Variants of Parseval’s formula and the Grothendieck inequality

Ron Blei

Department of Mathematics
University of Connecticut

October 8, 2013

The setting

Given a set A, take the Euclidean space

$$\ell^2 = \ell^2(A) := \{\mathbf{x} \in \mathbb{C}^A : \sum_{\alpha \in A} |\mathbf{x}(\alpha)|^2 < \infty\},$$

equipped with the usual dot product

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{\alpha \in A} \mathbf{x}(\alpha)\overline{\mathbf{y}(\alpha)}, \quad \mathbf{x} \in \ell^2,\ \mathbf{y} \in \ell^2, \tag{1}$$

and the resulting Euclidean norm

$$\|\mathbf{x}\|_2 := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \left(\sum_{\alpha \in A} |\mathbf{x}(\alpha)|^2 \right)^{\frac{1}{2}}, \quad \mathbf{x} \in \ell^2.$$

Let B_{ℓ^2} denote the closed unit ball,

$$B_{\ell^2} := \{\mathbf{x} \in \ell^2 : \|\mathbf{x}\|_2 \leqslant 1\}.$$

A generic presentation of $\ell^2(A)$

Take a sufficiently "large" σ-finite measure space (Ω, μ), and consider the Hilbert space

$$L^2(\Omega, \mu) = \{ f \in \mathcal{L}^0(\Omega) : \int_\Omega |f|^2 d\mu < \infty \}, \tag{2}$$

where $\mathcal{L}^0(\Omega)$ is the space of $\mathbb{C}$-valued measurable functions on Ω, with inner product

$$\langle f, g \rangle_{L^2} = \int_\Omega f\, \overline{g}\, d\mu, \quad f \in L^2(\Omega, \mu),\ g \in L^2(\Omega, \mu). \tag{3}$$

Select an orthonormal family of functions

$$F = \{f_\alpha\}_{\alpha \in A} \subset L^2(\Omega, \mu),$$

and define

$$U : \ell^2(A) \to L^2(\Omega, \mu) \tag{4}$$

by

$$U\mathbf{x} = \sum_{\alpha \in A} \mathbf{x}(\alpha) f_\alpha, \quad \mathbf{x} \in \ell^2(A). \tag{5}$$

Parseval's formula

Then,

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} \rangle = \int_\Omega U\mathbf{x}\; U\overline{\mathbf{y}}\; d\mu = \int_\Omega U\mathbf{x}\; \overline{U\mathbf{y}}\; d\mu \\ := \langle U\mathbf{x}, U\mathbf{y} \rangle_{L^2}, \quad \mathbf{x} \in \ell^2(A),\ \mathbf{y} \in \ell^2(A), \end{aligned} \tag{6}$$

i.e., U is a unitary map: a one-one, "angle" preserving linear map from the Euclidean space $\ell^2(A)$ onto

$$L^2_F := L^2\text{-closure of the span of } \{f_\alpha\}_{\alpha \in A}.$$

Note: U is automatically continuous...

The relation in (6) is called *Parseval's formula*, specifically in a harmonic-analytic framework, and sometimes also *Parseval's identity*, usually in more general contexts.

Beyond square-integrability, what more can be said about the ”size” of functions in L^2_F?

That depends on the choice of the orthonormal system

$$F = \{f_\alpha : \alpha \in A\} \subset L^2(\Omega, \mu).$$

If F is *complete*, then the answer is: nothing.

But if F is ”thin” – in some sense opposite to ”complete” – then we can expect improved integrability...

Harmonic analysis on dyadic groups

$\Omega_A = \{-1, 1\}^A$, a compact Abelian group with Haar measure $\mathbb{P}_A$.

Take the Rademacher system $R_A := \{r_\alpha : \alpha \in A\}$,

$$r_\alpha(\omega) = \omega(\alpha), \quad \omega \in \Omega_A, \;\; \alpha \in A \quad \text{(Rademacher characters)},$$

and adjoin to it $r_0 \equiv 1$ on Ω_A. The dual group of Ω_A is

$$\widehat{\Omega}_A = W_A = \bigcup_{k=0}^{\infty} W_{A,k} \quad \text{(Walsh characters)}, \tag{7}$$

where $W_{A,0} = \{r_0\}$, and for $k \in \mathbb{N}$,

$$W_{A,k} = \Big\{ \prod_{\alpha \in F} r_\alpha : F \subset A, \; |F| = k \Big\} \quad \text{(Walsh characters of order } k\text{)}.$$

W_A is a basis for $L^2(\Omega_A, \mathbb{P}_A)$, whereas R_A is a ”basis” for W_A.

We take the canonical unitary equivalence between $\ell^2(A)$ and $L^2_{R_A}$,

$$\mathbf{x} \to U\mathbf{x} = \sum_{\alpha \in A} \mathbf{x}(\alpha) r_\alpha, \qquad \mathbf{x} \in \ell^2(A), \quad U\mathbf{x} \in L^2_{R_A}. \tag{8}$$

Theorem 1 (Khintchin, 1924)

$$\mathbb{P}_A\big(|U\mathbf{x}| \geqslant t\big) \leqslant \exp(-t^2/2), \qquad \mathbf{x} \in B_{\ell^2}, \; t > 0, \tag{9}$$

which is optimal: 2 cannot be replaced by a larger exponent.

Question. Can we do better with other representations of $\ell^2(A)$?

E.g., are there probability spaces (Ω, μ), and continuous injections

$\Phi : \ell^2(A) \to L^2(\Omega, \mu)$, such that

$$\|\Phi(\mathbf{x})\|_{L^\infty} \leqslant K, \quad \mathbf{x} \in B_{\ell^2}, \qquad \text{and}$$

$$\sum_{\alpha \in A} \mathbf{x}(\alpha)\mathbf{y}(\alpha) = \int_\Omega \Phi(\mathbf{x})\Phi(\mathbf{y}) \, d\mu, \quad \mathbf{x} \in \ell^2(A), \; \mathbf{y} \in \ell^2(A)?$$

The Grothendieck inequality

There exists $1 < K < \infty$, such that for every finite scalar array (a_{jk}),

$$\sup\left\{\left|\sum_{j,k} a_{jk}\langle \mathbf{x}_j, \mathbf{y}_k\rangle\right| : \mathbf{x}_j, \mathbf{y}_k \in B_{\ell^2}\right\} \leqslant K \sup\left\{\left|\sum_{j,k} a_{jk} s_j t_k\right| : s_j, t_k \in [-1, 1]\right\}$$

An equivalent assertion had appeared in Grothendieck's 1953 *Resumé*, and remained unnoticed until its reformulation above in [Lindenstrauss and Pelczynski,1968]. Since its reformulation, known as *the Grothendieck inequality*, it has been applied in functional, harmonic, and stochastic analysis, and recently also in theoretical physics and theoretical computer science. (See [Pisier, 2012].)

The evaluation of the "smallest" K, denoted by $\mathcal{K}_G$ and dubbed *the Grothendieck constant*, is an open problem. For the latest on it, see [Braverman et al., 2011].

The dual statement

Take $\Omega_{B_{\ell^2}} = \{-1, 1\}^{B_{\ell^2}}$, and $R_{B_{\ell^2}} = \{r_{\mathbf{x}} : \mathbf{x} \in B_{\ell^2}\}$.

Proposition 1

The Grothendieck inequality holds $\Leftrightarrow$ *there exists a complex measure* $\lambda \in M\big(\Omega_{B_{\ell^2}} \times \Omega_{B_{\ell^2}}\big)$, *such that for all* $\mathbf{x} \in B_{\ell^2}$, $\mathbf{y} \in B_{\ell^2}$,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{\Omega_{B_{\ell^2}} \times \Omega_{B_{\ell^2}}} r_{\mathbf{x}}(\omega_1) r_{\mathbf{y}}(\omega_2) \lambda(d\omega_1, d\omega_2) = \widehat{\lambda}(r_{\mathbf{x}} \otimes r_{\mathbf{y}}),$$

The Grothendieck constant $\mathcal{K}_G$ *is the minimum of* $\|\lambda\|_M$ *over all representations of the dot product by* $\widehat{\lambda} \in \big(M(\Omega_{B_{\ell^2}} \times \Omega_{B_{\ell^2}})\big)^{\wedge}$.

A Parseval-like formula

Corollary 2

The Grothendieck inequality holds $\Leftrightarrow$ *if there exist a probability measure* μ *on* $\Omega := \Omega_{B_{\ell^2}} \times \Omega_{B_{\ell^2}}$, *and a one-one map*

$$\Phi : \ell^2(A) \to L^2(\Omega, \mu), \tag{10}$$

such that

$$\|\Phi(\mathbf{x})\|_{L^\infty} \leqslant K\|\mathbf{x}\|_2, \tag{11}$$

and

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_\Omega \Phi(\mathbf{x})\Phi(\overline{\mathbf{y}})d\mu, \quad \mathbf{x} \in \ell^2(A),\ \mathbf{y} \in \ell^2(A). \tag{12}$$

Proof.

"Polarize" the mappings supplied by Proposition 1. □

Still a question...

In the Parseval-like formula – the assertion equivalent to the Grothendieck inequality – the underlying measurable space Ω is huge(!), the probability measure μ is non-constructible, and the injection Φ is nowhere continuous (with respect to the norm as well as the weak topologies).

Is there a more tractable, "standard" probability space (Ω, μ), along with a constructible *continuous* injection

$$\Phi : \ell^2(A) \to L^2(\Omega, \mu),$$

such that

$$\|\Phi(\mathbf{x})\|_{L^\infty} \leqslant K\|\mathbf{x}\|_2, \quad \mathbf{x} \in \ell^2(A), \tag{13}$$

and

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_\Omega \Phi(\mathbf{x})\Phi(\overline{\mathbf{y}})d\mu, \quad \mathbf{x} \in \ell^2(A),\ \mathbf{y} \in \ell^2(A)? \tag{14}$$

The $L^1_{R_A} \hookrightarrow L^2$ - Khintchin inequality

Theorem 3 (Littlewood, 1930)

$$\sup\left\{ \frac{\|f\|_{L^2(\Omega_A,\mathbb{P}_A)}}{\|f\|_{L^1(\Omega_A,\mathbb{P}_A)}} : f \in L^2_{R_A},\ f \neq \mathbf{0} \right\} := \kappa_A \leqslant \sqrt{6} \tag{15}$$

Note

$$\kappa_A < \ \infty \Leftrightarrow\ L^1_{R_A} = L^2_{R_A}, \tag{16}$$

where $L^1_{R_A}$ is the $L^1(\Omega_A, \mathbb{P}_A)$-closure of the linear span of R_A.

The assertion in (15), with various upper estimates for the *Khintchin constant*

$$\kappa := \sup_A \kappa_A,$$

had been proved nearly a century ago, independently, by Littlewood, Orlicz, Steinhaus, and Zygmund. That $\kappa = \sqrt{2}$ was proved by Szarek in his 1976 Master's thesis.

The dual statement

Restated as

$$(L^2_{R_A})^* = (L^1_{R_A})^*,$$

the Khintchin inequality becomes the assertion (via Hahn-Banach, Riesz, and Parseval)

$$\ell^2(A) = \big(L^\infty(\Omega_A, \mathbb{P}_A)\big)^\wedge\big|_{R_A}.$$

That is, there exists a mapping

$$G = U + g : \ \ell^2(A) \ \to \ L^\infty(\Omega_A, \mathbb{P}_A), \tag{17}$$

where

$$U\mathbf{x} = \sum_{\alpha \in A} \mathbf{x}(\alpha) r_\alpha, \qquad g(\mathbf{x}) \in L^2_{W_A \setminus R_A},$$

and

$$\|G(\mathbf{x})\|_{L^\infty} \leqslant \sqrt{2}\|\mathbf{x}\|_2, \quad \mathbf{x} \in \ell^2(A). \tag{18}$$

(We refer to G as an *interpolant*, and to g as the *orthogonal perturbation* associated with it.)

Khintchin falls short

The Khintchin inequality guarantees the existence of an interpolant,

$$G : \ell^2(A) \to L^\infty(\Omega_A, \mathbb{P}_A),$$

such that

$$\big(G(\mathbf{x})\big)^\wedge(r_\alpha) = \mathbf{x}(\alpha), \quad \mathbf{x} \in \ell^2(A), \;\; \alpha \in A, \tag{19}$$

but does not guarantee its continuity, and that

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{\Omega_A} G(\mathbf{x}) G(\overline{\mathbf{y}}) d\mathbb{P}_A. \quad \mathbf{x} \in \ell^2(A), \;\; \mathbf{y} \in \ell^2(A). \tag{20}$$

Indeed, we have (via Parseval)

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{\Omega_A} G(\mathbf{x}) G(\overline{\mathbf{y}}) d\mathbb{P}_A - \int_{\Omega_A} g(\mathbf{x}) g(\overline{\mathbf{y}}) d\mathbb{P}_A, \tag{21}$$

but have no assurance that the second term on the right side of (21) vanishes.

But, an idea...

The application of Parseval's formula,

$$\int_{\Omega_A} G(\mathbf{x})G(\overline{\mathbf{y}})d\mathbb{P}_A = \sum_{\alpha\in A} \widehat{G(\mathbf{x})}(r_\alpha)\widehat{G(\overline{\mathbf{y}})}(r_\alpha) + \int_{\Omega_A} g(\mathbf{x})g(\overline{\mathbf{y}})d\mathbb{P}_A$$

$$\int_{\Omega_A} G(\mathbf{x})G(\overline{\mathbf{y}})d\mathbb{P}_A = \langle \mathbf{x}, \mathbf{y}\rangle + \int_{\Omega_A} g(\mathbf{x})g(\overline{\mathbf{y}})d\mathbb{P}_A, \tag{22}$$

suggests a recursive scheme:

The right side of (22) equals

$$\langle \mathbf{x}, \mathbf{y}\rangle \; + \; \text{"error"}, \tag{23}$$

where the "error" is a dot product of of two vectors in $\ell^2(W_A \backslash R_A)$. Apply the interpolant G to each of these two vectors, apply Parseval's formula again, subtract the result from (22), and repeat...

An iteration?

Assume A is infinite. Then, A and $W_A \backslash R_A$ have the same cardinality, and we fix a bijection

$$\tau : A \to W_A \backslash R_A. \tag{24}$$

Given $\mathbf{x} \in \ell^2(A)$, define $\mathbf{x}^{(j)} \in \ell^2(A)$ recursively:

$$\begin{aligned} \mathbf{x}^{(1)} &= \mathbf{x}, \\ \mathbf{x}^{(j)} &= \big(g(\mathbf{x}^{(j-1)})\big)^\wedge \circ \tau, \quad j \geqslant 2. \end{aligned} \tag{25}$$

Then, by an iteration of Parseval's formula, we formally(!) have

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^{\infty} (-1)^{j-1} \int_{\Omega_A} G(\mathbf{x}^{(j)}) G(\overline{\mathbf{y}}^{(j)}) \, d\mathbb{P}_A. \tag{26}$$

Uniformizability?

To guarantee convergence of the iteration, say, via a geometric series argument, we need the L^2-norm of the perturbation $g(\mathbf{x}),\ \mathbf{x} \in B_{\ell^2(A)}$, to be uniformly below 1.

Note: $\kappa = \sqrt{2}$ (= the Khintchin constant) implies, via the triangle inequality,

$$\|g(\mathbf{x})\|_{L^2} \leqslant 1, \quad \mathbf{x} \in B_{\ell^2(A)}, \tag{27}$$

and no better...

Question. Is there an interpolant G with an orthogonal perturbation g, such that

$$\|g(\mathbf{x})\|_{L^2} \leqslant \delta < 1, \quad \|G(\mathbf{x})\|_{L^\infty} \leqslant u(\delta) < \infty, \qquad \mathbf{x} \in B_{\ell^2(A)}? \tag{28}$$

We call such an interpolant G a *uniformizing interpolant*.

Uniformizing constants

Making matters precise, for $\delta > 0$ and $\mathbf{x} \in B_{\ell^2(A)}$, we let

$$u_A(\mathbf{x};\delta) = \inf\Big\{\big\|\sum_{\alpha\in A}\mathbf{x}(\alpha)r_\alpha + g(\mathbf{x})\big\|_{L^\infty} : g(\mathbf{x}) \in \big(L^2_{R_A}\big)^\perp,\ \|g(\mathbf{x})\|_{L^2} \leqslant \delta\Big\},$$

$$u_A(\delta) = \sup\big\{u_A(\mathbf{x};\delta) : \mathbf{x} \in B_{\ell^2(A)}\big\},$$

and

$$u(\delta) = \sup_A u_A(\delta).$$

We refer to $u(\delta)$, $\delta > 0$, as *uniformizing constants* (associated with the Rademacher system).

Problem. Compute $u(\delta)$, $\delta > 0$.

Whereas $u(1) = \sqrt{2}$ is immediate from $\kappa = \sqrt{2}$ (the Khintchin constant), it is not obvious that $u(\delta)$ is finite for $0 < \delta < 1$.

Continuity?

The dual formulation of the $(L^1 - L^2)$-Khintchin inequality guarantees existence of an interpolant, via the *axiom of choice* (Hahn-Banach, etc.), and implies nothing more.

Question. Can $G(\mathbf{x}),\ \mathbf{x} \in \ell^2(A),$ be chosen continuously with respect to the $\ell^2(A)$-norm (on its domain) and the $L^2(\Omega_A, \mathbb{P}_A)$-norm (on its range), and continuously also with respect to the weak topologies on its domain and range?

Answers to both questions (uniformizability and continuity) are affirmative. Both are obtained through the use of Riesz products.

Riesz products

Define the Riesz product

$$\mathfrak{R}_A(\mathbf{x}) \sim \prod_{\alpha \in A} \big(r_0 + \mathbf{x}(\alpha) r_\alpha\big), \quad \mathbf{x} \in \mathbb{C}^A, \tag{29}$$

to be the formal Walsh series

$$\mathfrak{R}_A(\mathbf{x}) \sim \sum_{k=1}^{\infty} \left(\sum_{\{\alpha_1,\ldots,\alpha_k\} \subset A} \mathbf{x}(\alpha_1) \cdots \mathbf{x}(\alpha_k) r_{\alpha_1} \cdots r_{\alpha_k} \right). \tag{30}$$

Basic question.
What does the Walsh series in (30) represent?

An L^∞-valued Riesz product

Define

$$Q_A(\mathbf{x}) := \Im m\, \Re_A(\mathrm{i}\mathbf{x}),$$

where $\mathrm{i} = \sqrt{-1}$, and $\Im m$ denotes the imaginary part.

Then,

$$Q_A(\mathbf{x}) \sim \sum_{k=1}^{\infty}(-1)^{k-1}\Bigg(\sum_{\{\alpha_1,\ldots,\alpha_{2k-1}\}\subset A} \mathbf{x}(\alpha_1)\cdots\mathbf{x}(\alpha_{2k-1})r_{\alpha_1}\cdots r_{\alpha_{2k-1}}\Bigg).$$

Key Lemma

If $\mathbf{x} \in \ell^2_{\mathbb{R}}(A)$ (= $\mathbb{R}$eal Euclidean space), then $Q_A(\mathbf{x})$ is the Walsh series of an element in $L^\infty(\Omega_A, \mathbb{P}_A)$, and

$$\|Q_A(\mathbf{x})\|_{L^\infty} \leqslant e^{\frac{\|\mathbf{x}\|_2^2}{2}}; \tag{31}$$

Key Lemma continued

for all $u > 0$,

$$\big(uQ_A(\mathbf{x}/u)\big)^\wedge(r_\alpha) = \mathbf{x}(\alpha), \quad \alpha \in A, \tag{32}$$

and

$$\|\big(uQ_A(\mathbf{x}/u)\big)^\wedge|_{W\setminus R_A}\|_2 \ \leqslant u\sqrt{\sinh(\|\mathbf{x}/u\|_2^2) - \|\mathbf{x}/u\|_2}. \tag{33}$$

Moreover, For $\boldsymbol{x} \in \ell^2_{\mathbb{R}}(A),\ \boldsymbol{y} \in \ell^2_{\mathbb{R}}(A)$,

$$\|Q_A(\mathbf{x}) - Q_A(\mathbf{y})\|_{L^2(\Omega_A,\mathbb{P}_A)} \ \leqslant\ \sqrt{2\cosh(2\rho^2)}\ \|\mathbf{x}-\mathbf{y}\|_2, \tag{34}$$

where $\rho = \max\{\|\mathbf{x}\|_2, \|\mathbf{y}\|_2\}$.

Sketch of proof

To verify that $Q_A(\mathbf{x}) \in L^\infty(\Omega_A, \mathbb{P}_A)$, take

$$\text{finite } F \subset \operatorname{spect}\big(Q_A(\mathbf{x})\big),$$

and estimate

$$\begin{aligned}
\big\|Q_A(\mathbf{x})\big\|_{L^\infty} &\leqslant \big\|\prod_{\alpha\in F}\big(r_0 + \mathrm{i}\mathbf{x}(\alpha)r_\alpha\big)\big\|_{L^\infty} = \Big(\prod_{\alpha\in F}\big(1+|\mathbf{x}(\alpha)|^2\big)\Big)^{\frac{1}{2}} \\
&= e^{\frac{1}{2}\sum_{\alpha\in F}\log(1+|\mathbf{x}(\alpha)|^2)} \\
&\leqslant\ e^{\frac{\|\mathbf{x}\|_2^2}{2}}.
\end{aligned} \tag{35}$$

Now take a sequence of finite sets (F_k) increasing to $\operatorname{spect}\big(Q_A(\mathbf{x})\big)$, and verify that $Q_{F_k}(\mathbf{x})$ converges in weak*-L^∞ to $Q_A(\mathbf{x})$ with the norm bound in (35).

Proof continued...

For every $u > 0$,

$uQ_A(\mathbf{x}/u) \sim$

$$\sim u \sum_{k=1}^{\infty} (-1)^{k-1} \left(\frac{1}{u^{2k-1}} \sum_{\{\alpha_1,\ldots,\alpha_{2k-1}\} \subset A} \mathbf{x}(\alpha_1) \cdots \mathbf{x}(\alpha_{2k-1}) r_{\alpha_1} \cdots r_{\alpha_{2k-1}} \right)$$

$$= \sum_{\alpha \in A} \mathbf{x}(\alpha) r_\alpha + \sum_{k=1}^{\infty} (-1)^{k-1} \left(\frac{1}{u^{2k}} \sum_{\{\alpha_1,\ldots,\alpha_{2k-1}\} \subset A} \mathbf{x}(\alpha_1) \cdots \mathbf{x}(\alpha_{2k-1}) r_{\alpha_1} \cdots r_{\alpha_{2k-1}} \right), \tag{36}$$

which verifies that $\left(uQ_A(\mathbf{x}/u)\right)^\wedge$ interpolates $\mathbf{x}$ on R_A.

$uQ_A(\cdot/u)$ uniformizes and is Lipschitz...

To verify the ℓ^2-bound on $\big(uQ_A(\mathbf{x}/u)\big)^\wedge\big|_{W_A\setminus R_A}$, we estimate

$$\sum_{\{\alpha_1,\ldots,\alpha_{2k-1}\}\subset A} |\mathbf{x}(\alpha_1)\cdots\mathbf{x}(\alpha_{2k-1})|^2 \;\leqslant\; \frac{1}{(2k-1)!}\bigg(\sum_{\alpha\in A}|\mathbf{x}(\alpha)|^2\bigg)^{2k-1},$$

and then,

$$\big\|\big(uQ_A(\mathbf{x}/u)\big)^\wedge|_{W_A\setminus R_A}\big\|_2 \;\leqslant u\left(\sum_{k=2}^{\infty}\frac{\|\mathbf{x}/u\|_2^{2(2k-1)}}{(2k-1)!}\right)^{\frac{1}{2}}$$

$$= u\sqrt{\sinh(\|\mathbf{x}/u\|_2^2) - \|\mathbf{x}/u\|_2}.$$

That $Q_A : \ell^2_{\mathbb{R}}(A) \to L^2(\Omega_A, \mathbb{P}_A)$ is a Lipschitz function follows from estimates...

Therefore...

Corollary

The uniformizing constants $u(\delta)$ *are* $\mathcal{O}\left(\frac{1}{\sqrt{\delta}}\right), \quad 0 < \delta < 1.$

In particular, $Q_A : \ell^2_{\mathbb{R}}(A) \to L^2(\Omega_A, \mathbb{P}_A)$ *is a uniformizing interpolant with*

$$\delta = \sqrt{\sinh(1) - 1} < 1,$$

and is norm-continuous as well as weakly continuous.

Returning to Grothendieck, we can now implement the iteration:

Assume A is infinite, and then let $\{A_j : j \in \mathbb{N}\}$ be a partition of A, such that each A_j has the same cardinality as A.

Then, for every $j \in \mathbb{N}$, A_{j+1} and $W_{A_j} \backslash R_{A_j}$ also have the same cardinality, and we fix bijections

$$\tau_1 : A_1 \to A, \qquad \tau_j : A_j \to W_{A_{j-1}} \backslash R_{A_{j-1}}, \;\; j \geqslant 2. \tag{37}$$

An iteration

Write $Q_{A_j} = U_j + g_j$. Given $\mathbf{x} \in \ell^2_{\mathbb{R}}(A)$, define $\mathbf{x}^{(j)} \in \ell^2(A_j)$ recursively:

$$\mathbf{x}^{(1)} = \mathbf{x} \circ \tau_1,$$
$$\mathbf{x}^{(j)} = \big(g_{j-1}(\mathbf{x}^{(j-1)})\big)^{\wedge} \circ \tau_j, \quad j \geqslant 2. \tag{38}$$

Then, by an iteration of Parseval's formula, we have

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^{\infty} (-\delta^2)^{j-1} \int_{\Omega_A} Q_{A_j}(\mathbf{x}^{(j)}/\delta^{j-1}) Q_{A_j}(\overline{\mathbf{y}}^{(j)}/\delta^{j-1}) d\mathbb{P}_A, \quad \mathbf{x}, \mathbf{y} \in B_{\ell^2_{\mathbb{R}}}(A),$$

which converges!

Because the A_j are disjoint, the Q_{A_j} are independent, and therefore,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{\Omega_A} \Big(\sum_{j=1}^{\infty} (-\delta)^{j-1} Q_{A_j}(\mathbf{x}^{(j)}/\delta^{j-1}) \Big) \Big(\sum_{j=1}^{\infty} \delta^{j-1} Q_{A_j}(\overline{\mathbf{y}}^{(j)}/\delta^{j-1}) \Big) d\mathbb{P}_A. \tag{39}$$

A Parseval-like formula

Define

$$\Phi_A(\mathbf{x}) = \sum_{j=1}^{\infty} (\mathrm{i}\delta)^{j-1} Q_{A_j}(\mathbf{x}^{(j)}/\delta^{j-1}), \qquad \mathbf{x} \in B_{\ell^2_{\mathbb{R}}}(A), \tag{40}$$

and let $\Phi_A(\mathbf{x}) = \Phi_A(\mathbf{u}) + \mathrm{i}\Phi_A(\mathbf{v})$, for $\mathbf{x} = \mathbf{u} + \mathrm{i}\mathbf{v}$, $\mathbf{u}, \mathbf{v} \in B_{\ell^2_{\mathbb{R}}(A)}$.

Theorem 4

The map

$$\Phi_A : B_{\ell^2(A)} \to L^\infty(\Omega_A, \mathbb{P}_A)$$

is a uniformly bounded injection that is $(\ell^2 \to L^2)$-continuous, as well as (weak-$\ell^2 \to$ weak-L^∞)-continuous. Moreover,*

$$\|\Phi_A(\mathbf{x})\|_{L^\infty} \leqslant K, \quad \mathbf{x} \in B_{\ell^2(A)}, \tag{41}$$

where $K > 1$ is a universal constant independent of A, and

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{\Omega_A} \Phi_A(\mathbf{x}) \Phi_A(\bar{\mathbf{y}})\, d\mathbb{P}_A, \quad \mathbf{x},\ \mathbf{y} \in B_{\ell^2(A)}. \tag{42}$$

A word of caution

The map Φ_A (or any other map with the same properties) does not commute with complex conjugation. In particular,

$$\left\{\Phi_A(\mathbf{x}) : \mathbf{x} \in B_{\ell^2_{\mathbb{R}}}(A)\right\}$$

must contain elements with non-zero imaginary parts.

Indeed, by [Kashin and Szarek, 2003], for every $N > 0$ there exist vectors $\mathbf{v}_1, \ldots, \mathbf{v}_N \in B_{\ell^2(A)}$, such that if $f_1, \ldots, f_N \in L^\infty(\Omega_A, \mathbb{P}_A)$, and

$$\langle \mathbf{v}_j, \mathbf{v}_k \rangle = \int_{\Omega_A} f_j \, \overline{f_k} \, d\mathbb{P}_A, \quad 1 \leqslant j < k \leqslant N, \tag{43}$$

then

$$\|f_j\|_{L^\infty} \geqslant K(logN)^{\frac{1}{4}}, \quad j = 1, \ldots, N, \tag{44}$$

where $K > 0$ is a universal constant.

Constants?

Let

$$\|\Phi_A\|_{\infty,L^\infty} = \sup\left\{\|\Phi_A(\mathbf{x})\|_{L^\infty} : \mathbf{x} \in B_{\ell^2(A)}\right\},$$

where Φ_A is the injection in Theorem 4, and let

$$\mathcal{K}_{GC} = \sup_{A} \inf_{\Phi_A} \left(\|\Phi_A\|_{\infty,L^\infty}\right)^2,$$

where *infimum* is taken over all continuous injections

$$\Phi_A : B_{\ell^2(A)} \to L^\infty(\Omega, \mu)$$

that satisfy Theorem 4, with (Ω, μ) in place of $(\Omega_A, \mathbb{P}_A)$.
Then,

$$\mathcal{K}_G \text{ (= the Grothendieck constant) } \leqslant \mathcal{K}_{GC}.$$

Question.

$$\mathcal{K}_G < \mathcal{K}_{GC}\,?$$

Thank you.