Algorithmic Regularity Lemmas and Applications

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Szemerédi's Regularity Lemma

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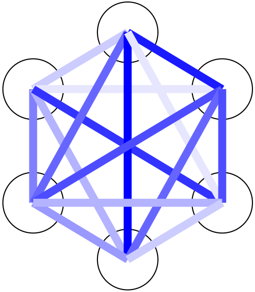

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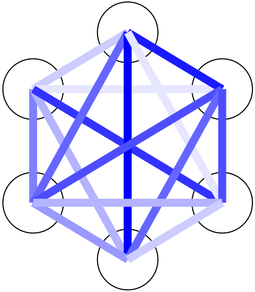

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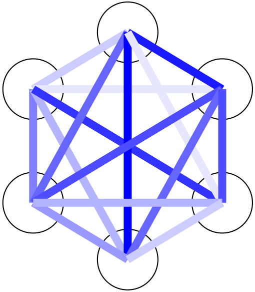

- Very important tool in graph theory
- Gives a rough structural result for all graphs

Regularity of Sets

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Definition

Given a graph G and two sets of vertices X and Y, we say the pair (X, Y) is ϵ-*regular* if for any $X' \subset X$ with $|X'| \geq \epsilon|X|$, $Y' \subset Y$ with $|Y'| \geq \epsilon|Y|$, we have

$$\left| d(X', Y') - d(X, Y) \right| \leq \epsilon.$$

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Roughly says graph between X and Y is "random-like".

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Szemerédi's regularity lemma

For every $\epsilon > 0$, there is an $M(\epsilon)$ such that for any graph $G = (V, E)$, there is an equitable, ϵ-regular partition of the vertices into at most $M(\epsilon)$ parts.

Regularity Lemma Proof Sketch

Definition

For a vertex partition $\mathcal{P} : V = V_1 \cup V_2 \cup \ldots \cup V_k$, define the *mean square density*:

$$q(\mathcal{P}) = \sum_{i,j} p_i p_j d(V_i, V_j)^2,$$

where $p_i = \frac{|V_i|}{|V|}$.

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- Between 0 and 1.
- If we refine the partition, it cannot decrease.
- If a partition into k parts is not ϵ-regular, can divide each piece into at most 2^{k+1} parts, according to worst case sets, to get an increase of ϵ^5 (then make equitable).

Algorithmic Regularity

Alon-Duke-Lefmann-Rödl-Yuster (1994)

If a pair (X, Y) is not ϵ-regular, find a pair of subsets that show they are not $\epsilon^4/16$-regular, in time $O_\epsilon(n^{\omega+o(1)})$. Implies tower height at most $T(\epsilon^{-20})$. $(\omega < 2.373)$

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Alon-Naor (2006)

Polynomial-time algorithm, at most $T(O(\epsilon^{-7}))$ parts.

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Folklore/Tao blog post (2010)

Randomized algorithm in time $O_{\epsilon}(1)$, ϵ-regular partition.

Finding a regular partition

Fox-L.-Zhao

An $O_{\epsilon,\alpha}(n^2)$-time *deterministic* algorithm which, given ϵ, α, k and a graph G on n vertices that has an ϵ-regular partition with k parts, gives a $(1+\alpha)\epsilon$-regular partition into k parts.

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- that (X, Y) are ϵ-regular.
- a pair of subsets $U \subset X$, $W \subset Y$ that show that (X, Y) are not $(1-\alpha)\epsilon$-regular, i.e. $|U| \geq (1-\alpha)\epsilon|X|$, $|W| \geq (1-\alpha)\epsilon|Y|$, and $|d(X, Y) - d(U, W)| > (1-\alpha)\epsilon$.

Frieze-Kannan (weak) regularity lemma

Definition

Given a partition $\mathcal{P} = \{V_1, V_2, ..., V_k\}$ of the set of vertices V, it is *Frieze-Kannan ϵ-regular* (FK-ϵ-regular) if for any pair of sets $S, T \subseteq V$, we have

$$\left| e(S,T) - \sum_{i,j=1}^{k} d(V_i, V_j)|S \cap V_i||T \cap V_j| \right| \leq \epsilon |V|^2$$

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Proof similar: refine by worst case sets, mean square density increases by ϵ^2.

Counting Lemma

Definition

Given two (possibly weighted) graphs G_1 and G_2 on the same vertex set V, we define their *cut distance*

$$d_{\square}(G_1, G_2) = \frac{1}{|V|^2} \max_{S,T \subseteq V} |e_{G_1}(S,T) - e_{G_2}(S,T)|.$$

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Partition $\mathcal{P}$ is FK-ϵ-regular if and only if $d_\square(G, G_\mathcal{P}) \leq \epsilon$.

Counting lemma

Given two graphs G_1 and G_2 on the same vertex set, for any graph H on k vertices, we have

$$|\hom(H, G_1) - \hom(H, G_2)| \leq e(H) d_\square(G_1, G_2) n^k.$$

Algorithmic Frieze-Kannan

Dellamonica-Kalyanasundaram-Martin-Rödl-Shapira

Give a deterministic algorithm which finds a Frieze-Kannan ϵ-regular partition

- in time $\epsilon^{-6} n^{\omega + o(1)}$ into at most $2^{O(\epsilon^{-7})}$ parts (2012)
- in time $O(2^{2^{\epsilon^{-O(1)}}} n^2)$ into at most $2^{\epsilon^{-O(1)}}$ parts (2015)

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Dellamonica-Kalyanasundaram-Martin-Rödl-Shapira

There is an $n^{\omega+o(1)}$-time algorithm which, given $\epsilon > 0$, an n-vertex graph G and a partition $\mathcal{P}$ of $V(G)$, either:

1. Correctly states that $\mathcal{P}$ is FK-ϵ-regular;
2. Finds sets S, T which witness the fact that $\mathcal{P}$ is not FK-$\epsilon^3/1000$-regular.

Algorithmic Frieze-Kannan

Corollary

There is an $\epsilon^{-O(1)} n^{\omega+o(1)}$-time algorithm which, given $\epsilon > 0$, an n-vertex graph G, outputs $t \leq \epsilon^{-O(1)}$, subsets $S_1, S_2, \ldots, S_t, T_1, T_2, \ldots, T_t \subset V(G)$ and real numbers $c_1, c_2, \ldots, c_t$ such that

$$d_{\square}(G, d(G)K_{V(G)} + c_1 K_{S_1,T_1} + c_2 K_{S_2,T_2} + \ldots + c_t K_{S_t,T_t}) \leq \epsilon.$$

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Can also do in time $2^{2^{\epsilon^{-O(1)}}} n^2$.

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How fast can we approximate the count within an additive $\epsilon n^{|V(H)|}$?

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Can use algorithmic regularity lemma.

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Can be done in time $O_H(\epsilon^{-O(e(H))} n + \epsilon^{-O(1)} n^{\omega+o(1)})$.

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Corollary

We can approximate the count of K_{1000} in a graph on n vertices within an additive $n^{1000-10^{-6}}$ in time $O(n^{2.4})$.

Counting subgraphs proof sketch

Fox-L.-Zhao (2017)

Can count the number of copies of a graph H on k vertices in a graph G on n vertices, up to an error of at most ϵn^k in time $O_H(\epsilon^{-O(e(H))} n + \epsilon^{-O(1)} n^{\omega + o(1)})$.

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Apply algorithmic Frieze-Kannan: In time $\epsilon^{-O(1)}n^{\omega+o(1)}$, get

$$G' = d(G)K_{V(G)} + c_1 K_{S_1,T_1} + c_2 K_{S_2,T_2} + \dots + c_t K_{S_t,T_t}$$

and $d_\square(G, G') \leq \epsilon/e(H)$, $t \leq \epsilon^{-O(1)}$.

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This means that the count is off by at most ϵn^k in G'.

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We can compute $\hom(H, G')$ by computing a sum of $(t+1)^{e(H)}$ terms.

Algorithmic regularity proof sketch

Fox-L.-Zhao

An $O_{\epsilon,\alpha}(n^2)$-time deterministic algorithm which, given ϵ, α and a graph G between sets X, Y of size n, outputs either

- that (X, Y) are ϵ-regular.
- a pair of subsets $U \subset X$, $W \subset Y$ that show that (X, Y) are not $(1 - \alpha)\epsilon$-regular.

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Algorithmic Frieze-Kannan: $t \le (\alpha\epsilon)^{-O(1)}$, G' with $d_\square(G, G') \le \alpha\epsilon^3/4$,

$$G' = d(G)K_{V(G)} + c_1 K_{S_1,T_1} + c_2 K_{S_2,T_2} + \ldots + c_t K_{S_t,T_t}.$$

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$$G' = d(G)K_{V(G)} + c_1 K_{S_1,T_1} + c_2 K_{S_2,T_2} + \ldots + c_t K_{S_t,T_t}.$$

Can check a bounded number of cases based on the sizes of the intersection of U, W with X, Y and each S_i, T_i. Check feasibility and whether the density is off.

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Corollary

An $O_{\epsilon,\alpha,k}(n^2)$-time algorithm which, given $\epsilon, \alpha, k > 0$, graph G on n vertices, and a k-part partition $\mathcal{P}$ of the vertices, either:

- correctly states that $\mathcal{P}$ is $(1 + \alpha)\epsilon$-regular.
- correctly states that $\mathcal{P}$ is not ϵ-regular.

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Apply algorithmic Frieze-Kannan to obtain $t \le (\alpha\epsilon/k)^{O(1)}$, G' such that $d_\square(G, G') \le \alpha\epsilon/(10k^2)$, and

$$G' = d(G)K_{V(G)} + c_1 K_{S_1,T_1} + c_2 K_{S_2,T_2} + \ldots + c_t K_{S_t,T_t}.$$

Algorithmic regularity proof sketch

Fox-L.-Zhao

An $O_{\epsilon,\alpha}(n^2)$-time deterministic algorithm which, given ϵ, α, k and a graph G on n vertices that has an ϵ-regular partition with k parts, gives a $(1+\alpha)\epsilon$-regular partition into k parts.

Apply algorithmic Frieze-Kannan to obtain $t \le (\alpha\epsilon/k)^{O(1)}$, G' such that $d_\square(G, G') \le \alpha\epsilon/(10k^2)$, and

$$G' = d(G)K_{V(G)} + c_1 K_{S_1,T_1} + c_2 K_{S_2,T_2} + \ldots + c_t K_{S_t,T_t}.$$

Can work with G'. Need to check $2^{2^{(k/\alpha\epsilon)^{O(1)}}}$ possible partitions. For each one, either get not $(1+\alpha/2)\epsilon$-regular, or $(1+3\alpha/4)\epsilon$-regular. Second case must happen for a partition.

Conclusion

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Questions

- Faster algorithmic regularity lemmas?
- With what additive error can we count subgraphs?