Discrete Ricci curvature via convexity of the entropy

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Starting point

McCann '94: beautiful connection between

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- Boltzmann-Shannon entropy

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- 2-Wasserstein metric from optimal transport

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Relative entropy

Let m be a reference measure on $\mathcal{X}$.

For $\mu \in \mathcal{P}(\mathcal{X})$: $$\mathrm{Ent}_m(\mu) = \begin{cases} \int_{\mathcal{X}} \rho(x) \log \rho(x) \, \mathrm{d}m(x), & \frac{d\mu}{dm} = \rho, \\ +\infty, & \text{otherwise.} \end{cases}$$

Optimal transport and entropy

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Definition of the 2-Wasserstein metric

$$W_2(\mu,\nu)^2 := \inf \left\{ \int_{\mathcal{X}\times\mathcal{X}} d(x,y)^2 \, \mathrm{d}\gamma(x,y) \ : \ \gamma \text{ with marginals } \mu \text{ and } \nu \right\}$$

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$$= \inf \left\{ \mathbb{E}[d(X,Y)^2] \ : \ \mathrm{law}(X) = \mu, \ \mathrm{law}(Y) = \nu \right\}$$

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Properties

- W_2 defines a metric on $\mathcal{P}_2(\mathcal{X})$.
- $(\mathcal{X}, d)$ is a geodesic space $\Rightarrow$ $(\mathcal{P}_2(\mathcal{X}), W_2)$ is a geodesic space.

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Theorem (McCann '94)

The entropy is convex along geodesics in $(\mathcal{P}_2(\mathbf{R}^n), W_2)$.

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Highly non-linear interpolation based on *geometry* of the underlying space!

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Theorem (Otto–Villani '00, Cordero–McCann–Schmuckenschläger '01, von Renesse–Sturm '05)

For a Riemannian manifold $\mathcal{M}$, TFAE:

1. Displacement κ-convexity of the entropy:

$$\mathrm{Ent}(\mu_t) \le (1-t)\mathrm{Ent}(\mu_0) + t\mathrm{Ent}(\mu_1) - \frac{\kappa}{2} t(1-t) W_2^2(\mu_0, \mu_1)$$

   for all 2-Wasserstein geodesics $(\mu_t)_{t\in[0,1]}$;

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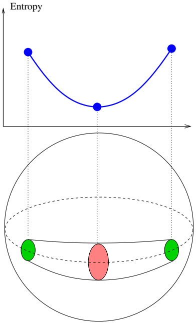

Optimal transport and Ricci curvature II

Definition (Sturm '06, Lott–Villani '09)

A metric measure space $(\mathcal{X}, d, m)$ satisfies $\mathrm{Ric}(\mathcal{X}) \geq \kappa$ if

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But..... what about discrete spaces?

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Conclusion: LSV-Definition does not apply to discrete spaces.

Ricci curvature of discrete spaces

Many approaches to discrete Ricci curvature:

- W_1-contraction à la Dobrushin
  Ollivier ('09); Sammer ('05), Joulin ('09), Jost, Bauer, Hua, Liu ('11-...)
- approximate W_2-geodesics
  Bonciocat, Sturm ('09), Ollivier, Villani ('12)
- modified Bakry-Émery criterion
  Lin, Lu, S.-T. Yau ('11-...), Bauer, Horn, Lin, Lippner, Mangoubi, S.-T. Yau ('13)
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Our goal: Find a notion of discrete Ricci curvature

- in the spirit of Lott-Sturm-Villani
- which allows to prove sharp functional inequalities.

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Gradient flows in $\mathbf{R}^n$

Let $\varphi : \mathbf{R}^n \to \mathbf{R}$ smooth and convex. For $u : \mathbf{R}_+ \to \mathbf{R}^n$ TFAE:

1. u solves the gradient flow equation $u'(t) = -\nabla\varphi(u(t))$.
2. u solves the evolution variational inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u(t) - y|^2 \leq \varphi(y) - \varphi(u(t)) \qquad \forall y \ .$$

(De Giorgi '93, Ambrosio–Gigli–Savaré '05)

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The heat flow is the gradient flow of the entropy w.r.t W_2 , i.e.,

$$\partial_t \mu = \Delta \mu \quad \Longleftrightarrow \quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} W_2(\mu_t, \nu)^2 \leq \mathrm{Ent}(\nu) - \mathrm{Ent}(\mu_t) \qquad \forall \nu \ .$$

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Heat flow as gradient flow of the entropy

Many extensions have been proved:

• $\mathbf{R}^n$	Jordan–Kinderlehrer–Otto
• Riemannian manifolds	Villani, Erbar
• Hilbert spaces	Ambrosio–Savaré–Zambotti
• Finsler spaces	Ohta–Sturm
• Wiener space	Fang–Shao–Sturm
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Question

Is there a version of the JKO-Theorem for *discrete* spaces?

Discrete setting

Setting

- $\mathcal{X}$: finite set
- $K : \mathcal{X} \times \mathcal{X} \to \mathbf{R}_+$ Markov kernel, $\forall x \;:\; \sum_y K(x,y) = 1$
- π: reversible measure, $\forall x, y \;:\; K(x,y)\pi(x) = K(y,x)\pi(y)$

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Heat flow

- Discrete Laplacian: $\Delta\psi(x) := \sum_y K(x,y)(\psi(y) - \psi(x))$
- Continuous time Markov semigroup: $P_t = e^{t\Delta}$

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Relative Entropy

- $\mathcal{P}(\mathcal{X}) := \left\{ \rho : \mathcal{X} \to \mathbf{R}_+ \ \middle| \ \sum_{x \in \mathcal{X}} \rho(x)\pi(x) = 1 \right\}$,
- $\mathrm{Ent}(\rho) := \sum_{x \in \mathcal{X}} \rho(x) \log \rho(x)\, \pi(x), \qquad \rho \in \mathcal{P}(\mathcal{X})$.

Simplest non-trivial example: 2-point space

$\mathcal{X} = \{0, 1\}$ $\qquad K(0,1) = K(1,0) = 1$ $\qquad \pi(0) = \pi(1) = \frac{1}{2}$

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Is the heat flow on $\{0,1\}$ the gradient flow of the entropy w.r.t. some other metric on $\mathcal{P}(\{-1,1\})$?

Answer

YES!

Proposition [M. 2011]

The heat flow is the gradient flow of Ent w.r.t. the metric $\mathcal{W}$, where

$$\mathcal{W}(\mu_\alpha, \mu_\beta) := \sqrt{2}\int_\alpha^\beta \sqrt{\frac{\mathrm{arctanh}(2r-1)}{2r-1}}\,\mathrm{d}r, \qquad 0 \le \alpha \le \beta \le 1.$$

Simplest non-trivial example: 2-point space

$\mathcal{X} = \{0, 1\}$ $\quad K(0,1) = K(1,0) = 1$ $\quad \pi(0) = \pi(1) = \frac{1}{2}$

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How to generalise this to the general discrete case?

Back to $\mathbf{R}^n$: W_2 as Riemannian metric (Otto '01)

- If $t \mapsto \rho_t$ is smooth, the continuity equation

$$\partial_t \rho + \mathrm{div}(\rho \Psi) = 0$$

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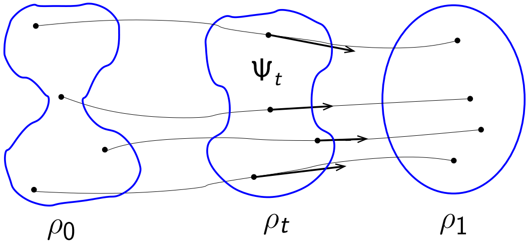

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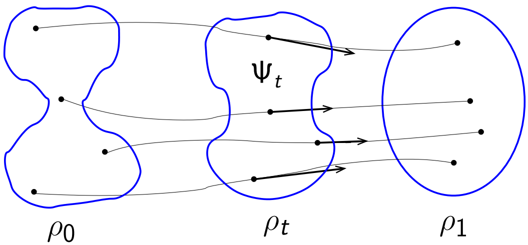

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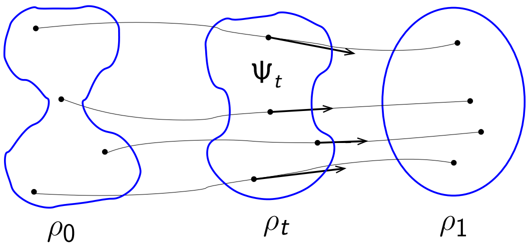

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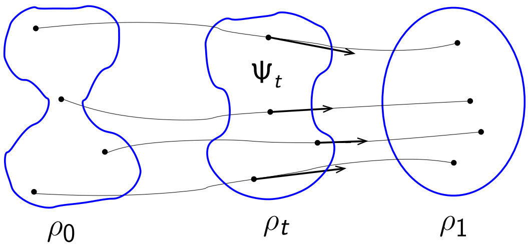

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Associated Riemannian distance (Benamou-Brenier formula)

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Definition of the metric $\mathcal{W}$

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Problem: ρ is defined on vertices, $\nabla \psi$ is defined on edges

$\rightsquigarrow$ No canonical way to define the product $\rho \cdot \nabla \psi$!

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Independent works by Chow–Huang–Li–Zhou '12 and Mielke '11.

Why the logarithmic mean?

Formal proof of the JKO-Theorem

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$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ent}(\rho_t) = -\langle \log \rho_t, \mathsf{div}(\rho_t \nabla \psi_t) \rangle = \langle \nabla \log \rho_t, \rho_t \nabla \psi_t \rangle \ .$$

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Logarithmic mean compensates for the lack of a discrete chain rule:

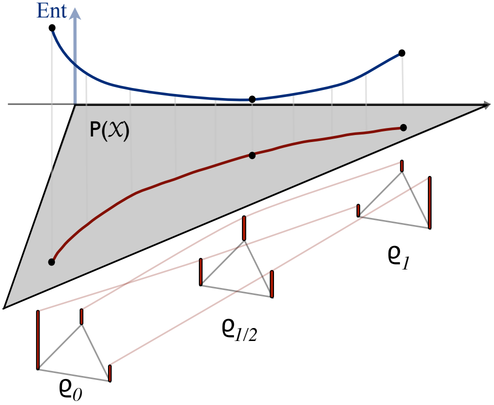
$$\rho(x) - \rho(y) = \hat{\rho}(x, y)\big(\log \rho(x) - \log \rho(y)\big)$$

Ricci curvature of Markov chains

Discrete analogue of Lott–Sturm–Villani:

Definition (Erbar, M. 2012)

We say that $(\mathcal{X}, K, \pi)$ has Ricci curvature bounded from below by $\kappa \in \mathbf{R}$ if the entropy is κ-convex along geodesics in $(\mathcal{P}(\mathcal{X}), \mathcal{W})$.



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Discrete Bakry-Émery Theorem (Erbar, M. '12)

If $\mathrm{Ric}(K) \geq \kappa > 0$, then the modified log-Sobolev inequality holds:

$$\mathrm{Ent}(\rho) \leq \frac{1}{2\kappa} \mathcal{I}(\rho) \qquad (\mathrm{mLSI}(\kappa))$$

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$$\mathrm{Ent}(P_t\rho) \leq e^{-2\kappa t}\,\mathrm{Ent}(\rho).$$

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Let $(\mathcal{X}, K, \pi)$ be a reversible Markov chain. Let $\kappa > 0$.

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- the T_1-inequality holds: $W_1(\rho, \mathbf{1})^2 \le \frac{1}{\kappa} \operatorname{Ent}(\rho)$.

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- Dissipative quantum mechanics (Carlen-M., Mielke)
  non-commutative analogue of $\mathcal{W}$ for density matrices

Thank you!