A structure theorem for Boolean functions with small total influences

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Influences

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- The influence of the j-th variable on f is

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- $I_f = \sum_{j=1}^{n} I_j(f)$ is called the total influence of f.

Some Examples

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- Let $X = (\{0,1\}, \mu)$ be the uniform distribution on $\{0,1\}$.
- Let $f : \{0,1\}^n \to \{0,1\}$ be parity

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- Total influence of f is $n/2$.

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- Total influence $I_f \leq 2(n-1)/n \approx 2$.

Main Question

What can we say about the structure of functions $f : X^n \to \{0, 1\}$ with $I_f = O(1)$?

Some background

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  - 3-colorability exhibits a sharp threshold.

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  - 3-SAT exhibits a sharp threshold.
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- Bourgain 2000: Partially extended this to general setting $f : X^n \to \{0,1\}$.

Phase Transitions

Erdös-Rényi graph

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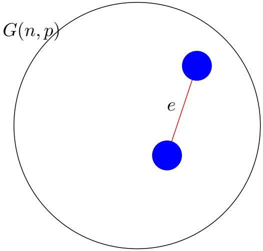
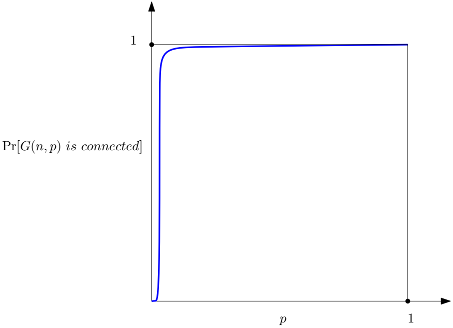
- Every edge is present independently with probability p.



$\Pr[e \in G(n, p)] = p$

Thresholds

They observed that some fundamental graph properties such as connectivity exhibit a threshold as p increases.



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One of the main questions that arises in studying phase transitions is:

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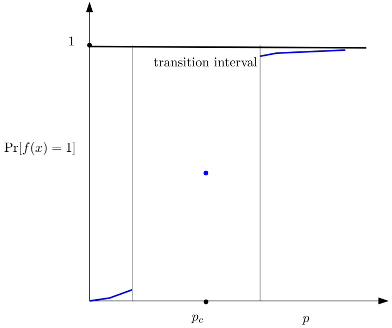
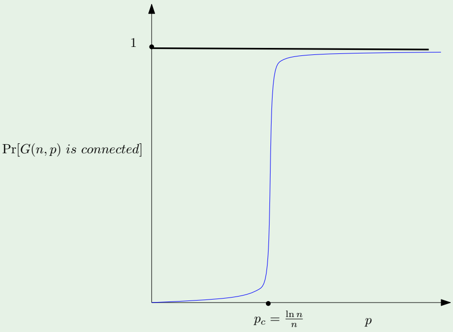
- That is how short is the interval in which the transition occurs.



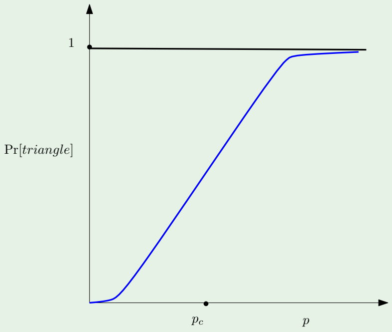
Example

Connectivity exhibits a sharp threshold.



Example

Containing a triangle does not exhibit a sharp threshold.



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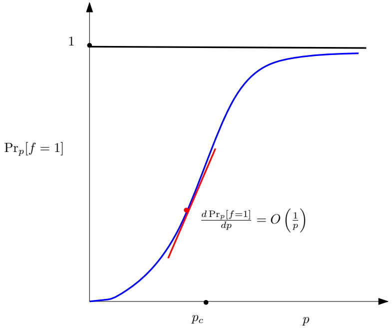
Is there a general approach to such questions?

Observation

If $f : \{0,1\}^n \to \{0,1\}$ does not exhibit a sharp threshold, then $\frac{d\,\Pr_p[f(x)=1]}{dp} = O\left(\frac{1}{p}\right)$, for some p in the transition interval.



Question [Coarse Threshold]

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The sharpness of the threshold is controlled by the total influence of the indicator function of the property:

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Question Rephrased

Which functions $f : (\{0,1\}^n, \mu_p) \to \{0,1\}$ satisfy $I_f = O(1)$?

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More general question

What is the structure of the functions $f : X^n \to \{0, 1\}$ with bounded total influence?

Bounded Total Influence

Functions with Small Total Influence

Juntas

Definition (Junta)

The value of $f(x_1, \ldots, x_n)$ depends on a small set of variables $\{x_{i_1}, \ldots, x_{i_k}\}$:

$$f(x) := g(x_{i_1}, \ldots, x_{i_k}).$$

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- $\sum I_i \leq k$.
- Juntas have total influence $O(1)$.

Direct theorem

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Theorem (Friedgut 98 Inverse Theorem)

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Theorem (More precisely)

Let $f : \{0,1\}^n \to \{0,1\}$ have total influence $O(1)$. Then for every $\epsilon > 0$, there exists a $O_\epsilon(1)$-junta $g : \{0,1\}^n \to \{0,1\}$ such that

$$\Pr[f(x) \neq g(x)] \leq \epsilon.$$

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- For the applications in phase transition, the range $p \lesssim n^{-c}$ is the most interesting case.
    - connectivity
    - satisfiability of 3-SAT
    - 3-colorability of graphs....

Pseudo-Juntas

$$f : X^n \to \{0, 1\}$$

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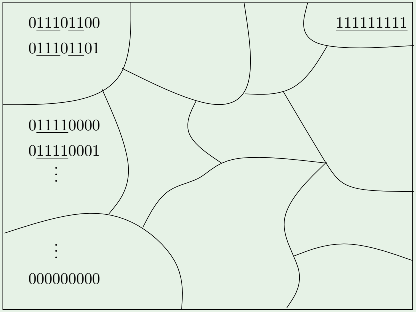

Definition (Pseudo-junta)

If $f : X^n \to \{0,1\}$ is measurable w.r.t. $\mathcal{F}_{\mathcal{J}}$, then f is called a k-pseudo-junta provided that

$$\mathbb{E}[\text{number of active coordiates of } x] \leq k.$$

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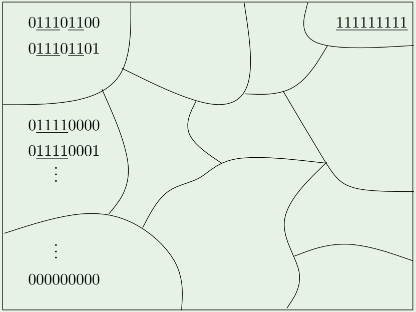

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- Suppose x and y differ only in one coordinate j.

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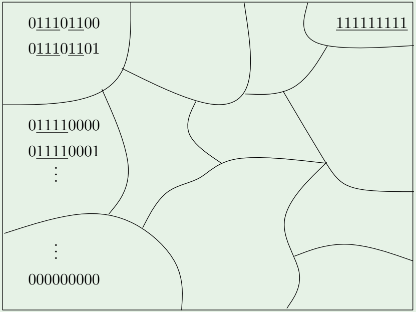

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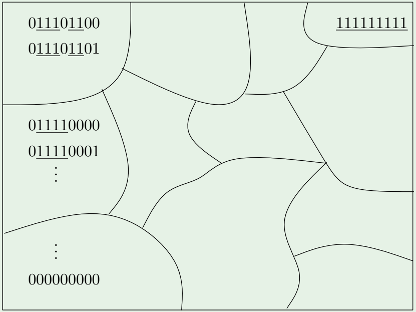

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- $j \notin J_{\mathcal{J}}(x) \cup J_{\mathcal{J}}(y) \Leftrightarrow x$ and y are atom-mates.

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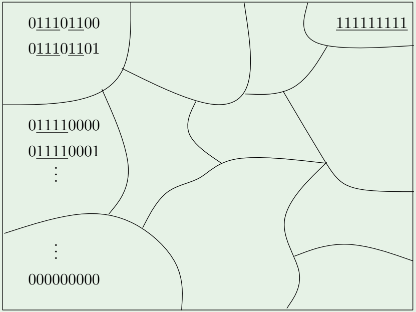

Theorem (Direct Theorem)

Let $f : X^n \to \{0,1\}$ be a k-pseudo-junta. Then $I_f \le 2k$.

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I_f &= \sum_{j\in[n]} \Pr[f(x_1,\ldots,x_j,\ldots,x_n) \neq f(x_1,\ldots,y_j,\ldots,x_n)] \\
&\le \sum_{j\in[n]} \Pr\left[j \in J_{\mathcal{J}}(x_1,\ldots,x_j,\ldots,x_n) \cup J_{\mathcal{J}}(x_1,\ldots,y_j,\ldots,x_n)\right]
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&\leq \sum_{j\in[n]} \Pr\left[j \in J_{\mathcal{J}}(x_1,\dots,x_j,\dots,x_n)\right] + \Pr\left[j \in J_{\mathcal{J}}(x_1,\dots,y_j,\dots,x_n)\right] \\
&\leq 2\sum_{j\in[n]} \Pr[j \in J_{\mathcal{J}}(x)] \leq 2\mathbb{E}|J_{\mathcal{J}}(x)| \leq 2k.
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Theorem (Recall - Direct Theorem)

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If $f : X^n \to \{0,1\}$ is a pseudo-junta, then the total influence of f is $O(1)$.

Theorem (Friedgut 2000 - Inverse Theorem for graphs)

If the total influence of a graph property f is $O(1)$ on $G(n,p)$, then f is essentially a pseudo-junta.

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- It is only applicable to p-biased distribution.

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- does not tell anything about the global structure of f.

Main Theorem

Theorem (H. 2011, Inverse Theorem)

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Theorem (More precisely:)

Let $f : X^n \to \{0,1\}$ *and* $\epsilon > 0$

- *There exists a* $\exp(10^{15}\epsilon^{-3}\lceil I_f \rceil^3)$*-pseudo-junta* $h : X^n \to \{0,1\}$ *such that*

$$\Pr[f(x) \neq h(x)] \leq \epsilon.$$

Proof Sketch

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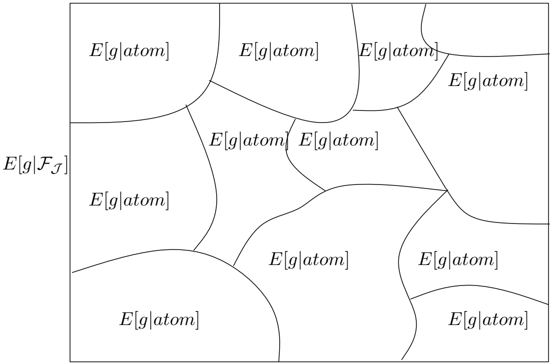
- Show $\|g - \mathbb{E}[g|\mathcal{F}_{\mathcal{J}}]\|_2$ is small.



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- So
$$f \approx \sum_{S:|S|<k} F_S =: g.$$

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Bourgain 2000

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- But we want

$$\int |f - \mathbb{E}[f|\mathcal{F}_{\mathcal{J}}]|^2 = \int \left| \sum (F_S - \mathbb{E}[F_S|\mathcal{F}_{\mathcal{J}}]) \right|^2 \approx 0.$$

Note bounding $\mathbb{E}[|J_{\mathcal{J}}|]$ is easy:

$$\begin{aligned}\mathbb{E}[|J_{\mathcal{J}}(x)|] &\leq \sum |S| \times \Pr[J_S(x) = 1] \\ &\leq k \sum \int J_S(x) \\ &\leq k \sum \int \epsilon_1^{-1} |F_S|^2 \leq \epsilon_1^{-1} k = O(1).\end{aligned}$$

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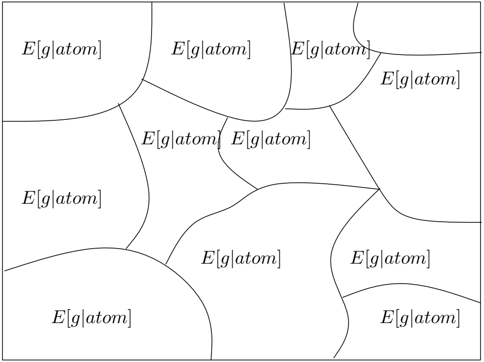
$E[g|\mathcal{F}_{\mathcal{J}}]$



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- To remedy this we define some auxiliary σ-algebras $\mathcal{F}_S$ (activate coordinates only if x_S activates them, E.g.

$$(\underbrace{1, 1, 0, 1}_{S}, 1, 0, 0) \qquad vs. \qquad (\underbrace{1, 1, 0, 1}_{S}, 1, 0, 0).$$

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  - $\mathcal{F}_S$ depends only on coordinates in S.
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- Since $\mathcal{F}_S$ is coarser than $\mathcal{F}_{\mathcal{J}}$:

$$\|g - \mathbb{E}[g|\mathcal{F}_{\mathcal{J}}]\|_2^2 \leq \left\| g - \sum \mathbb{E}[F_S|\mathcal{F}_S] \right\|_2^2 .$$

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$$\int \mathbb{E}[F_{S_1}|\mathcal{F}_{S_1}]\mathbb{E}[F_{S_2}|\mathcal{F}_{S_2}] = 0,$$

if $S_1 \cap S_2 = \emptyset$.

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- We get

$$\begin{aligned}\|g - \mathbb{E}[g|\mathcal{F}_{\mathcal{J}}]\|_2^2 \ &\lesssim \ \int \sum |F_S - \mathbb{E}[F_S|\mathcal{F}_S]|^2 \\ &+ \sum_{\substack{S_1,S_2 \in \mathcal{S} \\ S_1 \cap S_2 \neq \emptyset, S_1 \neq S_2}} \left| \int \mathbb{E}[F_{S_1}|\mathcal{F}_{S_1}]\mathbb{E}[F_{S_2}|\mathcal{F}_{S_2}] \right|.\end{aligned}$$

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- First sum is small by Bourgain's inequality.

$$\|g - \mathbb{E}[g|\mathcal{F}_{\mathcal{J}}]\|_2^2 \lesssim \int \sum |F_S - \mathbb{E}[F_S|\mathcal{F}_S]|^2 + \sum_{\substack{S_1, S_2 \in \mathcal{S} \\ S_1 \cap S_2 \neq \emptyset, S_1 \neq S_2}} \left| \int \mathbb{E}[F_{S_1}|\mathcal{F}_{S_1}]\mathbb{E}[F_{S_2}|\mathcal{F}_{S_2}] \right|.$$

- First sum is small by Bourgain's inequality.
- Second term is analyzed by considering $T := S_1 \cap S_2$:

$$\int \mathbb{E}[F_{S_1}(x_T, \cdot)|\mathcal{F}_{S_1}]\mathbb{E}[F_{S_2}(x_T, \cdot)|\mathcal{F}_{S_2}] =$$

$$\int \mathbb{E}[F_{S_1}(x_T, \cdot)|\mathcal{F}_{S_1}] \times \int \mathbb{E}[F_{S_2}(x_T, \cdot)|\mathcal{F}_{S_2}].$$

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$$J_T(y) := \begin{cases} 1 & \max_{R\subseteq T} \delta_0^{-2|T\setminus R|} \int \xi_T(y_R, x_{T\setminus R}) dx_{T\setminus R} \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

where

$$\xi_T(y) := \begin{cases} 1 & \sum_{R\subseteq T} \sum_{S\in\mathcal{S}:S\supseteq T} \int a_S(y_R, x_{S\setminus R}) dx_{S\setminus R} > \epsilon_2, \\ 0 & \text{otherwise.} \end{cases}$$

where

$$a_S(y) := 2^{3k}\delta^{-2k} \sum_{T\subseteq S} \int 1_{[|F_S(y_{S\setminus T}, x_T)|>\epsilon_1]} dx_T.$$

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  - $\sum \|F_S - G_S\|_2^2$ is small.
  - $\sum_{|S|\le k} \|G_S\|_1 = O(1)$.

Increasing Functions

Friedgut 2000

For an increasing graph property f, if $I_f = O(1)$, then there exists a small set of coordinates J such that

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If $I_f = O(1)$ for an increasing $f : (\{0,1\}^n, \mu_p) \to \{0,1\}$, then $\exists \delta > 0$ and a small J such that

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H 2011

Under the above assumptions

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Open problem

Conjecture[Friedgut]

If $I_f = O(1)$ for an increasing $f : (\{0,1\}^n, \mu_p) \to \{0,1\}$, then

$$f \approx O(1) - \text{Monotone DNF.}$$

Conjecture

If $f : [0,1]^n \to \{0,1\}$ is increasing, and $I_f = O(1)$, then there is $|J| = O_\epsilon(1)$ such that either

$$\mathbb{E}[f(x)|x_J = \vec{1}] \geq 1 - \epsilon,$$

or

$$\mathbb{E}[f(x)|x_J = \vec{0}] \leq \epsilon.$$

Thank you!