Composing Strategies in Pebble Games

Anuj Dawar

University of Cambridge Computer Laboratory

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Finite Structures

Fix a *finite relational vocabulary*: $\tau = (R_1, \ldots, R_m)$.

and consider finite τ-structures

$$\mathbb{A} = (A, R_1^{\mathbb{A}}, \ldots, R_m^{\mathbb{A}})$$

$$\mathbb{B} = (B, R_1^{\mathbb{B}}, \ldots, R_m^{\mathbb{B}})$$

As a special case, we have graphs, where τ consists of a single *binary* relation E.

Homomorphism and Isomorphism

$\mathbb{A} \stackrel{hom}{\rightarrow} \mathbb{B}$: there is $h : A \rightarrow B$ *s.t.* for any $\mathbf{a}$:

$$R^{\mathbb{A}}(\mathbf{a}) \Rightarrow R^{\mathbb{B}}(h(\mathbf{a})).$$

$\mathbb{A} \cong \mathbb{B}$: there is a *bijection* $h : A \rightarrow B$ *s.t.* for any $\mathbf{a}$:

$$R^{\mathbb{A}}(\mathbf{a}) \Leftrightarrow R^{\mathbb{B}}(h(\mathbf{a})).$$

Or, equivalently $\mathbb{A} \cong \mathbb{B}$ if there are $h : \mathbb{A} \stackrel{hom}{\rightarrow} \mathbb{B}$ and $g : \mathbb{B} \stackrel{hom}{\rightarrow} \mathbb{A}$ such that

$$h \circ g = \mathrm{id}_{\mathbb{B}} \qquad \text{and} \qquad g \circ h = \mathrm{id}_{\mathbb{A}}$$

Complexity of Homomorphism and Isomorphism

The problem of deciding, given $\mathbb{A}$ and $\mathbb{B}$, whether $\mathbb{A} \stackrel{hom}{\to} \mathbb{B}$ is NP-complete.

The problem of deciding, given $\mathbb{A}$ and $\mathbb{B}$, whether $\mathbb{A} \cong \mathbb{B}$ is

- not known to be NP-complete;
- not known to be in P;
- known to be in *quasi-polynomial* time **(Babai 2016)**

The *k-local consistency* test gives an algorithm, running in time $n^{O(k)}$ that gives an *approximate* test for $\mathbb{A} \stackrel{hom}{\to} \mathbb{B}$.

Finite Variable Logic

$\exists^{+,k}\mathrm{FO}$: *existential, positive* formulas of first-order logic, using no more than k distinct variables.

$$\exists x_1 \cdots \exists x_k \bigwedge_{i \neq j} E(x_i, x_j)$$

In $\exists^{+,k}\mathrm{FO}$ we can express the existence of a k-clique, but not a $(k+1)$-clique.

$$\exists x_1 \exists x_2 E(x_1, x_2) \wedge (\exists x_1 E(x_2, x_1) \wedge \cdots)$$

In $\exists^{+,2}\mathrm{FO}$, we can express the existence of a *path* of length n for any n.

k-local Consistency

Write $\mathbb{A} \Rrightarrow^k \mathbb{B}$ to denote that for any sentence φ of $\exists^{+,k}\mathrm{FO}$

$$\text{if} \quad \mathbb{A} \models \varphi \quad \text{then} \quad \mathbb{B} \models \varphi.$$

The k-local consistency test determines whether $\mathbb{A} \Rrightarrow^k \mathbb{B}$

$$\mathbb{A} \stackrel{hom}{\to} \mathbb{B} \quad \Leftrightarrow \quad \mathbb{A} \Rrightarrow^n \mathbb{B} \quad \Rightarrow \quad \mathbb{A} \Rrightarrow^k \mathbb{B}$$

where $|A| = n$ and $n > k$.

Pebble Games

The relation $\mathbb{A} \Rrightarrow^k \mathbb{B}$ has a *pebble game* characterization due to **(Kolaitis-Vardi 1992)**.

The game is played by two players—*Spoiler* and *Duplicator*—using k pairs of pebbles $\{(a_1, b_1), \ldots, (a_k, b_k)\}$.

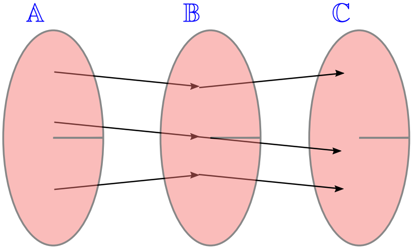
Spoiler moves by picking a pebble a_i and placing it on an element of $\mathbb{A}$.

Duplicator responds by placing b_i on an element of $\mathbb{B}$

Spoiler wins at any stage if the partial map from $\mathbb{A}$ to $\mathbb{B}$ defined by the pebble pairs is not a partial homomorphism

If Duplicator has a strategy to play forever without losing, then $\mathbb{A} \Rrightarrow^k \mathbb{B}$.

Composing Strategies



Duplicator can compose strategies witnessing $\mathbb{A} \equiv\rangle^k \mathbb{B}$ and $\mathbb{B} \equiv\rangle^k \mathbb{C}$ to get one for $\mathbb{A} \equiv\rangle^k \mathbb{C}$.

Strategies more formally

A *strategy* for $\mathbb{A} \Rrightarrow^k \mathbb{B}$ is a set H of pairs $(\mathbf{a}, \mathbf{b})$ where $\mathbf{a}$ and $\mathbf{b}$ are l-tuples of elements from $\mathbb{A}$ and $\mathbb{B}$ respectively for some $0 \leq l \leq k$, such that:

1. for each $(\mathbf{a}, \mathbf{b}) \in H$, the map $\mathbf{a} \mapsto \mathbf{b}$ is a partial homomorphism;
2. if $(\mathbf{a}, \mathbf{b}) \in H$, then $(\mathbf{a}', \mathbf{b}') \in H$ whenever $\mathbf{a}'$ and $\mathbf{b}'$ are obtained by deleting corresponding elements of $\mathbf{a}$ and $\mathbf{b}$; and
3. if $(\mathbf{a}, \mathbf{b}) \in H$ and $|\mathbf{a}| = |\mathbf{b}| = l < k$, then there is a function $f : A \to B$ so that for each $a \in A$, $(\mathbf{a}a, \mathbf{b}f(a)) \in H$.

$\mathrm{id}_{\mathbb{A}} : \mathbb{A} \Rrightarrow^k \mathbb{A}$ is the strategy consisting of all pairs $(\mathbf{a}, \mathbf{a})$.

Say that a strategy $H : \mathbb{A} \Rrightarrow^k \mathbb{B}$ is *injective* if the function f in (2) can always be chosen to be injective.

Invertible Strategies

The following are equivalent for any $\mathbb{A}$ and $\mathbb{B}$:

1. There are strategies $H : \mathbb{A} \Rrightarrow^k \mathbb{B}$ and $I : \mathbb{B} \Rrightarrow^k \mathbb{A}$ such that $I \circ H = \mathrm{id}_{\mathbb{A}}$ and $H \circ I = \mathrm{id}_{\mathbb{B}}$.
2. There are injective strategies $H : \mathbb{A} \Rrightarrow^k \mathbb{B}$ and $I : \mathbb{B} \Rrightarrow^k \mathbb{A}$.
3. There is a *bijective* strategy $H : \mathbb{A} \Rrightarrow^k \mathbb{B}$.

The last condition amounts to saying the *Duplicator* has a winning strategy in the *bijection game*. **(Hella 1996)**

Bijection Games

Hella's bijection game characterizes the equivalence $\mathbb{A} \equiv^k \mathbb{B}$, which says that the two structures cannot be distinguished by any sentence of C^k—k-variable first-order logic with *counting quantifiers*.

This equivalence relation has many *independent* characterizations.

$G \equiv^k H$ for a pair of graphs G, H *iff* they cannot be distinguished by the $(k-1)$-dimensional *Weisfeiler-Leman* method.

This is a much studied approximation of *graph isomorphism*.

Cores

A structure $\mathbb{A}$ is a *core* if there is no proper substructure $\mathbb{A}' \subseteq \mathbb{A}$ such that $\mathbb{A} \stackrel{hom}{\to} \mathbb{A}'$.

Every structure $\mathbb{A}$ has a core $\mathbb{A}' \subseteq \mathbb{A}$ such that $\mathbb{A} \stackrel{hom}{\to} \mathbb{A}'$.
Moreover, $\mathbb{A}'$ is unique up to *isomorphism*.

Say $\mathbb{A}'$ is a *k-core* of $\mathbb{A}$ if:

1. $\mathbb{A} \Rrightarrow^{k} \mathbb{A}'$;
2. $\mathbb{A}' \Rrightarrow^{k}_{\mathrm{inj}} \mathbb{A}$;
3. for any $\mathbb{B}$, if $\mathbb{A} \Rrightarrow^{k} \mathbb{B}$ and $\mathbb{B} \Rrightarrow^{k}_{\mathrm{inj}} \mathbb{A}$ then $\mathbb{A}' \Rrightarrow^{k}_{\mathrm{inj}} \mathbb{B}$.

Every structure $\mathbb{A}$ has a k-core and it is unique up to $\equiv^{k}$.

Some Questions

If $\mathcal{C}$ is a class of structures closed under $\equiv^k$ and *homomorphisms*, is it closed under $\Rrightarrow^k$; or $\Rrightarrow^{k'}$ for some k'?

Can we extract suitable *isomorphism tests* from other approximations of homomorphism given by algebraic constraint satisfaction algorithms?

Conversely, what homomorphism approximations do we get from group-theoretic methods for testing isomorphism?