Spectral gaps and geometric representations

Amir Yehudayoff (Technion)

Noga Alon (Tel Aviv) Shay Moran (Simons)

pseudorandomness

random objects have properties

difficult to describe
hard to compute
expanders

what are these properties?

find ‘explicit’ objects with these properties?

objects: sign matrices

let M be an $n \times n$ matrix with ± 1 entries

model

a linear map
a bi-partite graph
a class of boolean functions $f_1, \ldots, f_n$

the property: a complexity measure

let M be an $n \times n$ sign matrix

definition: dimension complexity

the minimum d in which M is realized as point-halfspace incidence matrix in d-dim euclidean geometry:

$$\exists p_1, \ldots, p_n \in \mathbb{R}^d \quad \& \quad \exists h_1, \ldots, h_n \text{ halfspaces in } \mathbb{R}^d$$

so that

$$M_{ij} = 1 \;\Leftrightarrow\; p_j \in h_i$$

halfspace $h = \{p : \langle p, a \rangle > b\}$ for $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$

background

1. dimension complexity is equivalent* to

i. $sign - rank(M) = \min\{rank(R) : sign(R) = sign(M)\}$

$M_{ij}R_{ij} > 0$ for all i, j

background

1. dimension complexity is equivalent* to

i. $sign - rank(M) = \min\{rank(R) : sign(R) = sign(M)\}$

$M_{ij} R_{ij} > 0$ for all i, j

ii. the unbounded error two-player communication complexity of M [Paturi-Simon]

alice gets i and bob gets j
they communicate and have shared randomness
need to output M_{ij} with probability $> 1/2$

background

1. dimension complexity is equivalent* to

i. $sign - rank(M) = \min\{rank(R) : sign(R) = sign(M)\}$

$M_{ij} R_{ij} > 0$ for all i, j

ii. the unbounded error two-player communication complexity of M [Paturi-Simon]

alice gets i and bob gets j
they communicate and have shared randomness
need to output M_{ij} with probability $> 1/2$

2. related to learning theory [Linial-Shraibman, ...]

example: the identity

let I be the $n \times n$ signed-identity

$$I_{ij} = -1 \iff i = j$$

the rank of I is full $(n > 2)$

what is its sign rank?

example: the identity

let I be the $n \times n$ signed-identity

$$I_{ij} = -1 \;\Leftrightarrow\; i = j$$

the rank of I is full ($n > 2$)

what is its sign rank?

3:

$$R_{ij} = (i - j)^2 - \frac{1}{2} = \left(i^2 - \frac{1}{2}\right) + \left(j^2\right) - \left(2ij\right)$$

more generally: the moment curve

claim [Alon-Frankl-Rodl]. if M be a matrix with at most Δ ones in each row then

$$sign - rank(M) \leq 2\Delta + 1$$

more generally: the moment curve

claim [Alon-Frankl-Rodl]. if M be a matrix with at most Δ ones in each row then

$$sign - rank(M) \leq 2\Delta + 1$$

proof.

(a) for each i, there is a polynomial $g_i(x)$ of degree at most 2Δ so that for all j,

$$g_i(j)M_{ij} > 0$$

(b) $g_i(j) = \langle a_i, b_j \rangle$ with $a_i, b_j \in \mathbb{R}^{2\Delta+1}$

random matrices

theorem [Alon-Frankl-Rodl]. the sign rank of most $n \times n$ sign matrices is at least $n/32$

at most $(1/2 + o(1))n$; the truth is unknown

theorem [Warren]. the zeros of polynomials do not partition real space to many parts

let $P_1, P_2, \ldots, P_m$ be real polynomials, each in ℓ variables and degree k. If $m \geq \ell$ then the number of connected components of $\mathbb{R}^\ell \setminus zeros(P)$ is at most $(4ekm/\ell)^\ell$

explicit constructions

theorem [Forster]. the Hadamard matrix H has sign rank $> \sqrt{n}$

$H_{x,y} = (-1)^{\langle x,y \rangle}$ for $x, y \in \mathbb{Z}_2^{\log n}$

explicit constructions

theorem [Forster]. the Hadamard matrix H has sign rank $> \sqrt{n}$

$H_{x,y} = (-1)^{\langle x,y \rangle}$ for $x, y \in \mathbb{Z}_2^{\log n}$

more generally. for every M

$$sign - rank(M) \geq \frac{n}{\|M\|}$$

where $\|M\| = \max_{\|x\| \leq 1} \|Mx\|$ is spectral norm

explicit constructions

theorem [Forster]. the Hadamard matrix H has sign rank $> \sqrt{n}$

$H_{x,y} = (-1)^{\langle x,y \rangle}$ for $x, y \in \mathbb{Z}_2^{\log n}$

more generally. for every M

$$r = sign - rank(M) \geq \frac{n}{\|M\|}$$

where $\|M\| = \max_{\|x\| \leq 1} \|Mx\|$ is spectral norm

idea. measure the correlation of M and R

$$\frac{n^2}{r} \underset{isotropy}{\leq} \langle M, R \rangle \underset{CS}{\leq} \|M\| n$$

normalized R

(isotropy)

lemma [Forster]. for every finite $X \subset \mathbb{R}^d$ in general position there is a linear map L so that $Y = LX$ is isotropic:

for every $v \in \mathbb{R}^d$

$$\mathop{\mathbb{E}}_{y \sim unif(Y)} \text{ projection of } v \text{ to } \mathrm{span}(y) = \frac{v}{d}$$

e.g. $\{e_1, \ldots, e_d\}$ are isotropic

on proof.
Forster: local changes + compactness
Hardt-Moitra (Lee): entropy optimality

spectral methods

the spectrum of a matrix defined by a combinatorial object provides a lot of data

expanders
Fourier
Forster

spectral gaps related to pseudorandomness

spectral gaps

theorem [Alon-Moran-Y]. if M is a Δ regular[1] sign matrix with $\Delta \leq n/2$ then

$$sign - rank(M) \geq \frac{\Delta}{\sigma_2}$$

where $d = \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$ are singular values of $bool(M)$

"spectral gaps yield non-trivial sign rank"

[1]there are Δ ones in every row and column

applications I: expanders

if M is sign-adjacency matrix of d-regular Ramanujan graph then

$$sign - rank(M) > \frac{\sqrt{d}}{2}$$

"expanders do not embed in low dim euclidean geometry"

we saw. the sign rank is at most $2d+1$

applications II: geometry

if M is $n \times n$ incidence matrix of finite projective plane

$$M_{PL} = 1 \Leftrightarrow P \in L$$

then

$$sign - rank(M) > n^{1/4} - 1$$

"finite geometries do not embed in low dim euclidean geometry"

applications III: communication complexity

explicit function $f : \{0,1\}^m \times \{0,1\}^m \to \{0,1\}$ so that

1. the unbounded error communication complexity of f is $\Omega(m)$

2. the distributional communication complexity of f under product distribution with error $1/3$ is at most $O(1)$ [Kremer-Nisan-Ron]

"product distributions are much easier"

alice gets P, bob gets L and they need to decide if $P \in L$

weaker versions [Sherstov]

summary

a pseudorandom property: $n/32 \leq$ sign rank $\leq n/2$ (open)

explicit constructions but not optimal (open)

methods: spectral & isotropy

applications: expanders, geometry, communication