Extended Formulations I
(Boot Camp)

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Organization

1. Thomas Rothvoss (1.x lectures): Introduction to LP Extended Formulations
2. Hamza Fawzi (1.x lectures): Introduction to SDP Extended Formulations
3. Prasad Raghavendra (1.x lectures): Lower bounds for LP/SDP lifts

Extended formulation

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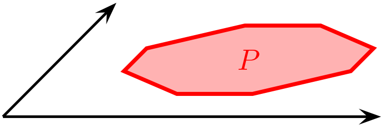
- Given polytope $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$



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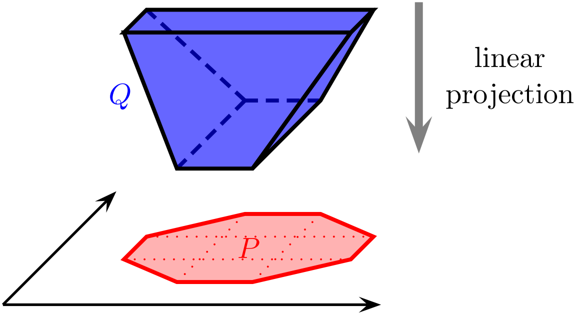
- Given polytope $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$
- Write $P = \{x \in \mathbb{R}^n \mid \exists y : Bx + Cy \leq d\}$



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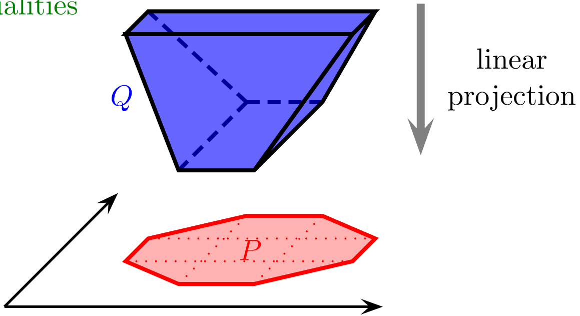

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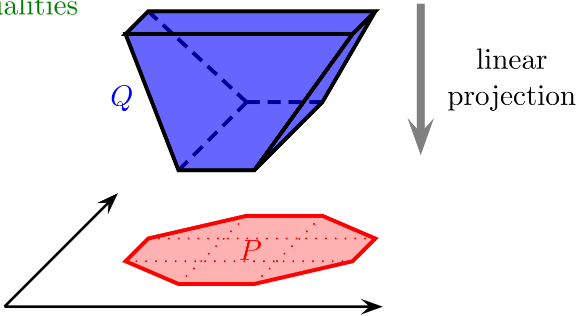

- The **extension complexity** of P is

$$\text{xc}(P) := \min \left\{ \#\text{facets of } Q \;\middle|\; \begin{array}{c} Q \text{ polyhedron} \\ p \text{ linear map} \\ p(Q) = P \end{array} \right\}$$

Motivation

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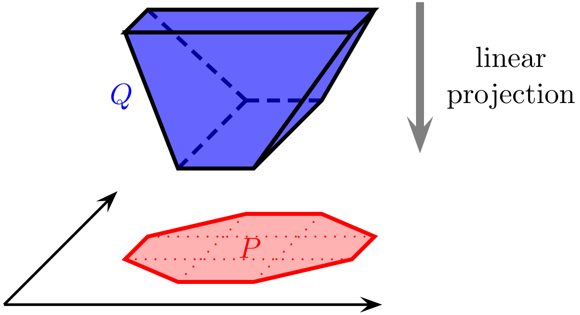
- xc(conv(Spanning trees)) $\leq O(n^3)$
  $\Rightarrow$ optimize over all Spanning trees with LP of size $O(n^3)$



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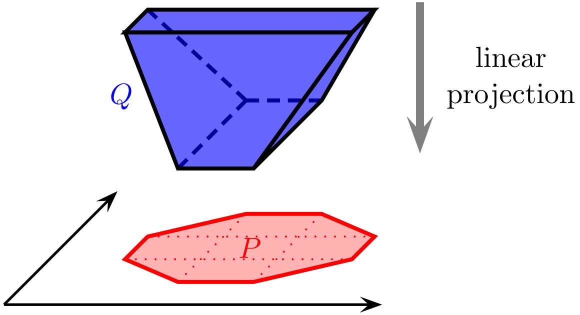

- **In reverse:** If $\mathrm{xc}(P)$ is **high** for TSP / MaxCut / Correlation / Matchings, then those problems cannot be solved with a single poly-size LP!

Part I

A non trivial example - Knapsack

Knapsack

- **Input:**
  - n objects with weight $w_i \in \mathbb{Z}_+$
  - profit $p_i \in \mathbb{Q}_+$
  - knapsack size $B \in \mathbb{Q}_+$
- **Goal:** Find subset of objects, maximizing the profit and not exceeding the weight bound:

$$OPT = \max_{I \subseteq \{1,\ldots,n\}} \left\{ \sum_{i \in I} p_i \mid \sum_{i \in I} w_i \leq B \right\}$$

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- weakly NP-hard
- Pseudo-polynomial time algorithm
- FPTAS

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Known:

- weakly NP-hard
- Pseudo-polynomial time algorithm
- FPTAS
- $\mathrm{xc}(\mathrm{conv}\{x \in \{0,1\}^n \mid \sum_{i=1}^n w_i x_i \leq B\}) \leq O(n \cdot B)$.

A dynamic program for KNAPSACK

Lemma

Knapsack can be solved in time $O(n \cdot B)$.

Algorithm

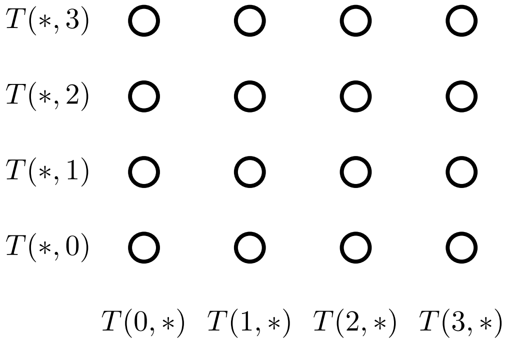
(1) Compute table entries

$$\begin{aligned} T(i,W) &= \max_{I \subseteq \{1,\dots,i\}} \Big\{ \sum_{j \in I} p_j \mid \sum_{j \in I} w_j = W \Big\} \\ &= \text{max. profit of weight } W \text{ subsets of first } i \text{ items} \end{aligned}$$

using dynamic programming

$$T(i,W) = \max \Big\{ \underbrace{T(i-1,W)}_{\text{don't take } i}, \underbrace{T(i-1,W-w_i)+p_i}_{\text{take } i} \Big\} \; \forall i \; \forall W \leq B$$

Knapsack (3)



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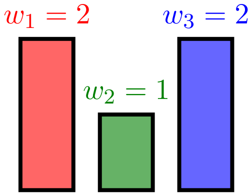

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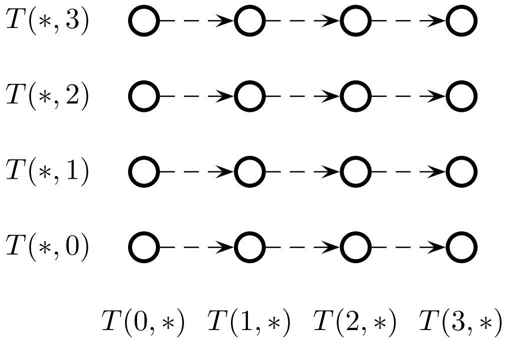

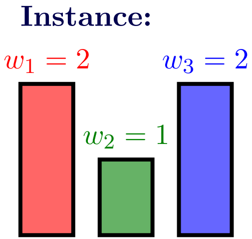

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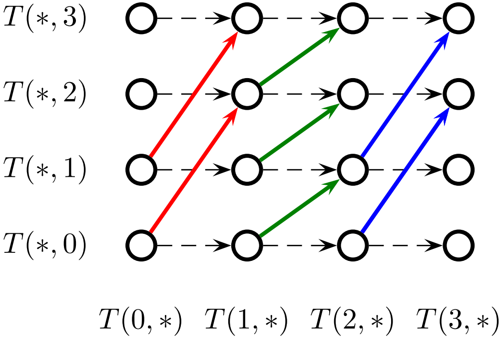

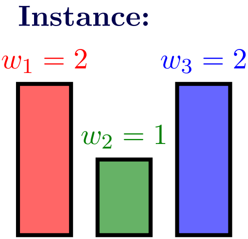

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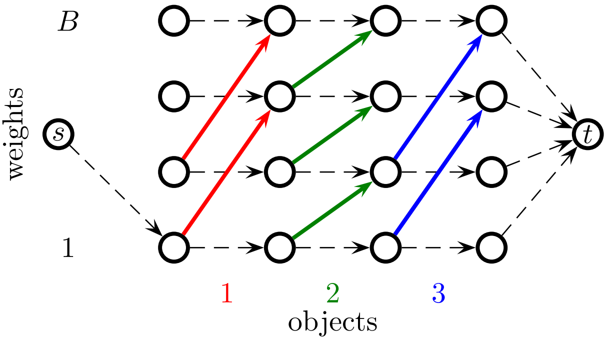

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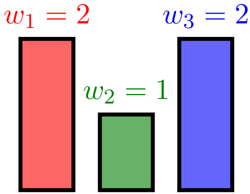

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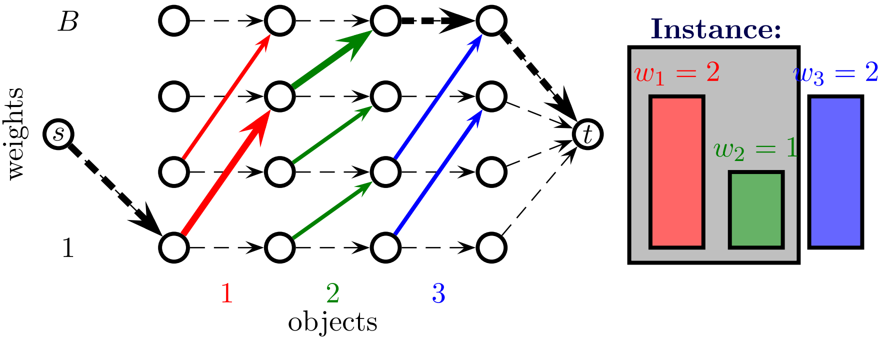

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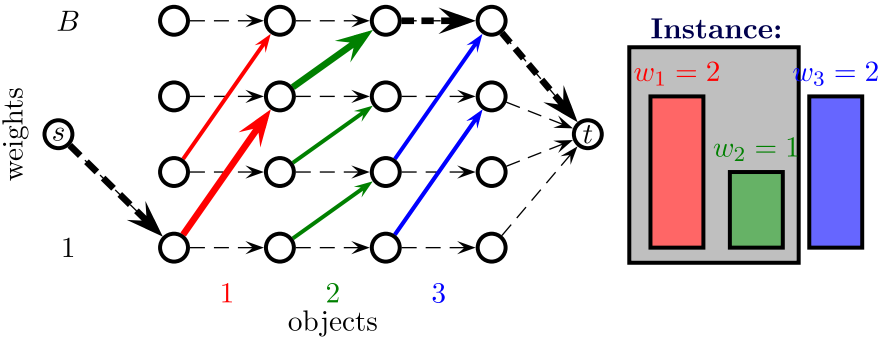

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Observations:

- s-t path $\longleftrightarrow$ Knapsack solution
- max cost s-t path = max profit packing

Knapsack (4)

- Let $G = (V, E)$ be network.
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Observation

The Knapsack polytope is the projection of Q with

$$\begin{aligned} x_i &= \sum_{e \in E_i} y(e) \quad \forall i \in [n] \\ y(\delta^+(v)) - y(\delta^-(v)) &= \begin{cases} 1 & v = s \\ -1 & v = t \\ 0 & \text{otherwise} \end{cases} \qquad \forall v \in V \\ x, y &\geq \mathbf{0} \end{aligned}$$

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Corollary

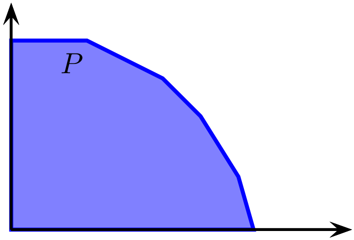
$\text{xc}(\text{Knapsack polytope}) \leq O(n \cdot B)$.

Knapsack (5)

Open problem

Consider a Knapsack polytope

$$P = \mathrm{conv}\Big\{x \in \{0,1\}^n \mid \sum_{i=1}^{n} w_i x_i \leq B\Big\}.$$



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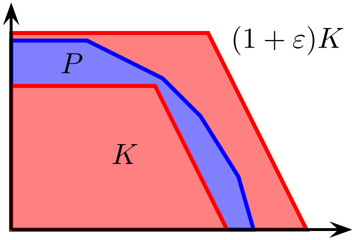
- $K \subseteq P \subseteq (1+\varepsilon)K$
- $xc(K) \leq \mathrm{poly}(n, \frac{1}{\varepsilon})$?



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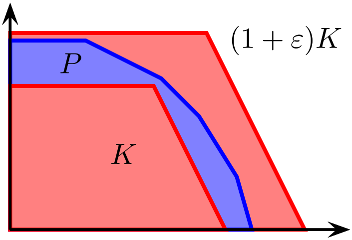
Known:

- $\text{xc}(K) \leq n^{O(1/\varepsilon)}$ possible (Bienstock)

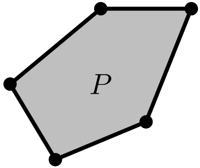


Part II

Slack-matrices, Yannakakis' Theorem and Communication Complexity

Slack-matrix

Write: $P = \mathrm{conv}(\{x_1, \ldots, x_v\}) = \{x \in \mathbb{R}^n \mid Ax \leq b\}$



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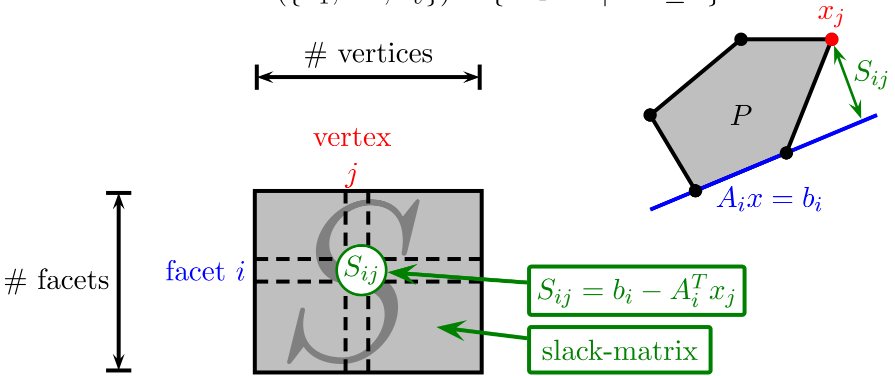

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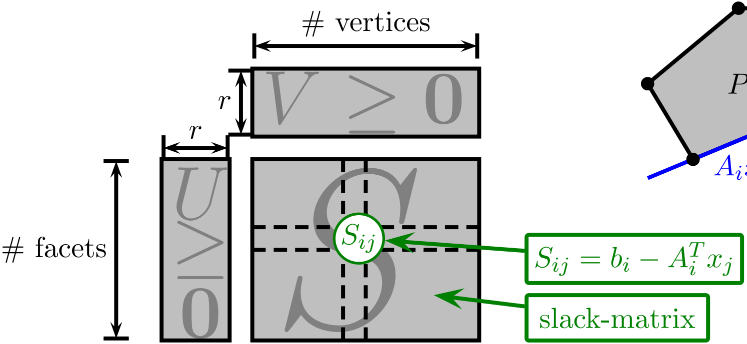

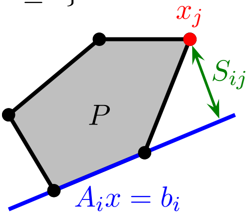

Non-negative rank:

$$\operatorname{rk}_+(S) = \min\{r \mid \exists U \in \mathbb{R}^{f \times r}_{\geq 0}, V \in \mathbb{R}^{r \times v}_{\geq 0} : S = UV\}$$

Yannakakis' Theorem

Theorem (Yannakakis '88)

If S is the **slack-matrix** for $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, then $\mathrm{xc}(P) = \mathrm{rk}_+(S)$.

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- $A_i x > b_i \Longrightarrow A_i x + \underbrace{U_i y}_{\geq 0} > b_i$.

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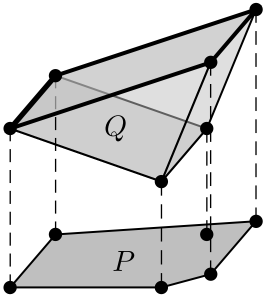

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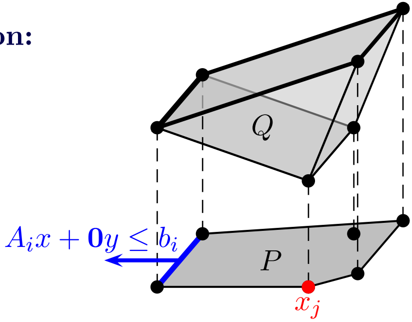

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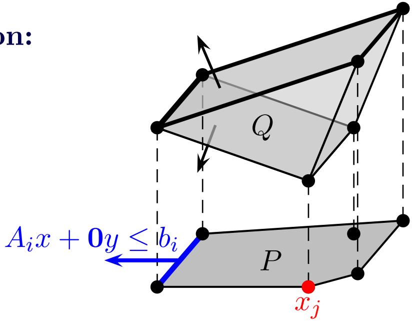

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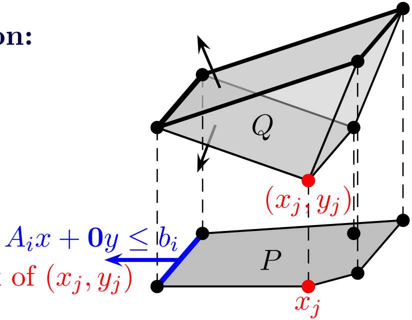

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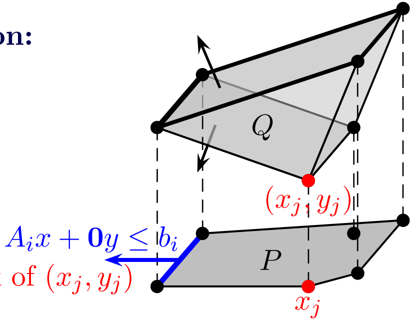

$$\langle u(i), v(j) \rangle = \underbrace{u(i)^T d}_{=b_i} - \underbrace{u(i)B}_{=A_i} x_j - \underbrace{u(i)C}_{=\mathbf{0}} y_j = S_{ij}$$

Rectangle covering lower bound

Observation

$\mathrm{rk}_+(S) \geq$ rectangle-covering-number(S).

Rectangle covering lower bound

$$V = \begin{pmatrix} 0 & 0 & 2 & 1 & 0 \\ 0 & 2 & 2 & 0 & 3 \end{pmatrix}$$

$$U = \begin{pmatrix} 3 & 2 \\ 1 & 1 \\ 0 & 2 \\ 0 & 0 \\ 2 & 0 \end{pmatrix} \qquad S = \begin{pmatrix} 0 & 4 & 10 & 3 & 5 \\ 0 & 2 & 4 & 1 & 3 \\ 0 & 4 & 4 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \end{pmatrix}$$

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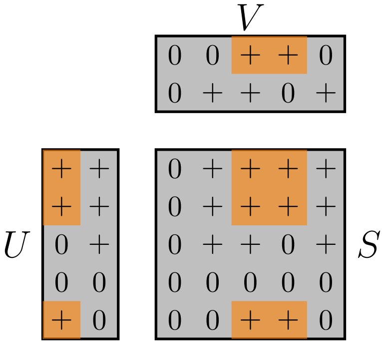

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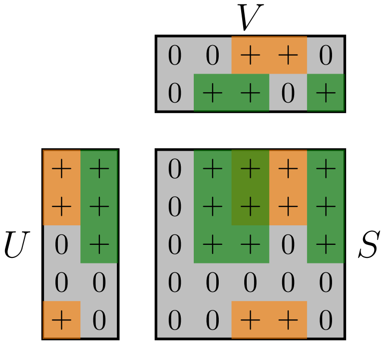

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Communication complexity

Setting:

- Function $f : X \times Y \to \mathbb{R}$

Alice **Bob**

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Alice	**Bob**
$x \in X$	$y \in Y$

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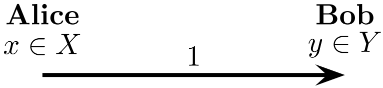

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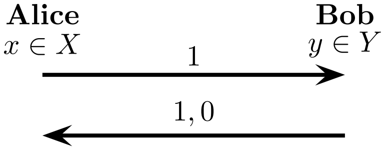

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- Function $f : X \times Y \to \mathbb{R}$
- Players Alice and Bob agree apriori on a deterministic communication protocoll
- Alice receives $x \in X$, Bob receives $y \in X$
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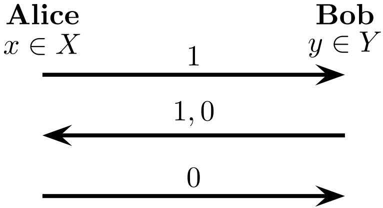

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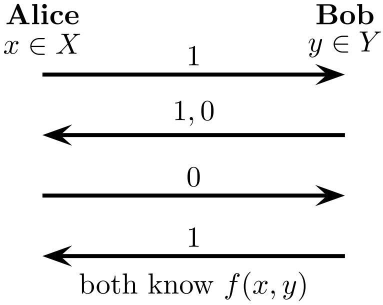

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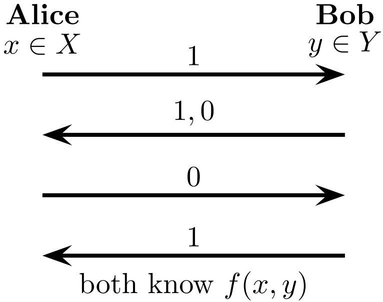
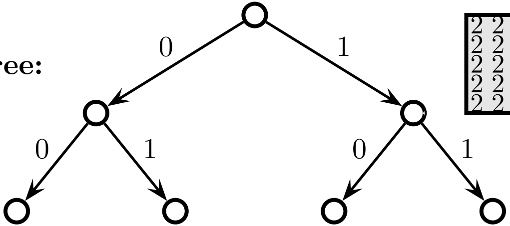


$$CC(f) = \min_{\text{protocoll}} \max_{x \in X, y \in Y} \{\text{bits to compute } f(x, y)\}$$

Communication complexity (3)

decision tree:



2	2	0	0	0
2	2	0	0	0
2	2	2	1	1
2	2	2	1	1
2	2	2	1	1

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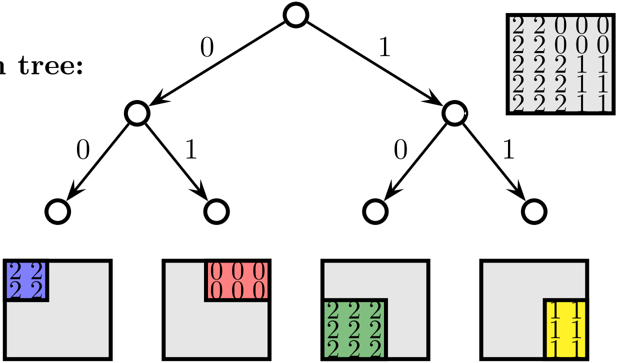

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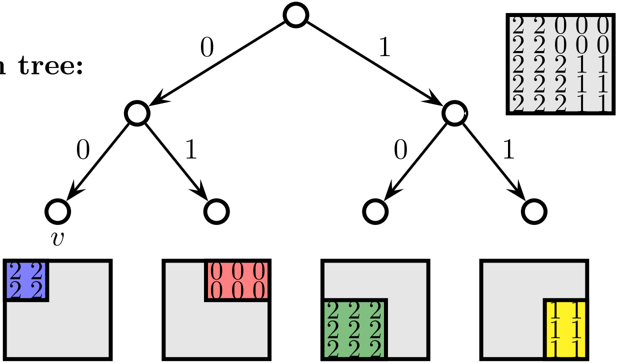

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- For a leave v of tree, $R_v := \{(x, y) : \text{protocoll ends in } v\}$ is a **monochromatic rectangle**

Communication complexity (3)

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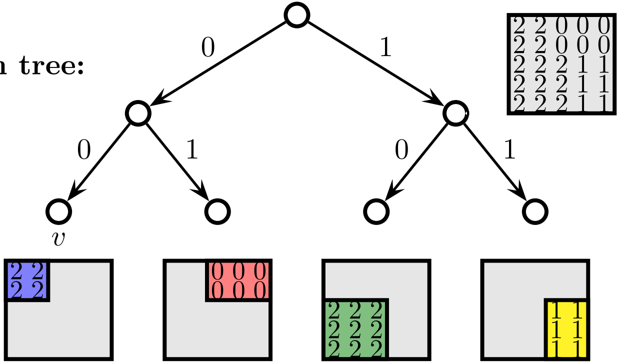

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  $\Rightarrow$ S is sum of 2^k rank-1 matrix

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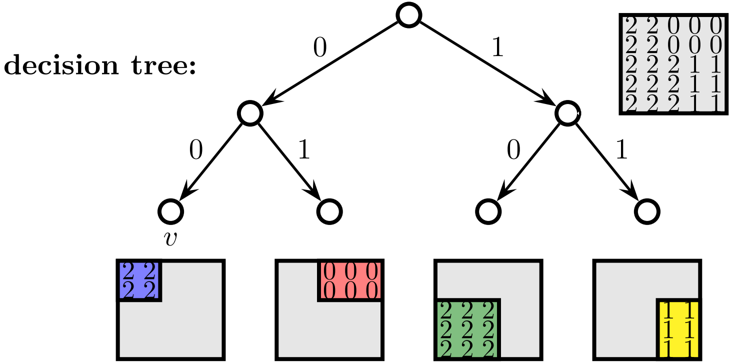

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- $\text{xc}(\textit{polytope}) \leq 2^{CC(\text{slack matrix})}$

An exact model (1)

We have seen:

$$2^{CC_{\text{nondet}}(\text{supp. of slack matrix})} \leq \text{xc}(\text{polytope}) \leq 2^{CC(\text{slack matrix})}$$

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Let $CC_{\text{RAND}}(S)$ be min. # bits. Then

$$\text{xc}(\text{polytope}) = 2^{CC_{\text{RAND}}(\text{slack matrix } S)}$$

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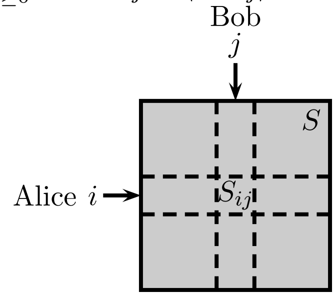
Let $u_i, v_j \in \mathbb{R}^r_{\geq 0}$ with $S_{ij} = \langle u_i, v_j \rangle$.



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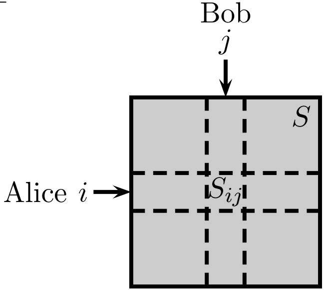

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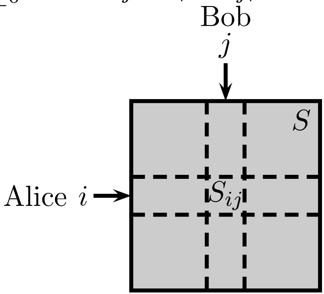

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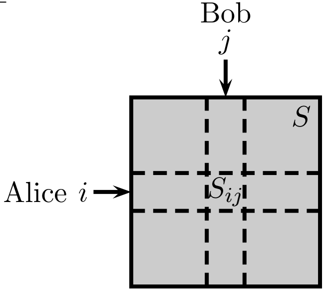

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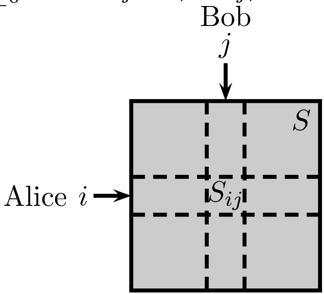

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Then

$$\mathbb{E}[\text{protocoll}(i,j)] = \sum_{k=1}^{r} v_i(k) \cdot u_j(k) = \langle u_i, v_j \rangle$$

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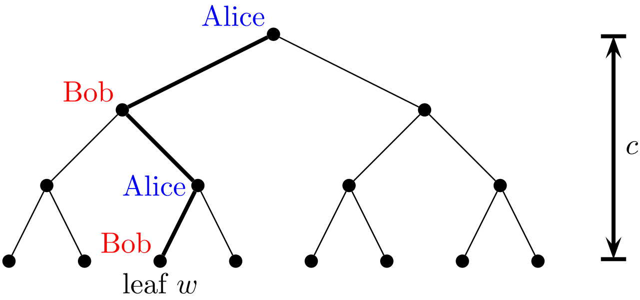
Consider depth c protocoll tree:



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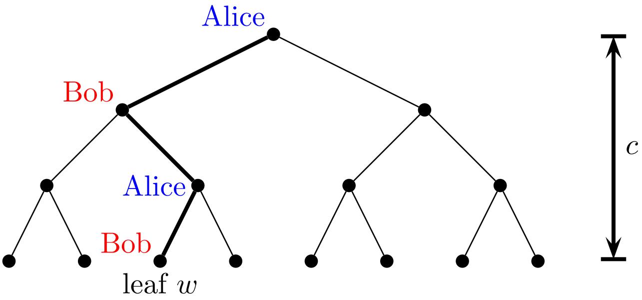

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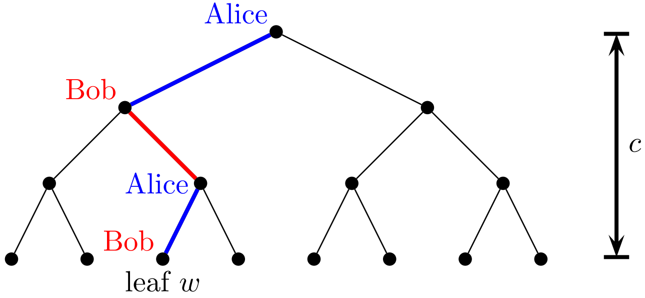

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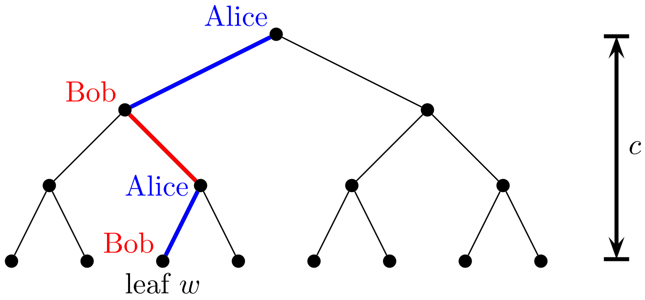

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$\Rightarrow 2^c$-size non-neg factorization

Part III

The Lower Bound on the Correlation Polytope

Correlation polytope (1)

The **correlation polytope** is

$$COR = \text{conv}\{bb^T : b \in \{0,1\}^n\}$$

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$$COR = \text{conv}\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

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Observation: The polytope is **NP**-hard.
For graph $G = ([n], E)$ with adjacency matrix A_G

$$\mathrm{maxcut}(G) = \max_{x \in \{0,1\}^n} (D_G - A_G) \bullet xx^T = \sum_{(i,j) \in E} \underbrace{(x_i + x_j - 2x_i x_j)}_{=1 \text{ if } x_i \neq x_j, 0 \text{ o.w.}}$$

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Theorem (Fiorini, Massar, Pokutta, Tiwary, de Wolf '12)

$\text{xc}(\text{COR}) \geq 2^{\Omega(n)}$.

- Here: Simplified proof by Kaibel and Weltge

Correlation polytope (2)

Lemma

For all $a \in \{0,1\}^n$, $(2\text{diag}(a) - aa^T) \bullet Y \leq 1$ is a feasible inequality for $Y \in COR$.

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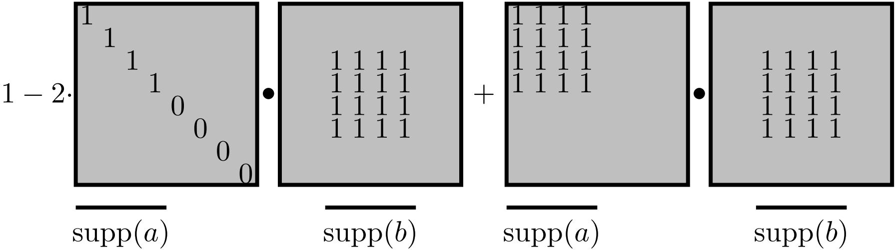

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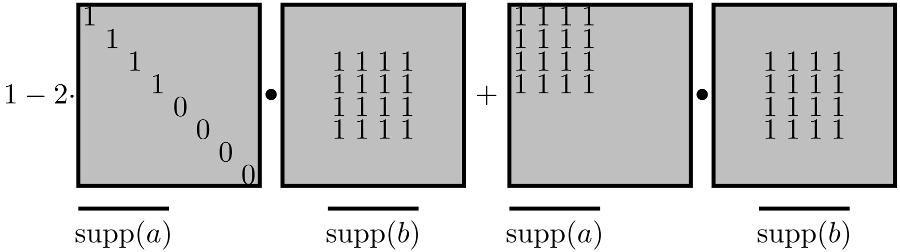
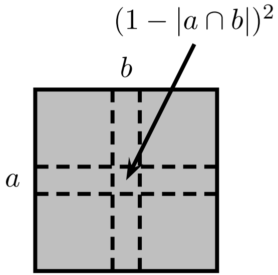
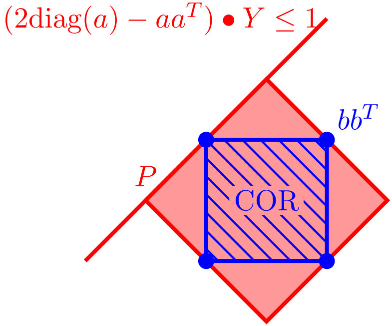


$$= 1 - 2|a \cap b| + |a \cap b|^2 = (1 - |a \cap b|)^2 \geq 0$$

Correlation polytope (3)



slack matrix S



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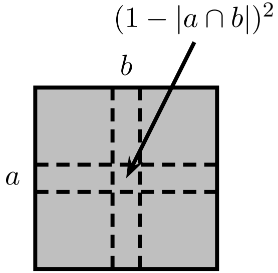

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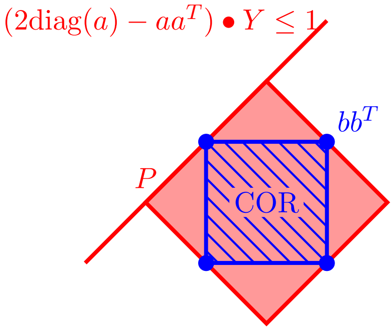

Observations:

- S is a submatrix of the “real” slack-matrix

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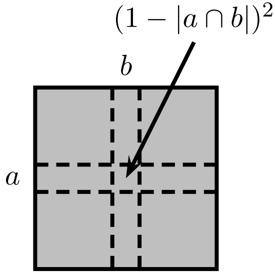

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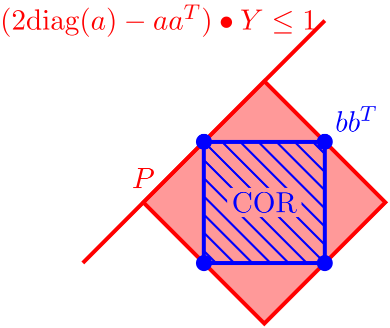

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- S is a submatrix of the "real" slack-matrix
- We have

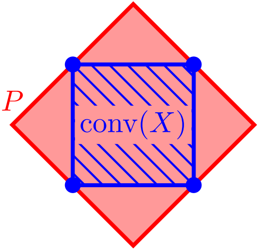
$$S_{ab} = \begin{cases} 1 & |a \cap b| = 0 \\ 0 & |a \cap b| = 1 \end{cases}$$

Incomplete slack matrices

Lemma

For a polytope $P = \{x \mid Ax \leq b\}$ and $X = \{x_1, \ldots, x_v\} \subseteq P$ define a matrix S with $S_{i,j} := b_i - A_i x_j$. Then

$$\mathrm{rk}_{\geq 0}(S) = \min\{\mathrm{xc}(Q) : X \subseteq Q \subseteq P\}$$



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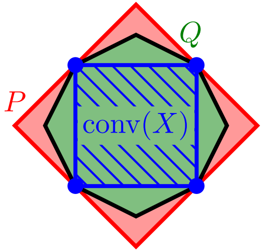

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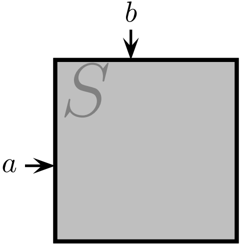

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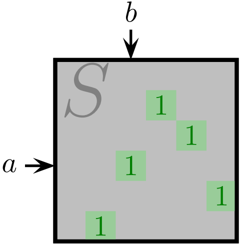

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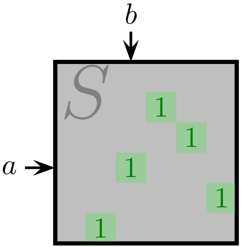

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Claim

$|\mathcal{P}_0| = 3^n$.

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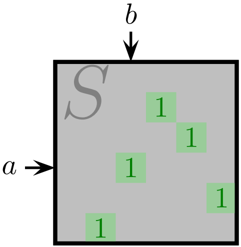

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- For disjoint pair (a,b), for coordinate i there are 3 options
  - $a_i = 0,\ b_i = 0$
  - $a_i = 1,\ b_i = 0$
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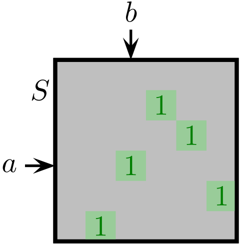
Correlation polytope (4)



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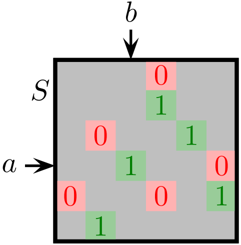

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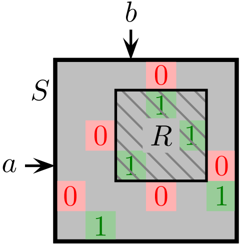

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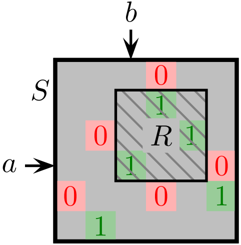

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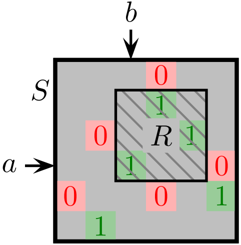
Lemma

Any rectangle R without forbidden pairs has $|R \cap \mathcal{P}_0| \leq 2^n$.

Correlation polytope (4)



$$S_{ab} = \begin{cases} 1 & |a \cap b| = 0 \\ 0 & |a \cap b| = 1 \end{cases}$$

- **disjoint pairs** $\mathcal{P}_0 := \{(a,b) : |a \cap b| = 0\}$
- **forbidden pairs** $\mathcal{P}_1 := \{(a,b) : |a \cap b| = 1\}$

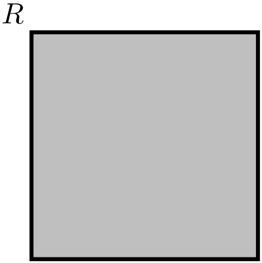
Lemma

Any rectangle R without forbidden pairs has $|R \cap \mathcal{P}_0| \leq 2^n$.

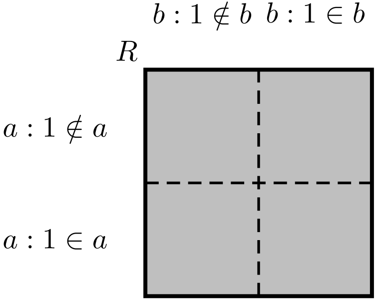
- By rectangle covering lower bound

$$\text{xc}(\text{COR}) \geq \frac{|\mathcal{P}_0|}{2^n} = \left(\frac{3}{2}\right)^n$$

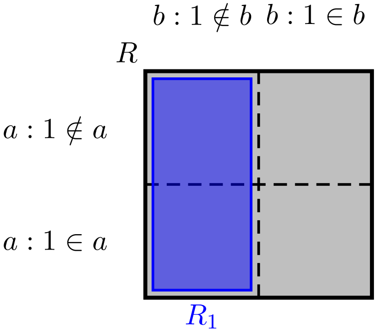
Correlation polytope (5)



Correlation polytope (5)

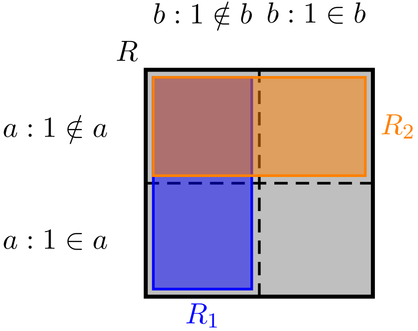


Correlation polytope (5)



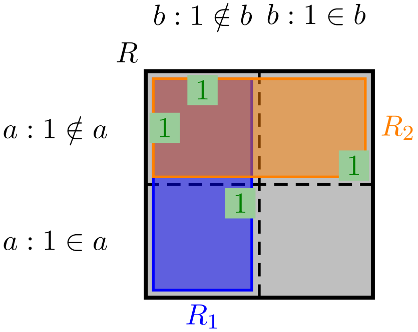
- Define rectangles $R_1 = \{(a, b) \in R : 1 \notin b\}$

Correlation polytope (5)



- Define rectangles $R_1 = \{(a, b) \in R : 1 \notin b\}$ and $R_2 = \{(a, b) \in R : 1 \notin a\}$

Correlation polytope (5)



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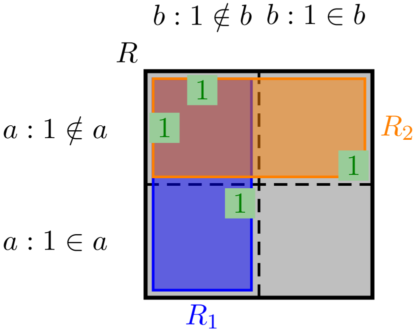

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  $|\mathcal{P}_0 \cap R| \leq |\mathcal{P}_0(n-1) \cap R_1| + |\mathcal{P}_0(n-1) \cap R_2| \leq 2 \cdot 2^{n-1} = 2^n$

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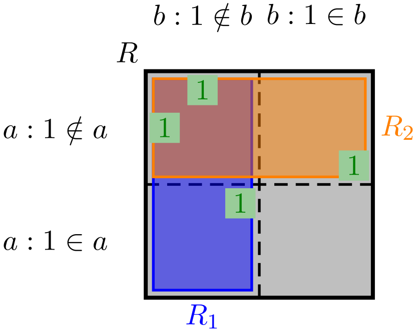

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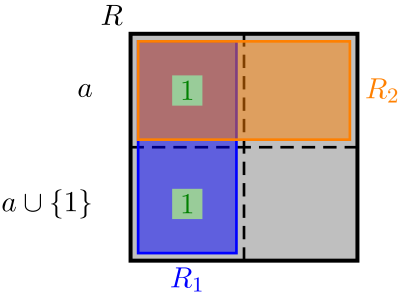

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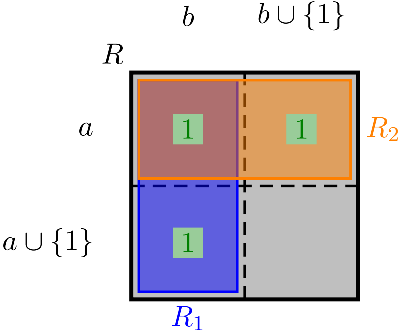

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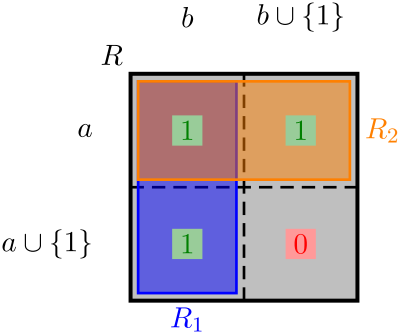

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The end

Thanks for your attention