Derandomizing Isolation Lemma: A geometric approach

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Based on joint works with Stephen Fenner and Thomas Thierauf

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  - Disjoint Paths(s_1, t_1, s_2, t_2) in RP [BH14]

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- For example,
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- Randomized arguments show existence for such families.

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- Common Independent sets two matroids (quasi-poly) [GT17].
- Minimum vertex covers in a bipartite graph (quasi-poly).

Polytope of a family

- For a set $S \subseteq E$, define $x^S \in \mathbb{R}^E$

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- Its corners are exactly $\{x^S \mid S \in \mathcal{B}\}$.

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- $w \cdot x^S = w(S)$, for any $S \subseteq E$.

Observation

w is isolating for $\mathcal{B}$

$$\Updownarrow$$

$w \cdot x$ has a unique minima over $P(\mathcal{B})$.

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- In each round, slightly modify the current weight function to get a smaller minimizing face.

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- In each round, slightly modify the current weight function to get a smaller minimizing face.
- We stop when we reach a zero-dimensional face.

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Claim

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- We will have $\log n$ rounds.

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- Significant reduction in the dimension: choose many vectors.

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- $F_{\log m}$: no length-m vectors, hence, the face is a corner .

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For all faces F of $P(\mathcal{B})$,

Number of vectors in L_F of length $\leq 2\lambda_1(L_F)$ is $\mathrm{poly}(m)$.

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Number of Cycles

Lemma

For a graph H with n nodes,

$$\text{No cycles of length} \leq r$$
$$\Downarrow$$
$$\text{number of cycles of length upto } 2r \text{ is} \leq n^4.$$

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- $\mathcal{B} =$ set of common bases.

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- $I \subseteq [m]$ is a common base if $\text{rank}(A_I) = \text{rank}(B_I) = n$.
- Question: is there a common base?
- $\mathcal{B} =$ set of common bases.
- $P(\mathcal{B})$ is given by [Edmonds 1970]

$$
\begin{aligned}
x_e &\geq 0 \quad e \in E, \\
\sum_{e \in S} x_e &\leq \text{rank}(A_S) \quad S \subseteq [m], \\
\sum_{e \in S} x_e &\leq \text{rank}(B_S) \quad S \subseteq [m], \\
\sum_{e \in [m]} x_e &= n.
\end{aligned}
$$

Faces of $P(\mathcal{B})$

- For any face F there exist
  - $[m] = A_0 \sqcup S_1 \sqcup S_2 \sqcup \cdots \sqcup S_p$
  - $[m] = A_0 \sqcup T_1 \sqcup T_2 \sqcup \cdots \sqcup T_q$ and
  - positive integers $n_1, n_2, \ldots, n_p$ and $m_1, m_2, \ldots, m_q$
  - with $\sum_i n_i = \sum_j m_j = n$

$$\begin{aligned} x_e &= 0 \ \forall e \in A_0 \\ \sum_{e \in S_i} x_e &= n_i \ \forall i \in [p] \\ \sum_{e \in T_j} x_e &= m_j \ \forall i \in [q] \end{aligned}$$

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$$\begin{aligned} x_e &= 0 \; \forall e \in A_0 \\ \sum_{e \in S_i} x_e &= 0 \quad \forall i \in [p] \\ \sum_{e \in T_j} x_e &= 0 \quad \forall i \in [q] \end{aligned}$$

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