Inferring Structural Properties: Bridges and Graphs

Anna Gilbert Jae Young Park Michael Wakin

University of Michigan
Colorado School of Mines

Structural health monitoring

I-35W Mississippi River Bridge (2007)

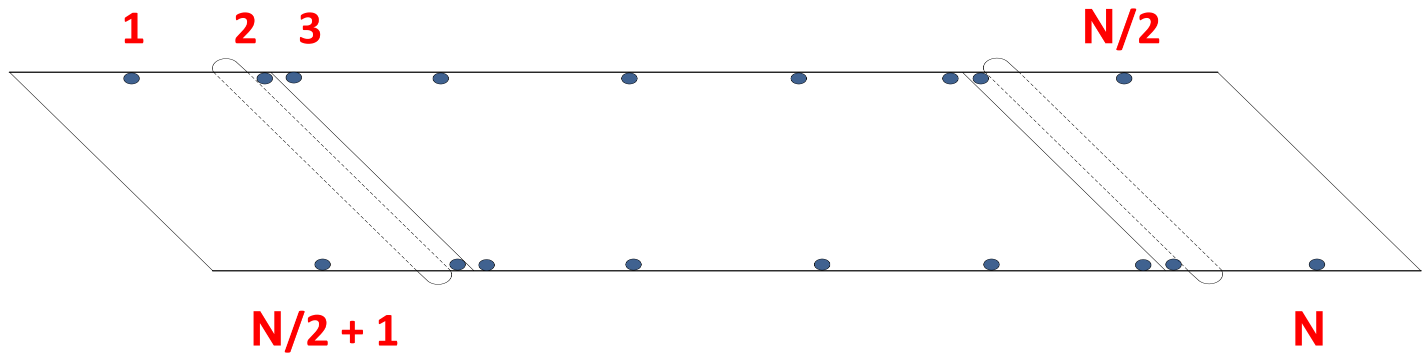
Sampoong Department Store (1995)

Dual of big data = resource constrained data collection, processing

- Automated monitoring of buildings
- Wireless sensors
  - acquire vibration data,
  - transmit to central node
- **Goal:** maximize battery life, accurately assess health, build accurate model of structure

Structural Dynamics

N sensors on structure



- Each sensor observes displacement data $x_l(t)$
- Concatenate to get: $[x(t)] = [x_1(t), x_2(t), \ldots, x_N(t)]^T$
- An N-degree-of-freedom structure with ***no damping*** can be modeled by:

$$[M]\left[\frac{d^2x(t)}{dt^2}\right] + [K][x(t)] = [0(t)] \qquad [M],[K] : \text{unknown}$$

$N \times N$ mass matrix

$N \times N$ stiffness matrix

free decay

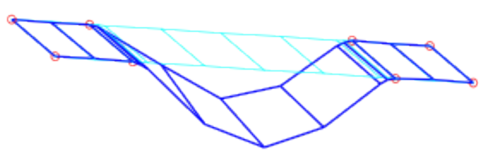
Structural Dynamics

- Homogeneous solution:

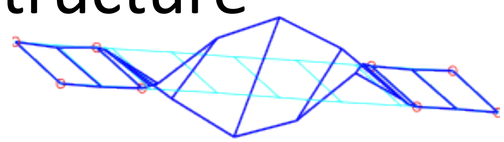
$$[x(t)] = \sum_{n=1}^{N} \rho_n \sin(\underbrace{\omega_n t}_{\text{modal frequency}} + \theta_n)[\underbrace{\psi_n}_{N \times 1 \text{ mode shape}}]$$

Generalized eigenvectors
$([K] - \lambda^2[M])\psi = 0$

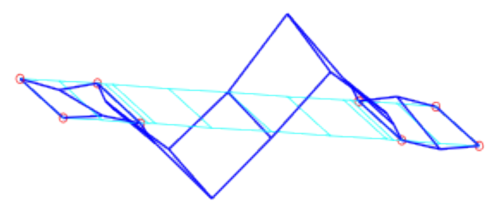
- $[\psi_n]$ are orthonormal, independent of time, physical information about structure



2.44 Hz



2.83 Hz



10.25 Hz

- **Modal analysis:**
  - Extract modal frequencies, mode shapes, etc., from $[x(t)]$

Data Collection

- Recall:

$$[x(t)] = \sum_{n=1}^{N} \rho_n \sin(\underbrace{\omega_n t}_{\text{modal frequency}} + \theta_n)[\underbrace{\psi_n}_{N \times 1 \text{ mode shape}}]$$

- Consider analytic signal:

$$[x(t)] = \sum_{n=1}^{N} A_n \mathrm{e}^{\mathrm{i}\omega_\mathrm{n} \mathrm{t}} [\psi_\mathrm{n}]$$

- Sample $[x(t)]$ at times $t_1, t_2, \ldots, t_M$
- Stack samples into $M \times N$ matrix $[X]$.

$$[X] = \begin{bmatrix} x_1(t_1) & x_2(t_1) & \cdots & x_N(t_1) \\ x_1(t_2) & x_2(t_2) & \cdots & x_N(t_2) \\ \vdots & & & \vdots \\ x_1(t_M) & x_2(t_M) & \cdots & x_N(t_M) \end{bmatrix}$$

SVD for Modal Analysis

$$[X] = \begin{bmatrix} x_1(t_1) & x_2(t_1) & \cdots & x_N(t_1) \\ x_1(t_2) & x_2(t_2) & \cdots & x_N(t_2) \\ \vdots & & & \vdots \\ x_1(t_M) & x_2(t_M) & \cdots & x_N(t_M) \end{bmatrix}$$

recall

$$[x(t)] = \sum_{n=1}^{N} A_n \mathrm{e}^{\mathrm{i}\omega_\mathrm{n}\mathrm{t}} [\psi_\mathrm{n}]$$

$$\begin{bmatrix} \mathrm{e}^{\mathrm{i}\omega_1 t_1} & \mathrm{e}^{\mathrm{i}\omega_2 t_1} & \cdots & \mathrm{e}^{\mathrm{i}\omega_N t_1} \\ \mathrm{e}^{\mathrm{i}\omega_1 t_2} & \mathrm{e}^{\mathrm{i}\omega_2 t_2} & \cdots & \mathrm{e}^{\mathrm{i}\omega_N t_2} \\ \vdots & & & \vdots \\ \mathrm{e}^{\mathrm{i}\omega_1 t_M} & \mathrm{e}^{\mathrm{i}\omega_2 t_M} & \cdots & \mathrm{e}^{\mathrm{i}\omega_N t_M} \end{bmatrix} \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_N \end{bmatrix} [[\psi_1] \quad [\psi_2] \quad \cdots \quad [\psi_N]]$$

UΣV*

sampled sinusoids
can make *nearly* orthogonal

diagonal
amplitudes

unitary
mode shapes

SVD for Modal Analysis

- **Key idea:**

  right singular vectors of $[X] \approx$ true mode shapes
- Accuracy depends on
  - strategy for choosing sample times $t_1, t_2, \ldots, t_M$
  - number of samples M
  - total sampling duration T
  - minimum separation between modal frequencies

    $$\delta_{\min} = \min_{l \neq n} |\omega_l - \omega_n|$$
  - maximum separation between modal frequencies

    $$\delta_{\max} = \max_{l \neq n} |\omega_l - \omega_n|$$

Uniform Sampling

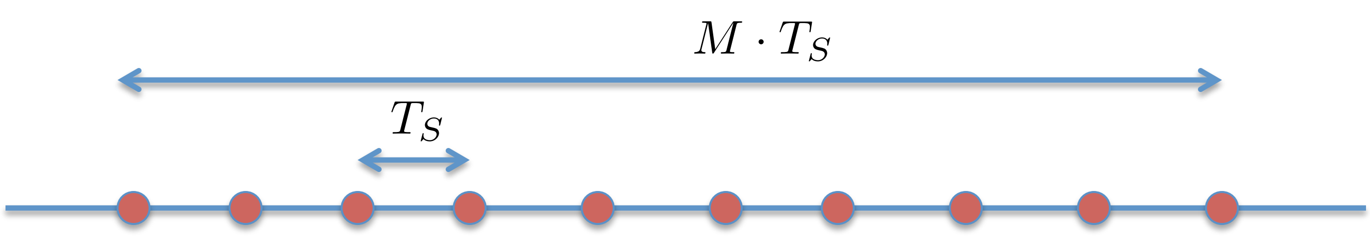
- **Theorem 1:**

  Suppose $t_1, t_2, \ldots, t_M$ are uniformly spaced with sampling interval $T_s = \frac{\pi}{\delta_{\max}}$ and

$$M \sim \max\left(\frac{\log N}{\epsilon} \cdot \frac{\delta_{\max}}{\delta_{\min}}, \; N\right).$$

  Then

$$\|\{\psi_n\} - \{\hat{\psi}_n\}\|_2 \leq C \cdot \epsilon \cdot \max_{l \neq n} \frac{|A_l||A_n|}{\min\limits_{c \in [-1,1]} \{||A_l|^2 - |A_n|^2(1 + c\epsilon)|\}}$$



Random Sampling

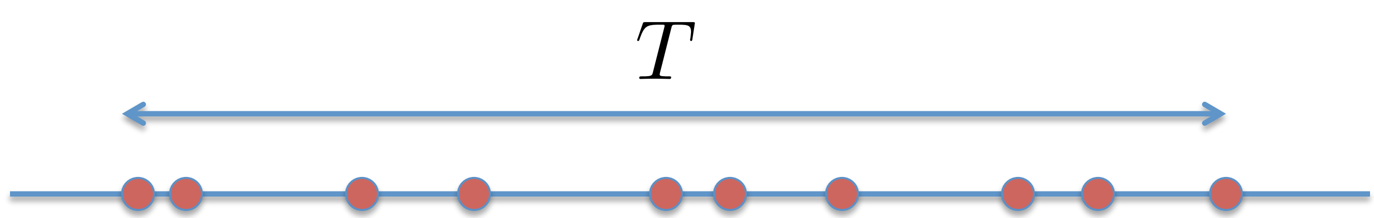
- **Theorem 2:**

  Suppose $t_1, t_2, \ldots, t_M$ are chosen uniformly at random over $[0, T]$ with

$$T \sim \frac{\log N}{\epsilon \cdot \delta_{\min}} \qquad \text{and} \qquad M \sim \frac{N \log N}{\epsilon^2}.$$

  Then with exponentially small failure probability,

$$\|\{\psi_n\} - \{\hat{\psi}_n\}\|_2 \leq C \cdot \epsilon \cdot \max_{l \neq n} \frac{|A_l||A_n|}{\min_{c \in [-1,1]} \{||A_l|^2 - |A_n|^2(1 + c\epsilon)|\}}$$

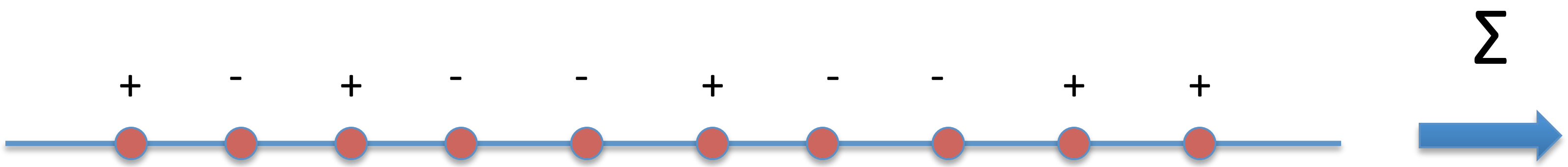


Practical sampling in HW

- **Goal:** reduce transmission, save batteries/use solar power

- Uniform samples possible, generates too much data
- Uniformly random in time too hard to implement

- Uniform samples but randomly “reduced” or sketched



Uniform Sampling with Random Matrix Multiplication

- **Theorem 3:** Suppose $t_1, t_2, \ldots, t_M$ are uniformly spaced with sampling interval $T_s = \frac{\pi}{\delta_{\max}}$ and

$$M \sim \max\left(\frac{\log N}{\epsilon} \cdot \frac{\delta_{\max}}{\delta_{\min}},\ N\right).$$

Let $[Y] = [\Phi][X]$ with $[\Phi]$ random JLT with $m \sim \frac{\text{rank}[X]}{\epsilon'^2}$ rows.

For the right singular vectors of $[Y]$, with high probability,

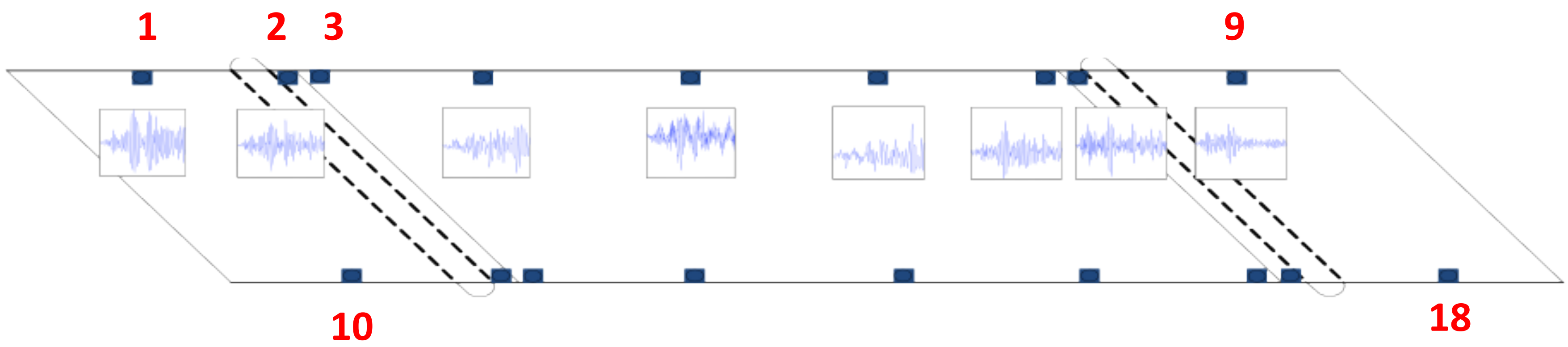
$$\|\{\psi_n\} - \{\tilde{\psi}_n\}\|_2 \leq C \cdot \epsilon \cdot \max_{l \neq n} \frac{|A_l||A_n|}{\min\limits_{c \in [-1,1]} \{||A_l|^2 - |A_n|^2(1 + c\epsilon)|\}}$$

$$+ C \cdot \epsilon' \cdot \max_{l \neq n} \frac{\sigma_l \sigma_n}{\min\limits_{c \in [-1,1]} \{|\sigma_l^2 - \sigma_n^2(1 + c\epsilon')|\}}$$

Spectral gap in [X]

Grove Street bridge, Ypsilanti, MI

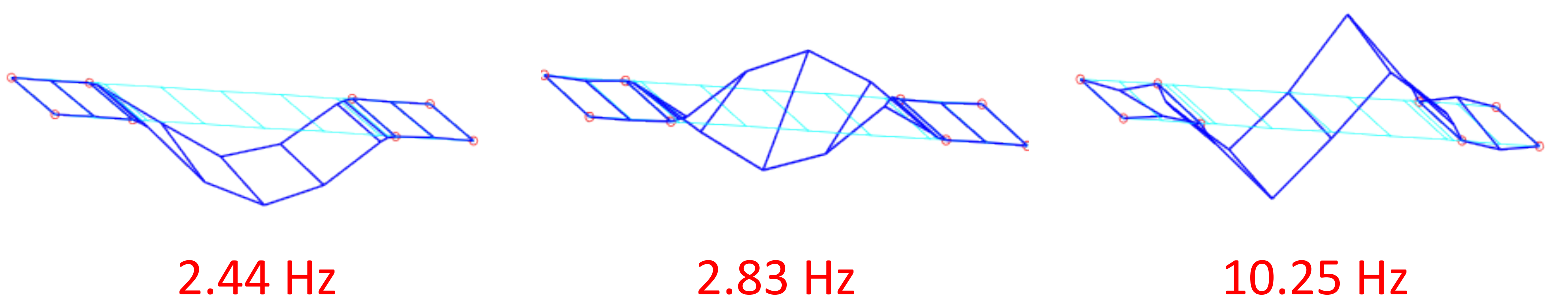
Simulations: Grove Street bridge data



- $N = 18$ sensor nodes acquire $M = 3000$ uniform time samples

$$[x_1], [x_2], \ldots, [x_{18}] \in \mathbb{R}^{3000}$$

- 3 dominant mode shapes in dataset



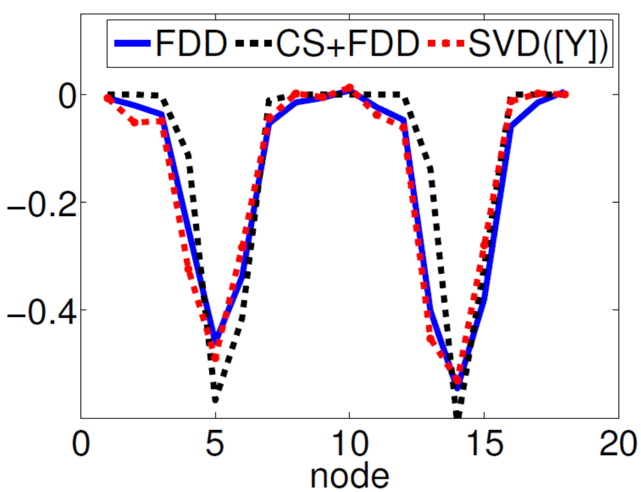
- $m = 50$ Gaussian measurements at each node

Simulations: Grove Street Bridge Data

FDD: popular modal analysis algorithm

CS+FDD: reconstruct each signal, then pass through FDD

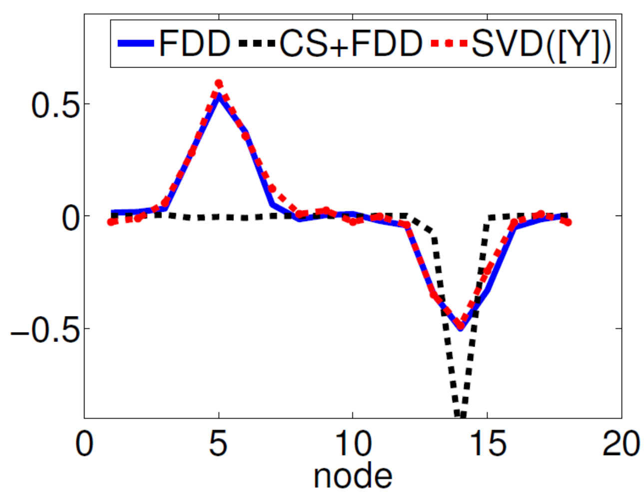
SVD([Y]): our proposed method



$$\|\{\psi_1\} - \{\psi_1^{'}\}\|_2$$

CS+FDD=0.35

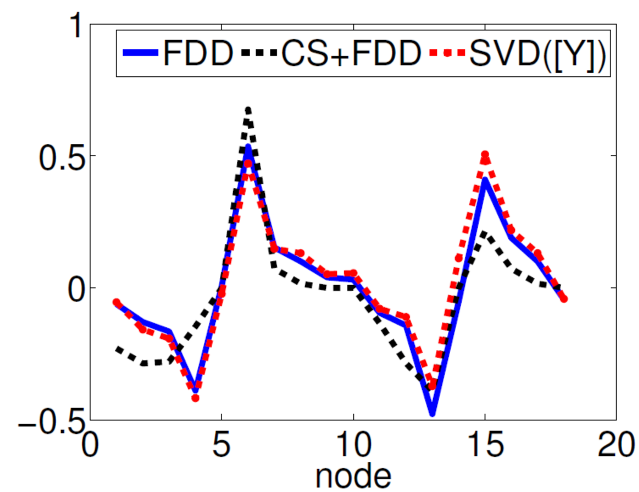
SVD([Y])=0.16



$$\|\{\psi_2\} - \{\psi_2^{'}\}\|_2$$

CS+FDD=0.96

SVD([Y])=0.14



$$\|\{\psi_3\} - \{\psi_3^{'}\}\|_2$$

CS+FDD=0.50

SVD([Y])=0.19

Estimating Modal Frequencies

- Recall:

$$\begin{bmatrix} e^{i\omega_1 t_1} & e^{i\omega_2 t_1} & \cdots & e^{i\omega_N t_1} \\ e^{i\omega_1 t_2} & e^{i\omega_2 t_2} & \cdots & e^{i\omega_N t_2} \\ \vdots & & & \vdots \\ e^{i\omega_1 t_M} & e^{i\omega_2 t_M} & \cdots & e^{i\omega_N t_M} \end{bmatrix} \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_N \end{bmatrix} [[\psi_1] \quad [\psi_2] \quad \cdots \quad [\psi_N]]$$

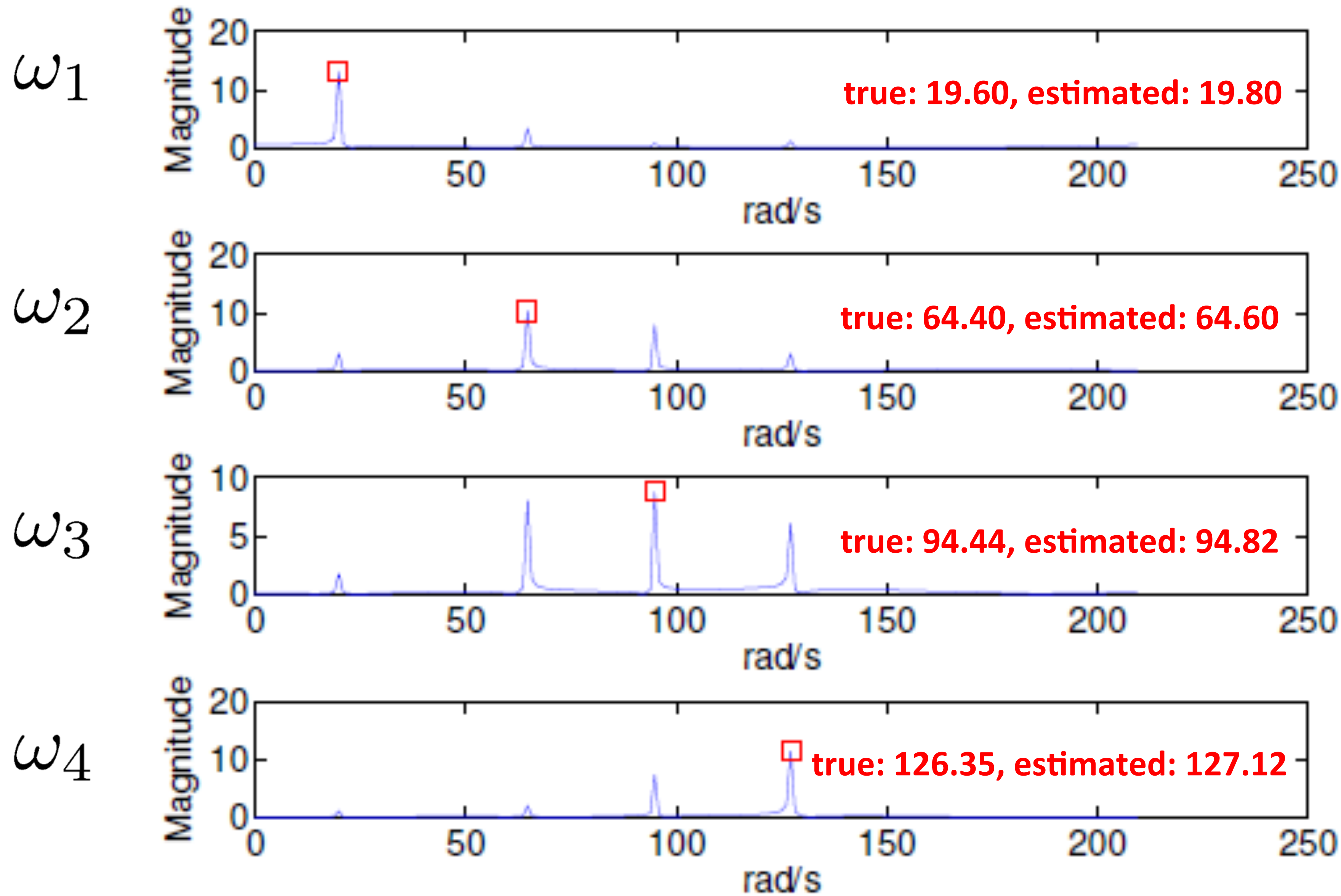
sampled sinusoids

diagonal amplitudes

unitary mode shapes

- **Idea:** estimate modal frequencies by taking FFT of *left* singular vectors of $[X]$.

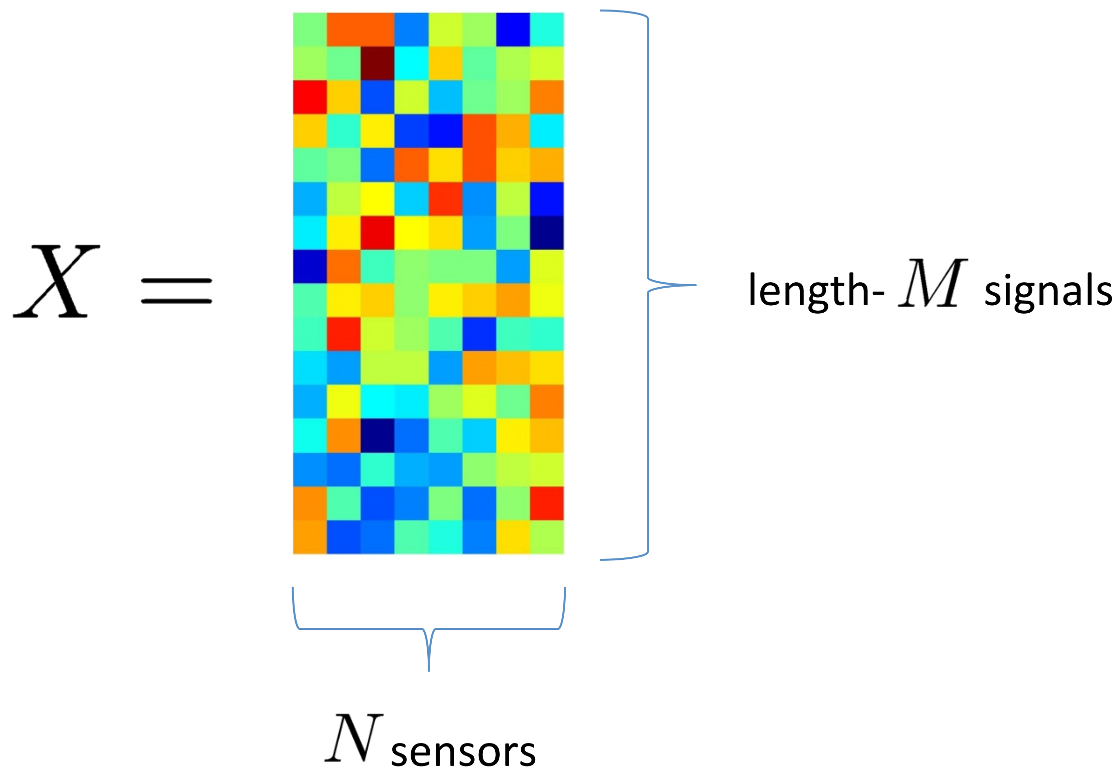
Simulations: Synthetic Data



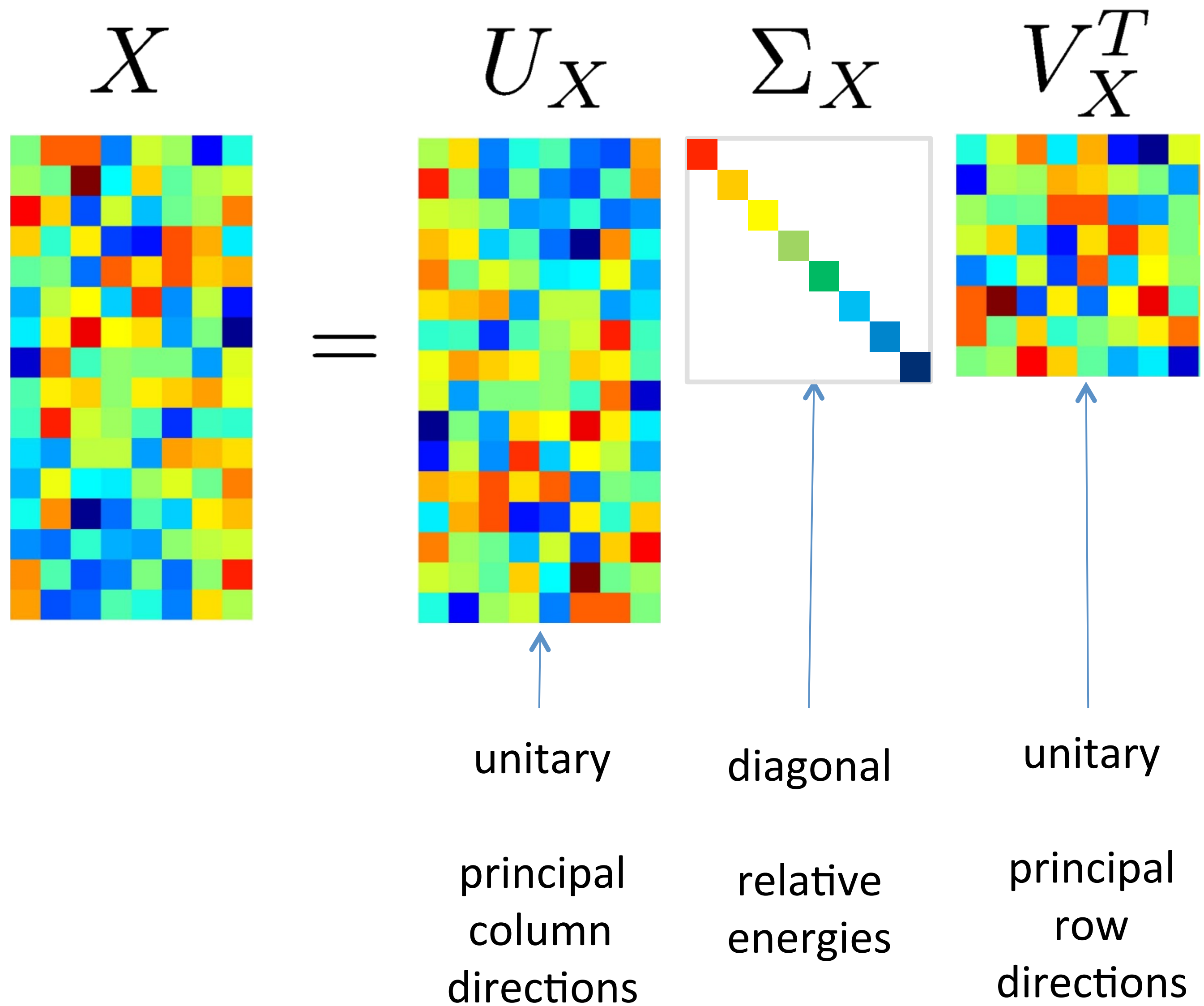
Not-so-hidden theory: Sketched SVD

Consider a Data Matrix

- Data matrix X of size $M \times N$ ($M \geq N$)
  - each column represents a signal/document/time series/etc.
  - recordings are distributed across N nodes or sensors



SVD of Data Matrix



Spectral Analysis

- SVD of X:

$$X = U_X \Sigma_X V_X^T$$

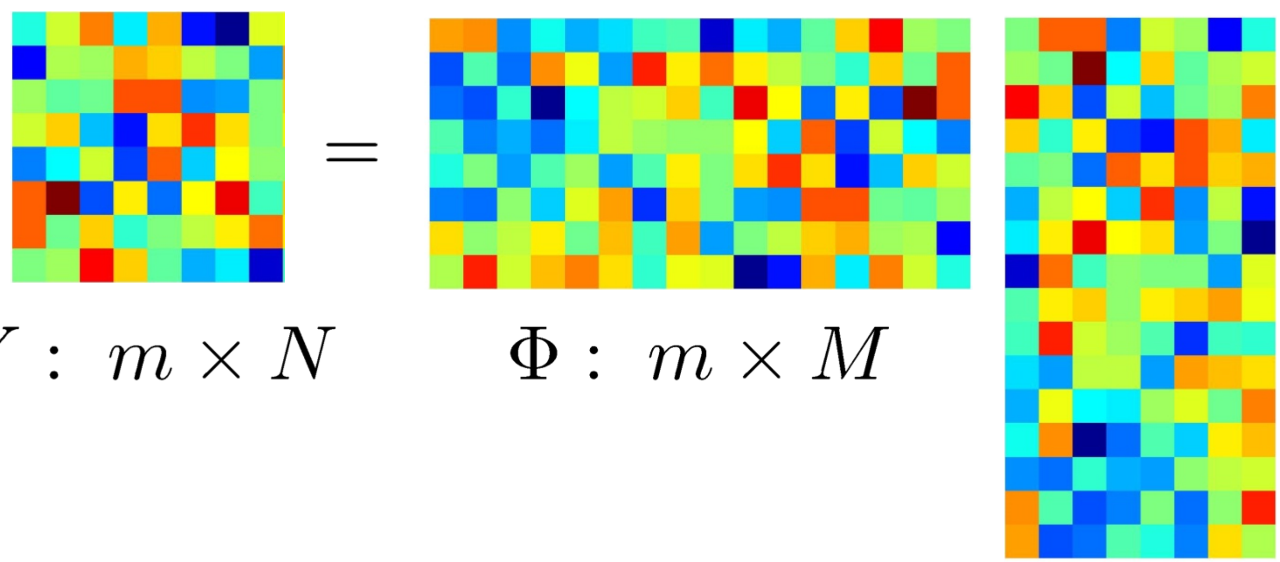
- Our interest: Σ_X and V_X , from which we can obtain
  - principal directions of rows of X (but not columns)
  - KL transform: inter-signal correlations (but not intra-signal)
  - NOT subspace spanned by (right) singular vectors

Challenge:

Obtaining X and computing SVD(X) when M is large.

Sketching

- Data matrix X of size $M \times N$ ($M \geq N$)
- Construct random $m \times M$ sketching matrix (JL matrix) Φ
- Collect a one-sided sketch $Y = \Phi X$
  - can be obtained column-by-column (“sensor-by-sensor”)
  - easily updated dynamically if X changes



$Y: \; m \times N$ $\Phi: \; m \times M$ $X: \; M \times N$

Sketched SVD

- Sketched matrix of size $m \times N$:

$$Y = \Phi X = \Phi U_X \Sigma_X V_X^T$$

- We simply compute the SVD of Y:

$$Y = U_Y \Sigma_Y V_Y^T$$

- Suppose X is rank k for some small k. If

$$m = O(k\epsilon^{-2})$$

then with high probability, $\Sigma_Y \approx \Sigma_X$ and $V_Y \approx V_X$.

Sketched SVD

- More formally, for j = 1,2,…, k,
  - singular values are preserved [Magen and Zouzias, 2010]

$$(1-\epsilon)^{1/2} \leq \frac{\sigma_j(Y)}{\sigma_j(X)} \leq (1+\epsilon)^{1/2}$$

  - right singular vectors are preserved

$$\|v_j(X) - v_j(Y)\|_2 \leq \frac{\epsilon\sqrt{1+\epsilon}}{\sqrt{1-\epsilon}} \max_{i \neq j} \frac{\sqrt{2}\sigma_i(X)\sigma_j(X)}{\min_{c\in[-1,1]} \{|\sigma_i^2(X) - \sigma_j^2(X)\cdot(1+c\epsilon)|\}}$$

roughly ϵ

small if $\sigma_j(X)$ is well separated from other singular values of X

Sketch of Proof

- Relies on arguments from matrix perturbation theory
- Recall:

$$Y = \Phi X = \Phi U_X \Sigma_X V_X^T$$

- Then:

$$Y^T Y = X^T \Phi^T \Phi X = V_X \Sigma_X U_X^T \Phi^T \Phi U_X \Sigma_X V_X^T$$

- Defining $\Delta_\Phi := \Phi^T \Phi - I$, we have

$$\begin{aligned} Y^T Y &= V_X \Sigma_X U_X^T (I + \Delta_\Phi) U_X \Sigma_X V_X^T \\ &= \underbrace{V_X \Sigma_X^2 V_X^T}_{\text{“original”}} + \underbrace{V_X \Sigma_X U_X^T \Delta_\Phi U_X \Sigma_X V_X^T}_{\text{“perturbation”}} \end{aligned}$$

Sketch of Proof

- Recall

$$Y^T Y = \overbrace{V_X \Sigma_X^2 V_X^T}^{\text{“original”}} + \overbrace{V_X \Sigma_X U_X^T \Delta_\Phi U_X \Sigma_X V_X^T}^{\text{“perturbation”}}$$

- Using concentration of measure arguments,

$$\Delta_\Phi := \Phi^T \Phi - I$$

will have small norm on colspan(U_X).

- Singular value bound follows from [Barlow and Demmel, 1980]. Singular vector bound follows from [Mathias and Veselic, 1998].

Related Work: Randomized Linear Algebra

- Compressive PCA [Fowler, 2009], [Qi and Hughes, 2012]
  - interested in *left* singular vectors rather than right
  - different aspect ratio for data matrix
  - utilize different random projections for different columns

- Subspace approximation, low-rank approximation [*many!*]
  - focused on *subspaces* rather than individual singular vectors
  - can require multiple passes over data matrix
  - many talks yesterday…

Conclusion

- Data application with embedded theory problems
  - Fourier sampling questions
  - Compressive SVD
  - Actual hardware platform for experimentation

- Future work
  - modal analysis of systems with damping
  - estimation bounds for modal frequencies
  - more sophisticated estimation strategies

  - Graph analogs for PDEs

EXTRAS

Related Work: Matrix Perturbation Theory

- Absolute bounds
  - absolute error in eigenvalues, absolute separation between eigenvalues [Davis and Kahan, 1970], [Golub and van Loan, 1996]
- Relative bounds
  - relative error in eigenvalues, relative separation between eigenvalues [Eisenstat and Ipsen, 1995], [Li, 1996], [Li, 1998]
  - useful even for small eigenvalues

SVD, eigenvectors of Laplacian, and spectral graph theory…?

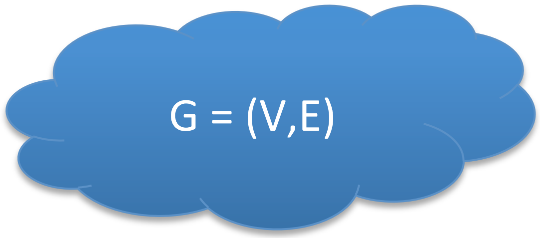
- Recall $[\psi_n]$ are eigenvectors of (discrete, generalized) Laplacian on line graph with N vertices
- Generalize to graph Laplacian, L
- Eigenvectors of L = singular vectors of incidence matrix
  - Well-approximated by sketch of incidence matrix

- One use of spectral graph theory: solve Poisson problems on graphs, construct Green's function = invert Laplacian

$$Lu = f \quad \text{on } V$$

$$u = 0 \quad \text{on } \partial V$$

Dirichlet boundary conditions

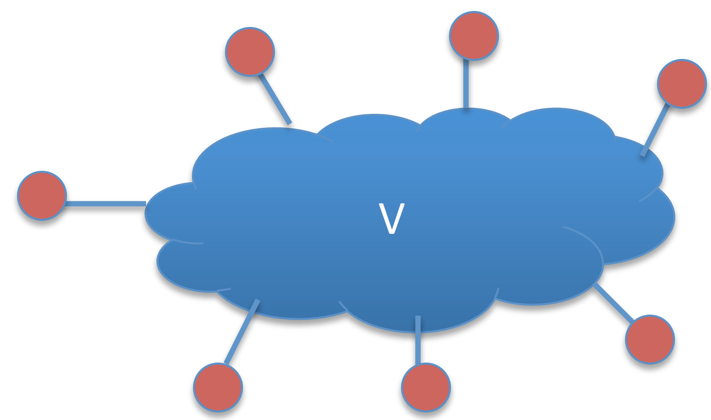


Graph Analogs for PDEs

Absorption at vertices in V

$$Lu + \alpha u = f \quad \text{on } V$$
$$tu + \nabla u = g \quad \text{on } \partial V$$

Generalized boundary conditions



- Forward problem: solve for *u* (discrete Green's function)
- Inverse problem: given sources and observations on boundary, find α

Joint work with Jeremy Hoskins, John Schotland (Umich)