counting the ordered eigenvalues of $\mathrm{Id}-M$). The definition of conductance can also be extended to all non-negligible and measurable $A \in \mathcal{S}$:

$$j(A) \quad := \quad \frac{\mu(\mathbb{1}_A M(\mathbb{1}_{A^c}))}{\mu(A)}$$

so that the connectivity spectrum $(\iota_n(M))_{n\in\mathbb{N}}$ can be defined as before.

Markovian higher order Cheeger's inequalities

Proposition

With $\eta > 0$ the universal constant of Theorem [LGT], we have

$$\forall\ n \in \mathbb{N}, \qquad \frac{\eta}{n^8} \frac{\iota_n^2(M)}{|M|} \leqslant \lambda_n(M) \leqslant \iota_n(M)$$

Extension to Markovian generators? Let L be the generator of self-adjoint Markovian semi-group $(P_t)_{t \geqslant 0}$. If we define for any $n \in \mathbb{N}$,

$$\lambda_n(L) \quad := \quad \inf_{H\,:\,\dim(H)=n} \max \left\{ \frac{\mu[f(-L)[f]]}{\mu[f^2]} \;:\; f \in H \setminus \{0\} \right\}$$

then we have

$$\forall\ n \in \mathbb{N}, \qquad \lambda_n(L) \quad = \quad \lim_{t \to 0_+} \frac{1 - \exp(-t\lambda_n(L))}{t}$$

$$= \quad \lim_{t \to 0_+} \frac{\lambda_n(P_t)}{t}$$

A similar definition of the connectivity spectrum $(\iota_n(L))_{n\in\mathbb{N}}$ through approximation via small times doesn't work properly. It is convenient to introduce the Dirichlet connectivity spectrum, intermediary between the usual spectrum and the connectivity spectrum. We come back to a general self-adjoint Markov operator M. To any non-negligible $A\in\mathcal{S}$, we associate its first Dirichlet eigenvalue $\lambda_0(M,A)$ given by

$$\left\{\begin{array}{rcl} \lambda_0(M,A) & := & \inf_{f\in\mathcal{D}(A)} \frac{\mu[f(\mathrm{Id}-M)[f]]}{\mu[f^2]} \\ \mathcal{D}(A) & := & \left\{f\in\mathbb{L}^2(\mu)\,:\, f=0\ \mu\text{-a.s. on } A^{\mathrm{c}}\right\} \end{array}\right.$$

The Dirichlet connectivity spectrum $(\Lambda_n(M))_{n\in\mathbb{N}}$ of M is defined by

$$\forall\ n\in\mathbb{N},\qquad \Lambda_n(M) \quad := \quad \min_{(A_1,\ldots,A_n)\in\mathcal{D}_n}\ \max_{k\in[\![n]\!]} \lambda_0(M,A_k)$$

Usual and Dirichlet connectivity spectra

In the finite setting, Lee, Gharan and Trevisan also proved:

Theorem (Lee, Gharan and Trevisan)

There exists a universal constant $\widehat{\eta} > 0$ such that for any finite self-adjoint Markov operator M, we have

$$\forall\ n \in \mathbb{N}, \qquad \frac{\widehat{\eta}}{n^6}\Lambda_n(M) \leqslant \lambda_n(M) \leqslant \Lambda_n(M)$$

There is no difficulty in applying the spatial approximation procedure to the Dirichlet connectivity spectrum, so that the finiteness assumption can be indeed removed.
But due to its spectral features, the Dirichlet connectivity spectrum also well-behaves under small times approximations.

Dirichlet conditions

For any non-negligible $A \in \mathcal{S}$, resorting to the Dirichlet semi-group

$$\forall\ t \geqslant 0, \quad P_{A,t}\ :\ \mathbb{L}^2(\mu) \ni f \quad \mapsto \quad \mathbb{1}_A P_t[\mathbb{1}_A f]$$

it can be proven that

$$\begin{aligned} \lim_{t \to 0+} \frac{\Lambda_n(P_t)}{t} &= \Lambda_n(L) \\ &:= \min_{(A_1,\ldots,A_n)\in\mathcal{D}_n} \max_{k\in[\![n]\!]} \lambda_0(L, A_k) \end{aligned}$$

where

$$\begin{cases} \lambda_0(L, A) &:= \inf_{f\in\mathcal{D}(L,A)} \frac{\mu[f(-L)[f]]}{\mu[f^2]} \\ \mathcal{D}(L, A) &:= \{f \in \mathcal{D}(L)\ :\ f = 0\ \mu\text{-a.s. on } A^{\mathrm{c}}\} \end{cases}$$

With these definitions, Theorem [LGT2] extends to the generator L.

Let us return to the Riemannian setting of Cheeger [1970]. The state space S is a compact Riemannian manifold and L is the associated Laplacian $\triangle$ operator. Due to the regularizing properties of the corresponding heat semi-group, $\mathcal{D}_n$ can be restricted to contain only subsets with smooth boundaries. If A is such a subset, a minimizer function exists for $\lambda_0(L, A)$, so going through the proof of original Cheeger's inequality, via co-area formula, it appears that we can find a subset $B \subset A$ with smooth boundary ∂B such that

$$\left(\frac{\sigma(\partial B)}{\mu(B)}\right)^2 \leqslant 4\lambda_0(\triangle, A)$$

σ is the $(\dim(S) - 1)$-dimensional measure associated to μ, the Riemannian probability. This observation leads to define the connectivity spectrum $(\iota_n(\triangle))_{n\in\mathbb{N}}$ of $\triangle$ through

$$\forall\ n \in [\![N]\!], \qquad \iota_n(\triangle) := \min_{(A_1,\ldots,A_n)\in\widehat{\mathcal{D}}_n} \max_{k\in[\![n]\!]} \frac{\sigma(\partial A_k)}{\mu(A_k)}$$

Cheeger and Buser inequalities

The Riemannian higher order Cheeger inequalities follow:

Theorem

There exists a universal constant $\widehat{\eta} > 0$ *such that for any compact Riemannian manifold* S*, we have*

$$\forall\ n \in \mathbb{N}, \qquad \lambda_n(\triangle) \ \geqslant\ \frac{\widehat{\eta}}{n^6} \iota_n^2(\triangle)$$

Adapting the proof of Ledoux [1994] to the context of Dirichlet semi-groups, we can also extend the inequalities of Buser [1982]:

Theorem

there exists a constant $C > 0$ *depending only on the dimension of* S*, such that for all* $n \in \mathbb{N}$*,*

$$\lambda_n(\triangle) \ \leqslant\ C(\sqrt{K}\iota_n(\triangle) + \iota_n^2(\triangle))$$

where $-K \leqslant 0$ *is a lower bound on the Ricci curvature of* S*.*

But it seems that the inequalities of Theorem [LGT2] are more interesting that those of Cheeger. For instance they can be used to deduce the exponential behavior of the small eigenvalues of Witten Laplacians at small temperature. Still on a compact Riemannian manifold, Witten Laplacians have the form $L_\beta = \triangle \cdot - \beta \langle \nabla U, \nabla \cdot \rangle$, where $\beta \geqslant 0$ is the inverse temperature and $U : S \to \mathbb{R}$ is a $\mathcal{C}^1$ potential.
An open and connected set $B \subset S$ is said to be a well, if U is constant on ∂B and if for any $x \in B$, $U(x) < U(\partial B)$. The height of a well B is given by $h(B) = U(\partial B) - \min_B U$. For $n \in \mathbb{N}$, let l_n the highest $l \geqslant 0$ such that n disjoint wells of height l can be found in S. Then we have

$$\forall \; n \in \mathbb{N}, \qquad l_n \; := \; - \lim_{\beta \to +\infty} \beta^{-1} \ln(\lambda_n(L_\beta))$$

Consider the traditional $\mathbb{L}^2$ to $\mathbb{L}^4$ hypercontractivity. Wang [2004] has shown that if $\|M\|^4_{\mathbb{L}^2(\mu)\to\mathbb{L}^4(\mu)} < 2$, then the conjecture of Høegh-Krohn and Simon is true and it is possible to deduce a quantitative lower bound on the spectral gap. Next result shows that this last part cannot be extended:

Proposition

For any $K \geqslant 2$ and for any $\epsilon \in (0,1)$, we can find a self-adjoint ergodic Markov operator M whose spectral gap is ϵ and which is such that $\|M\|^4_{\mathbb{L}^2(\mu)\to\mathbb{L}^4(\mu)} = K$.

Indeed, the example is very simple and is constructed on the two points state space $S := \{0,1\}$, endowed with the probability measure $\mu := (\eta, 1-\eta)$, $\eta \in (0, 1/2]$, and with the self-adjoint Markov operator in $\mathbb{L}^2(\mu)$ given by

$$M \quad := \quad (1-\epsilon)\mathrm{Id} + \epsilon\mu$$

where $\epsilon \in (0,1]$, whose spectral gap is ϵ.

Changing the indices

To get quantitative bounds, we need to look at other eigenvalues than the spectral gap. Here is another result of Lee, Gharan and Trevisan:

Theorem (Lee, Gharan and Trevisan)

There exists a universal constant $\eta > 0$ such that if $\widetilde{c}(n) := \eta/\ln(1+n)$, then for any finite self-adjoint Markov operator,

$$\forall\ n \in \mathbb{N}, \qquad \lambda_{2n} \quad \geqslant \quad \widetilde{c}(n)\iota_n^2$$

Again there is no difficulty to extend this result to general self-adjoint Markov operators.

Lee, Gharan and Trevisan checked that the logarithmic order of $\widetilde{c}(n)$ is optimal, by applying hypercontractivity results due to Bonami [1970] and Beckner [1975] to noisy hypercubes. We will recover this optimality via the hypercontractivity of the Ornstein-Uhlenbeck process.

A quantitative version

Consider a Young function Φ satisfying

$$\lim_{r\to+\infty} \frac{\Phi(r)}{r^2} = +\infty$$

meaning that the Orlicz's norm of $\mathbb{L}^\Phi$ is stronger than the $\mathbb{L}^2$ norm. Define

$$\forall\ n\in\mathbb{N}, \qquad k(n) = \frac{\sqrt{n}}{\Phi^{-1}(n)}$$

A careful examination of the previous proofs leads to:

Theorem

Let M be a self-adjoint Markov operator. If $n\in\mathbb{N}$ is such that $k(n)\geqslant 2\sqrt{2}\,\|M\|_{\mathbb{L}^2\to\mathbb{L}^\Phi}$, then we are assured of

$$\lambda_{2n}(M) \geqslant \frac{\widetilde{c}(n)}{16}$$

In particular the top of the spectrum of M consists of $2n$ eigenvalues 1, $1-\lambda_2(M)$, ..., $1-\lambda_{2n}(M)$ (with multiplicities).

Ornstein-Uhlenbeck generator

Let $(P_t)_{t\geqslant 0}$ be the self-adjoint Markov semi-group associated to the Ornstein-Uhlenbeck generator defined by

$$\forall\ f \in \mathcal{C}_b^2(\mathbb{R}),\ \forall\ x \in \mathbb{R}, \qquad L[f](x) \quad := \quad f''(x) - xf'(x)$$

which is essentially self-adjoint on $\mathbb{L}^2(\gamma)$, where γ is the normal centered Gaussian distribution.

It is well-known (see Nelson [1973] and Gross [1975]) that for any $p > 2$,

$$\|P_t\|_{\mathbb{L}^2(\gamma)\to\mathbb{L}^p(\gamma)} \quad = \quad \begin{cases} +\infty & \text{, if } t < \frac{1}{2}\ln(p-1) \\ 1 & \text{, if } t \geqslant \frac{1}{2}\ln(p-1) \end{cases}$$

(for a less radical transition, consider the Young function given by $\Phi(r) := (r^2+1)\ln(r^2+1) - r^2$, then $\|P_t\|_{\mathbb{L}^2(\gamma)\to\mathbb{L}^\Phi(\gamma)}$ is of order $1/\sqrt{t}$ for $t > 0$ small).

A disappointing application?

Let us apply the previous theorem to $M = P_t$ for $t > 0$ and relatively to the usual Lebesgue space $\mathbb{L}^p(\gamma)$ with $p > 2$. More precisely, for $n \in \mathbb{N}$, consider

$$
\begin{aligned}
p_n &:= \frac{2\ln(n)}{\ln(n) - 2\ln(2\sqrt{2})} \\
t_n &:= \frac{1}{2}\ln(p_n - 1) \sim 2\ln(2\sqrt{2})/\ln(n)
\end{aligned}
$$

and let $n_0 \in \mathbb{N}$ be the smallest integer such that $p_n > 2$. We obtain

$$
n \geqslant n_0 \implies \lambda_n(L) \geqslant \frac{c(n)}{16 t_n} \tag{2}
$$

The r.h.s. is of order 1, which is quite disappointing, since it is well-known that $\lambda_n(L) = n - 1$ for all $n \in \mathbb{N}$!
Could our quantitative estimates (or Theorem [LGT3]) be improved to get a lower bound of $\lambda_n(L)$ going to infinity with n?

Tensorization

This is not possible, because the hypercontractivity property is stable by tensorization. More precisely, for $N \in \mathbb{N}$, consider the semi-group $(P_t^{\otimes N})_{t \geqslant 0}$ acting on $\mathbb{L}^2(\gamma^{\otimes N})$. The generator $L^{(N)}$ of $(P_t^{\otimes N})_{t \geqslant 0}$ corresponds to the sum of N copies of L, each acting on different coordinates of $\mathbb{R}^N$. In particular, we get

$$\forall \; n \in [\![2, N+1]\!], \qquad \lambda_n(L^{(N)}) \;=\; \lambda_2(L)$$

This forbids a lower bound going to infinity with n in (2) and shows the optimality of the previous estimates.