Logic of Local Inference for Contextuality and Paradoxes

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Based primarily on arXiv:1502.03097 and arXiv:1605.08949
(with S. Abramsky, R. Barbosa,
R. Lal, and S. Mansfield)

Workshop on Compositionality
Simons Institute
Dec. 8, 2016

Contextuality? Why Should We Care?

- It is a distinctively non-classical feature of QM.
- It is probably a key resource for quantum computation, as suggested by recent examples:
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Outline

1. Topological approach (Abramsky and Brandenburger 2011, etc.): Contextuality = "global inconsistency" + "local consistency".
2. Gives a logical method unifying existent contextuality proofs; but local consistency is missing from the picture.
3. Novel type of logic, and semantics with completeness.

What is Contextuality?
A Topological Idea

Spaces of variables and of their values.

- measurements and outcomes
- attributes and data values
- sentences and truth values
- questions and answers

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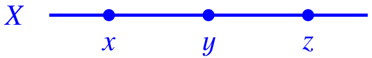

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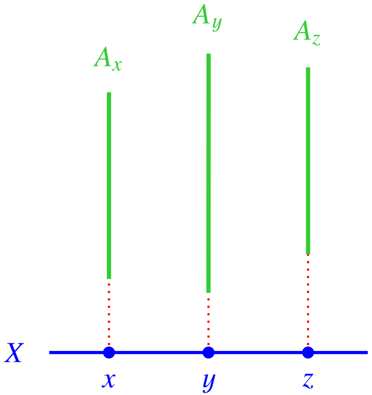

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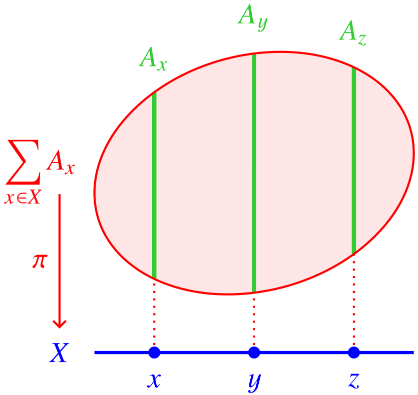
"Bundle" $\sum_{x \in X} A_x$
$= \{ (x, v) \mid x \in X, v \in A_x \}$



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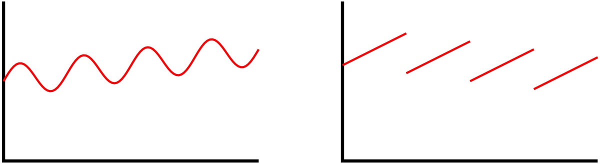

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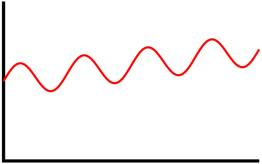
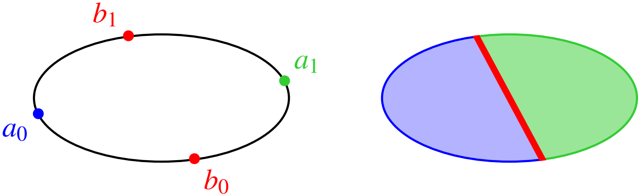


2. How one can move around:



When we ask several questions at once,
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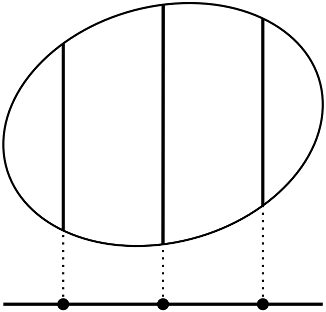

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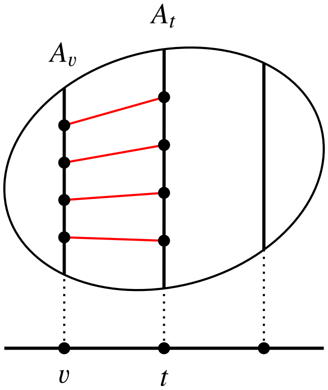

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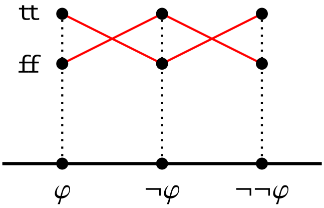

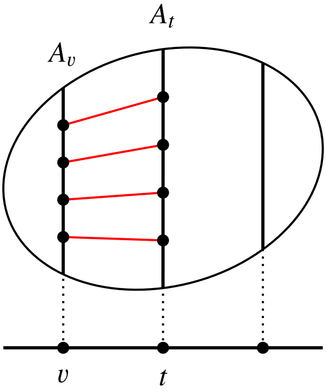

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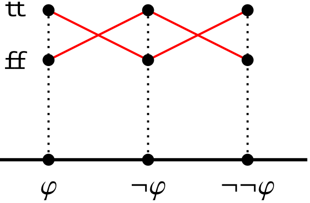

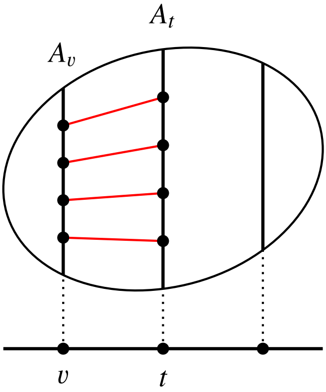

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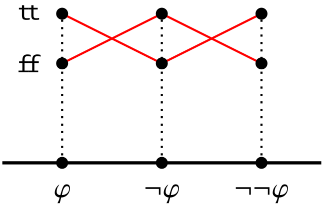

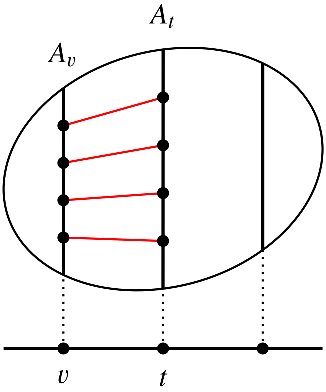

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Models distinguish good and bad ways of connecting dots in bundles
. . . just like continuous sections!

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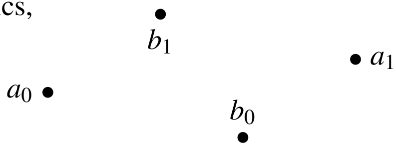

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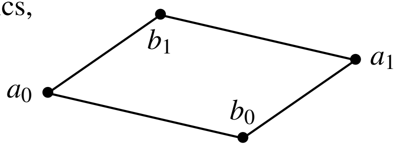

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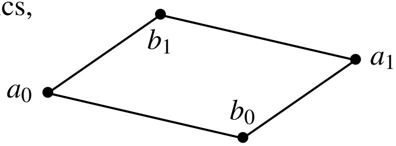

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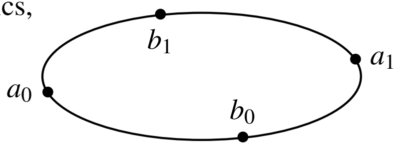

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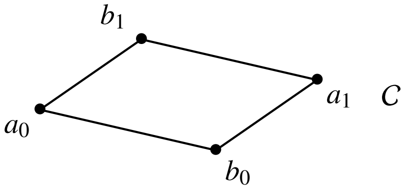

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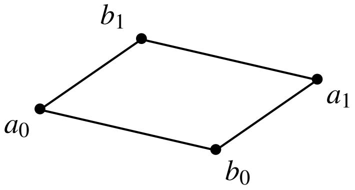

$$\begin{array}{c} \mathbf{Sets} \\ \big\uparrow A \\ C^{\mathrm{op}} \end{array}$$

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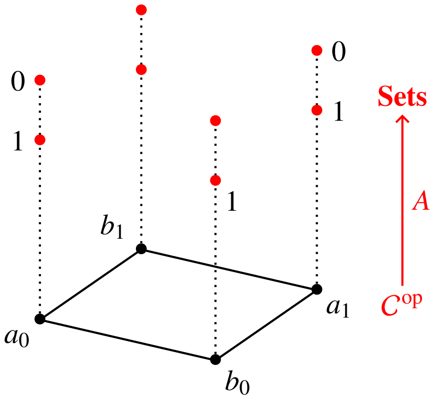

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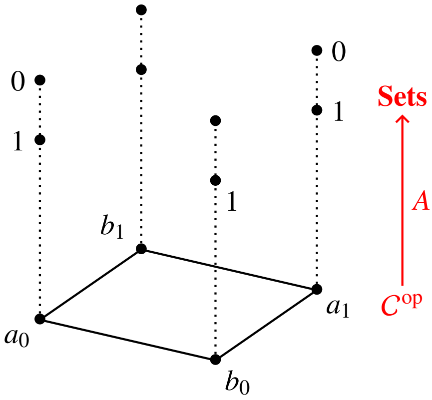

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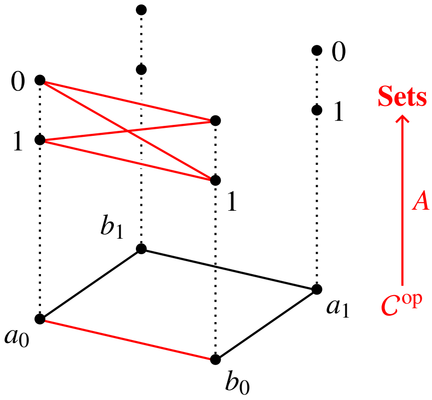

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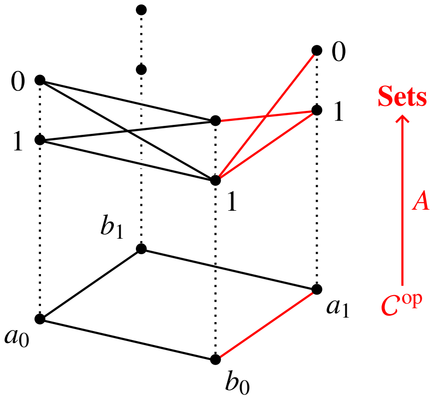

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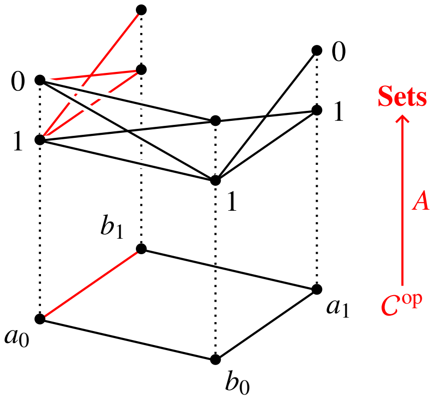

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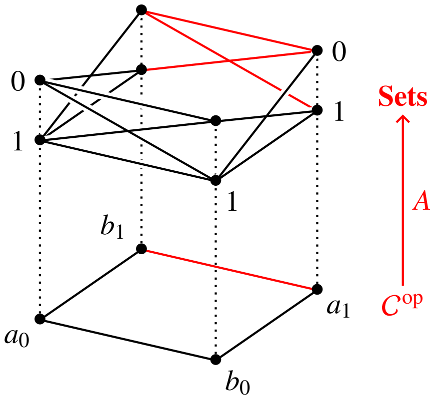

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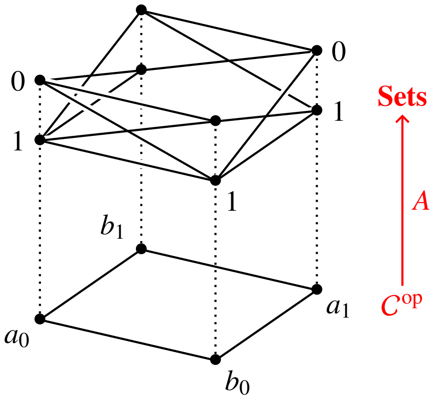

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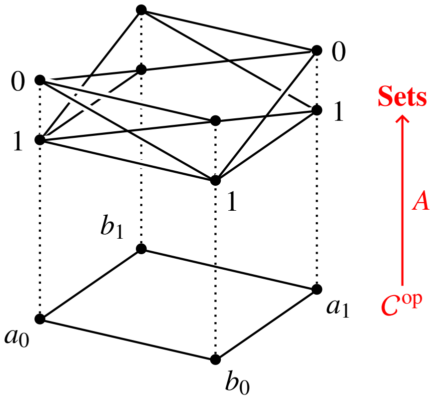

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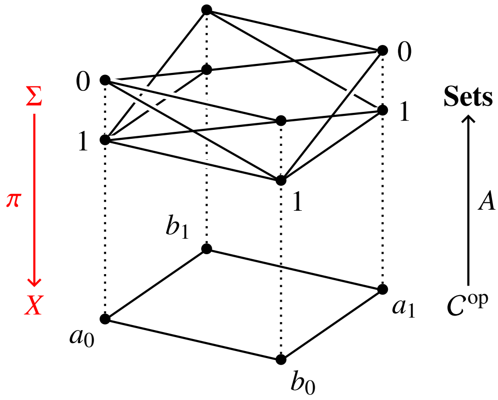

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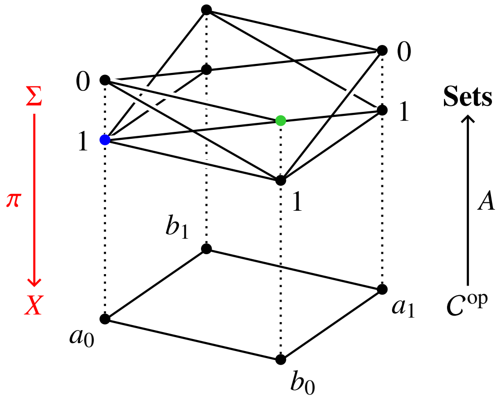

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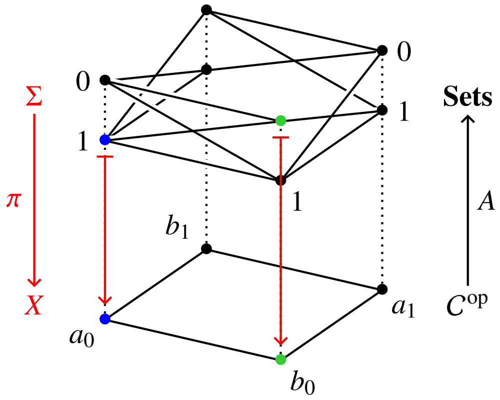

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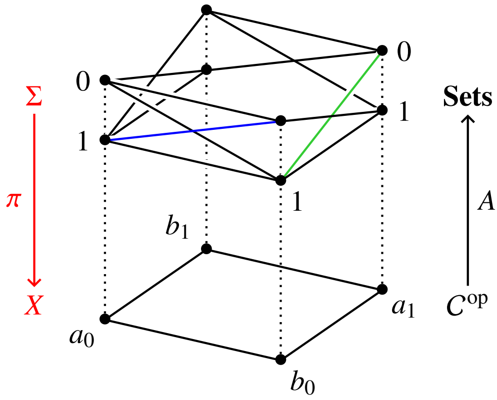

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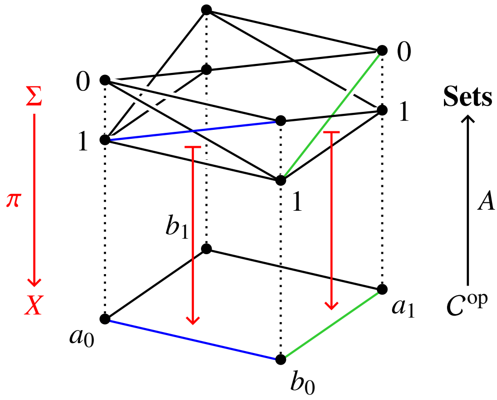

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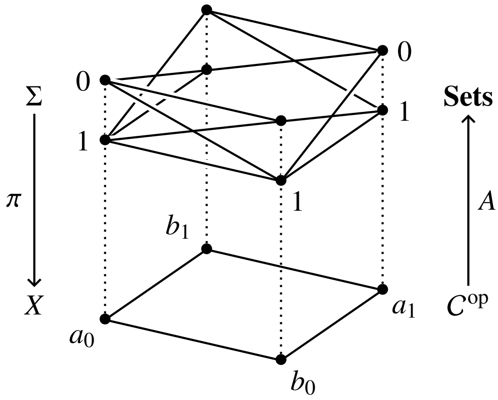

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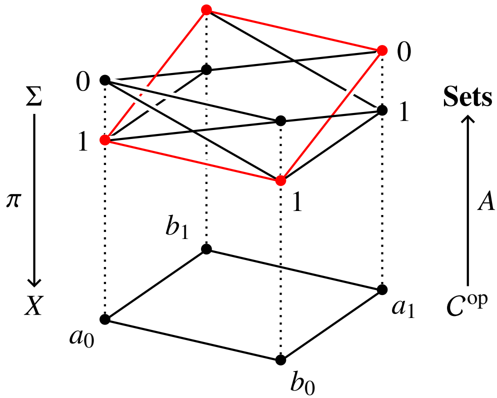

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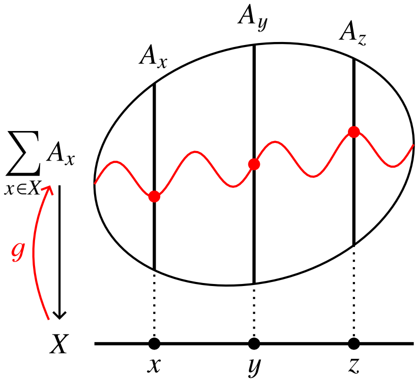
- assigns answers to all the questions, i.e. $\pi \circ g = 1$,
- and satisfies all the constraints, i.e.

$$g{\restriction}_U \in A_U \text{ for all } U \in C.$$



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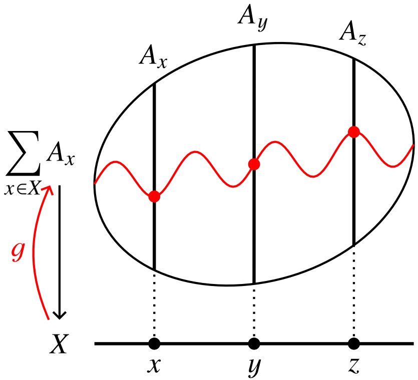

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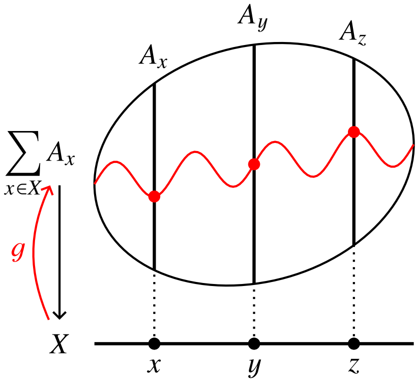

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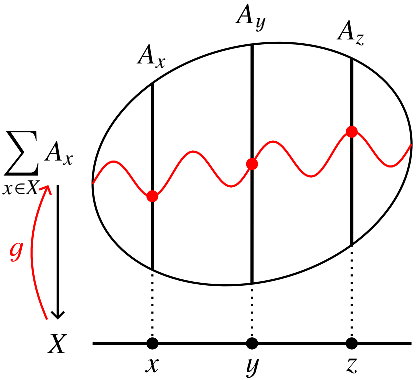


Def. A global section is a function $g : X \to \sum_{x\in X} A_x$ that

- assigns answers to all the questions, i.e. $\pi \circ g = 1$,
- and satisfies all the constraints, i.e.

$$g{\restriction_U} \in A_U \text{ for all } U \in C.$$

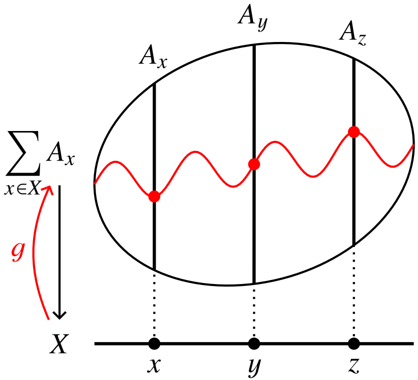
E.g.

- Models of classical logic.

  x is consistent $\iff$ $(x \mapsto \mathtt{tt})$ extends to a global section.

- States of a physical system?

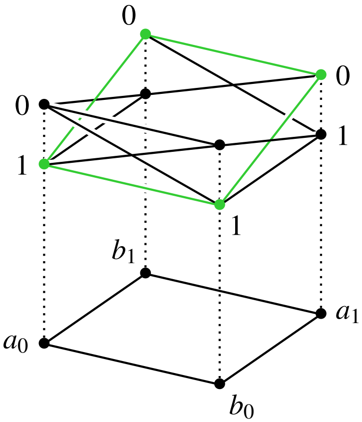
  . . . Classically yes, but no in QM!



Hardy 1993:

Some global sections, e.g.

$(a_0, a_1, b_0, b_1) \mapsto (1, 0, 1, 0)$;



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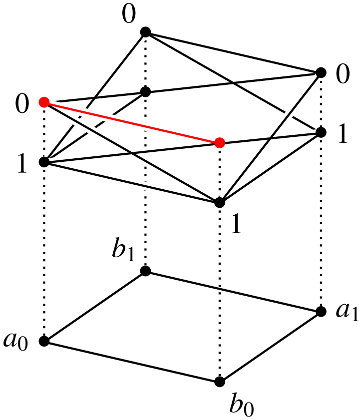

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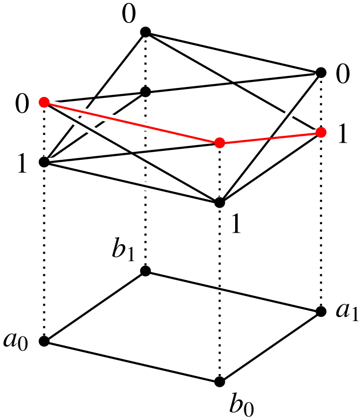

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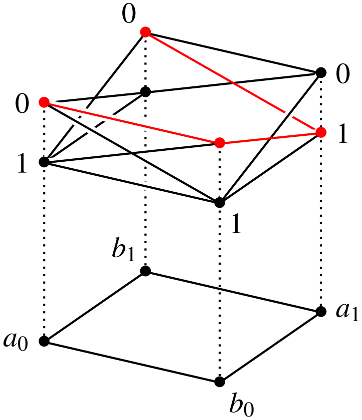

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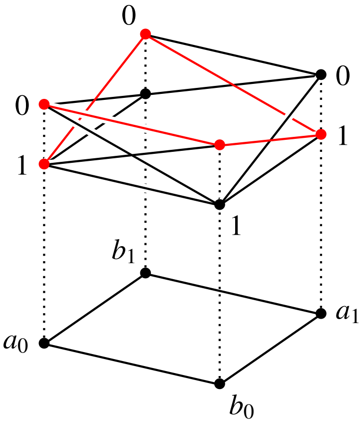

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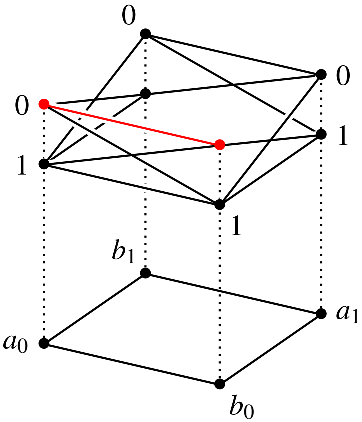

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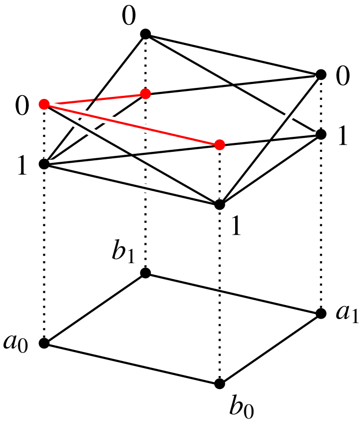

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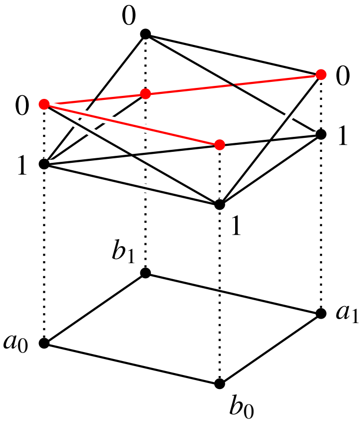

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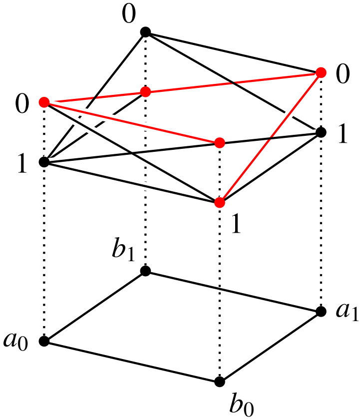

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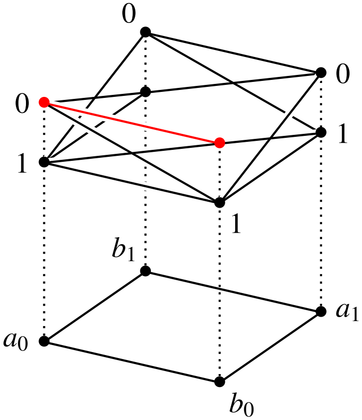

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(Logical) contextuality: Not all sections extend to global ones.

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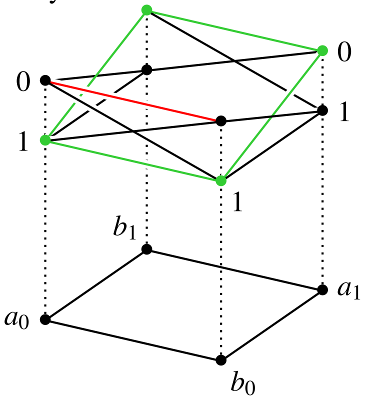
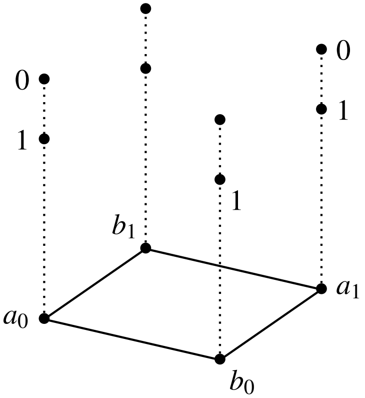


PR box:



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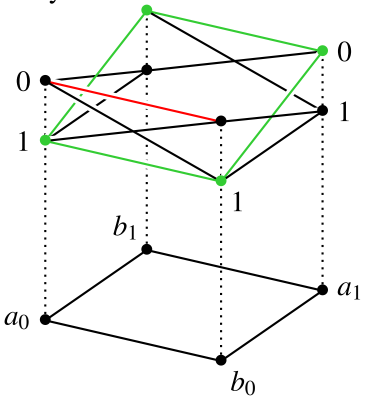

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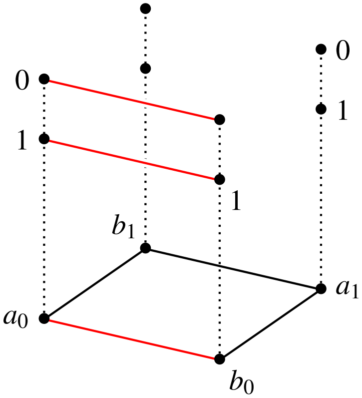

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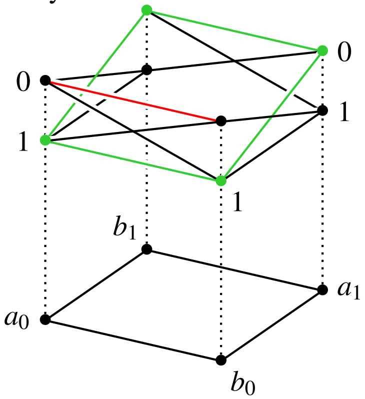

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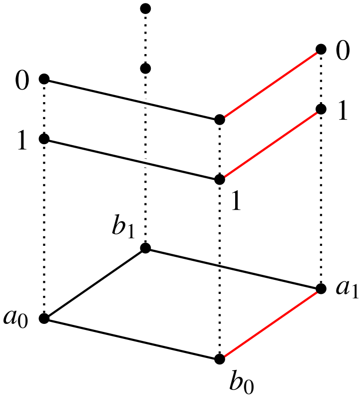

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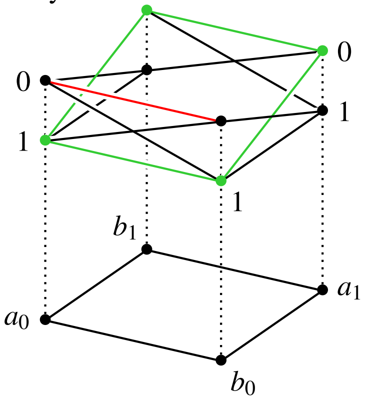

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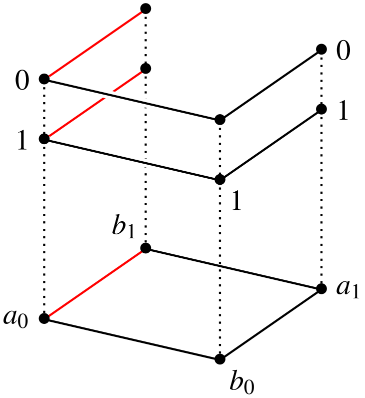

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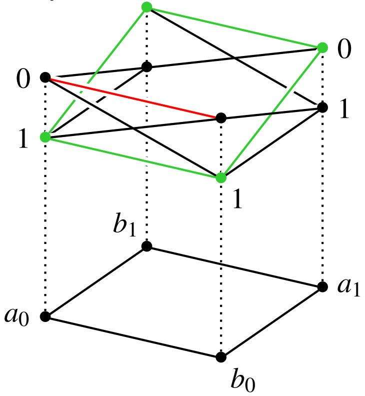

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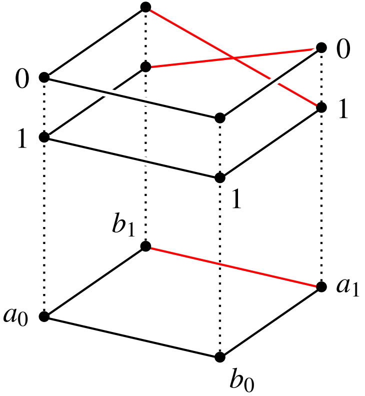

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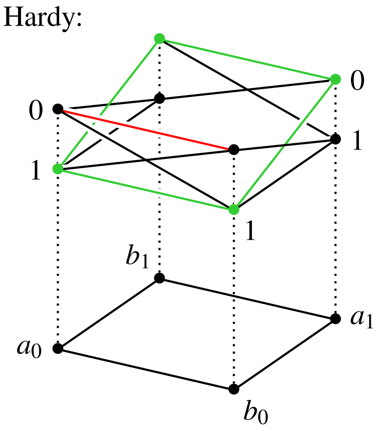

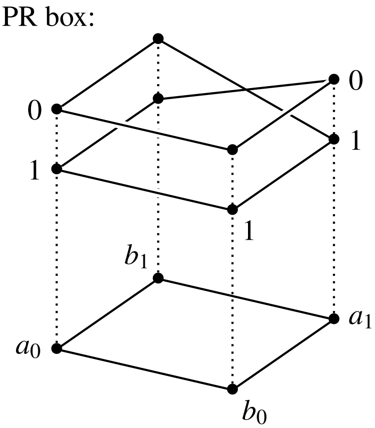

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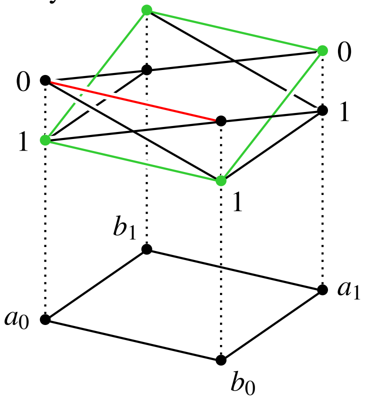

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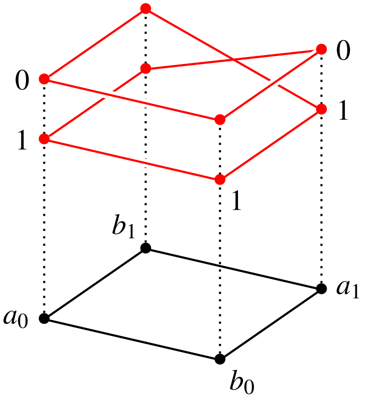

Logical contextuality: Not all sections extend to global ones.

Strong contextuality: No global section at all.

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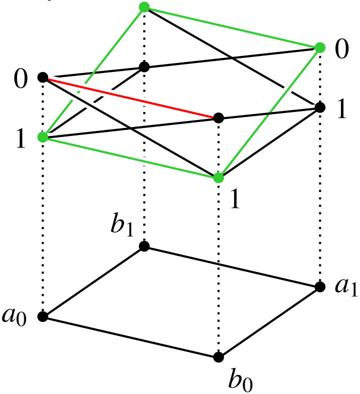

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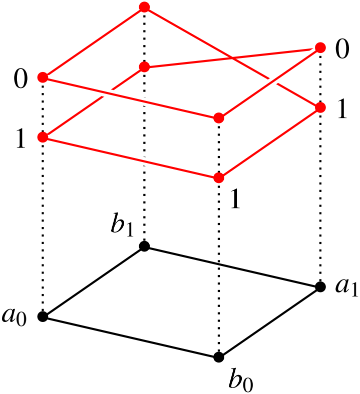

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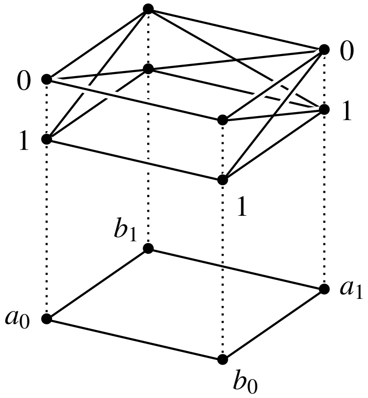
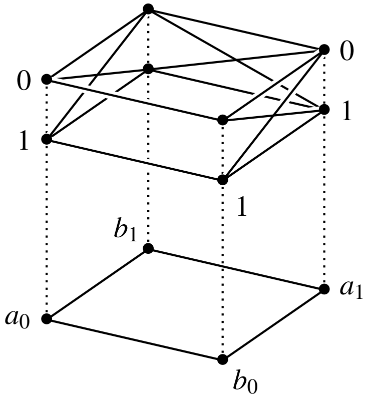
Contextuality = local consistency + global inconsistency

- Non-locality is just a special case of contextuality, involving spatially separated systems.



- Non-locality is just a special case of contextuality, involving spatially separated systems.
- A model A is not logically contextual
  $\iff$ every section extends to a global one



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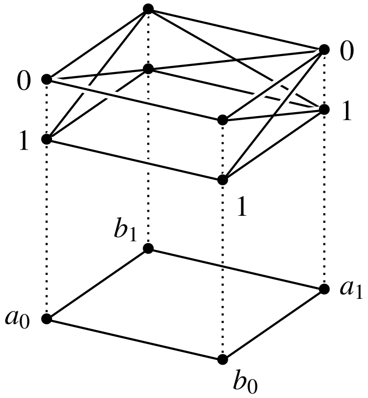

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Probabilistic contextuality / non-locality

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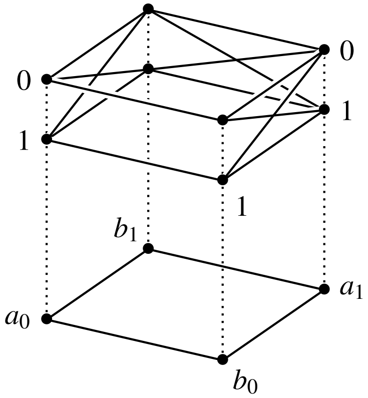

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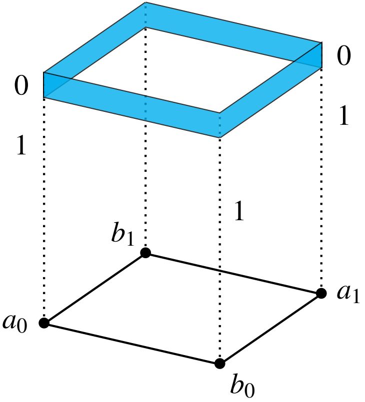

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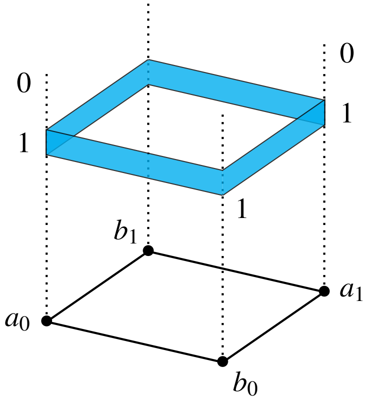

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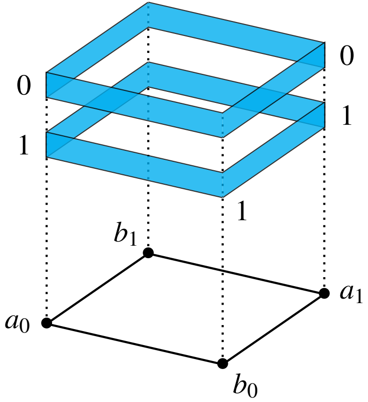

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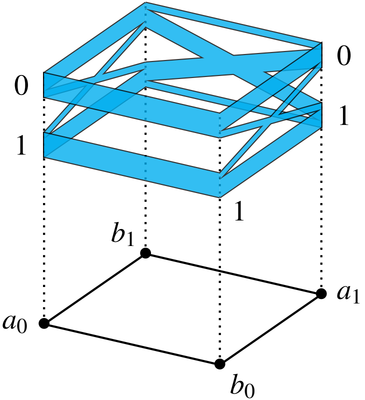
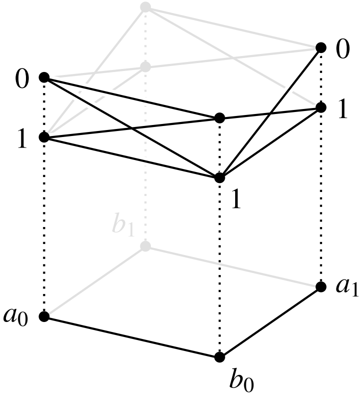


No-Signalling

Even if contextual, a quantum model must satisfy...

Def. We say a separated presheaf A is no-signalling if each restriction $A_{U \subseteq V} : A_V \to A_U$ is onto,



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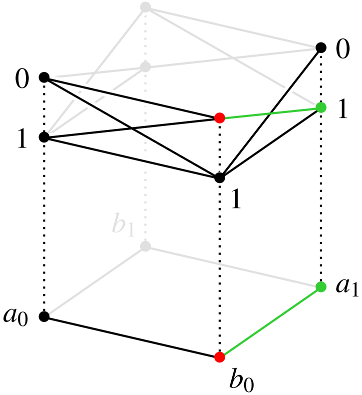

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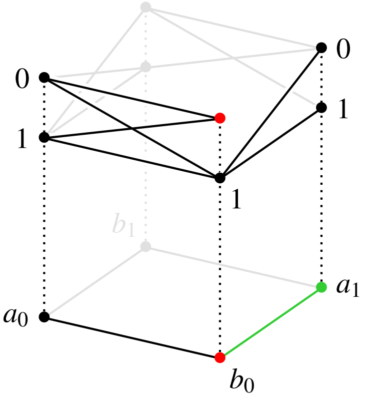

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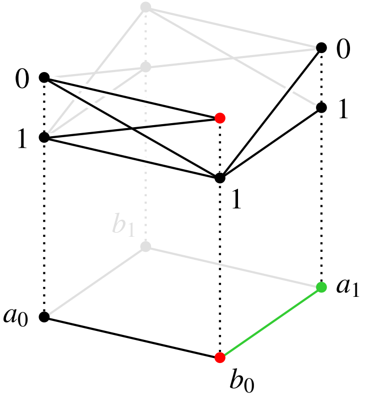
E.g.

- Relativity-ish principle.



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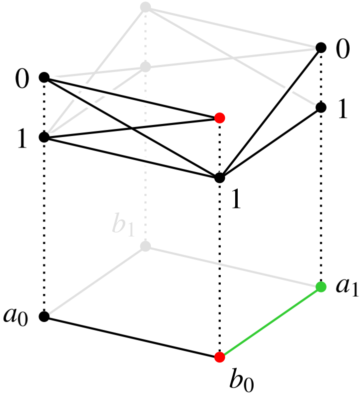

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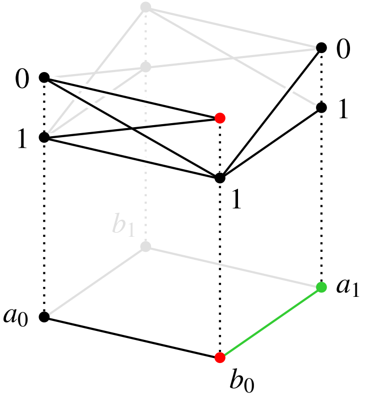
—Part of local consistency!



Contextuality in Relational Databases

(Abramsky 2012/13, arXiv:1208.6416)

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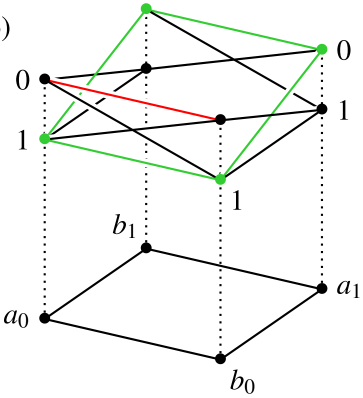
$$A_U \subseteq \prod_{x \in U} A_x$$

to each context $U \in C$.



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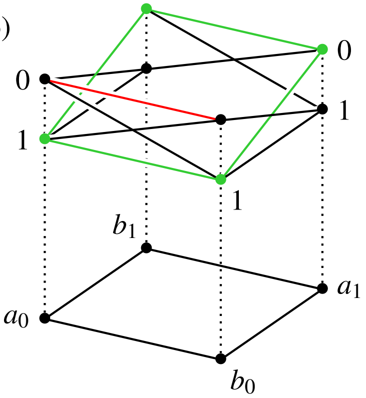

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- The natural join

$$\bowtie_{U \in C} A_U = \{ g \in \prod_{x \in X} A_x \mid g{\restriction}_U \in A_U \text{ for all } U \in C \}$$

of the relations A_U is, by definition, the set of global sections.

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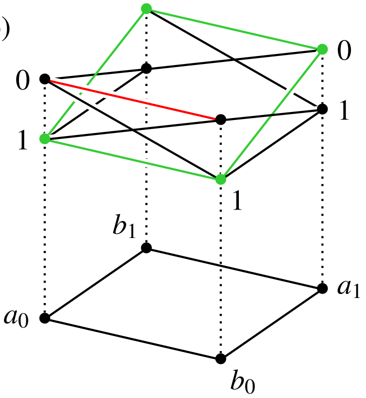

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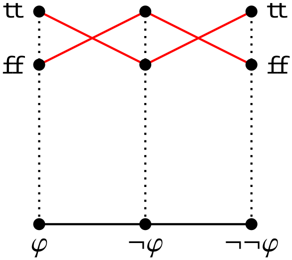
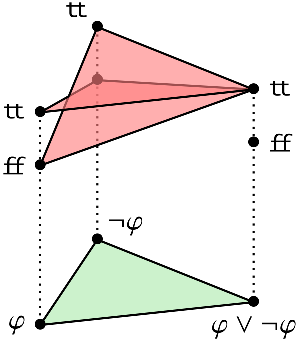
of the relations A_U is, by definition, the set of global sections.
- Contextuality amounts exactly to the absence of universal relations.

Contextuality and Logical Paradoxes

Topology is given along the composition of sentences:





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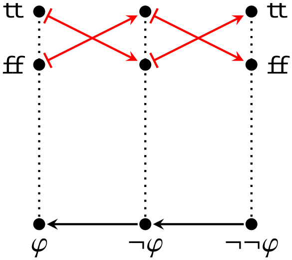

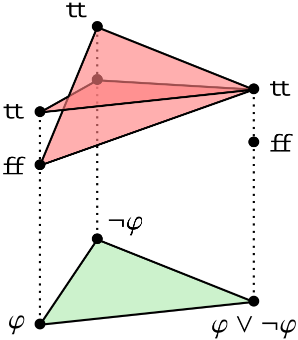

N.B. Constraints are functions (direction opposite to parsing).

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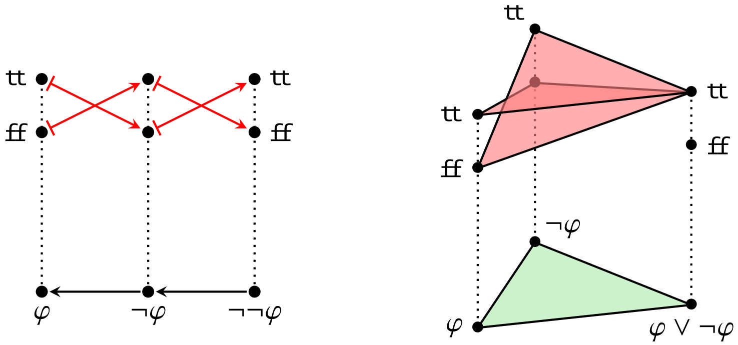
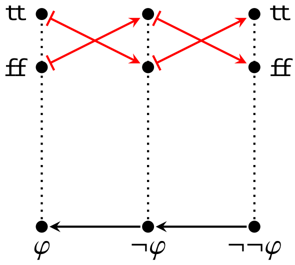
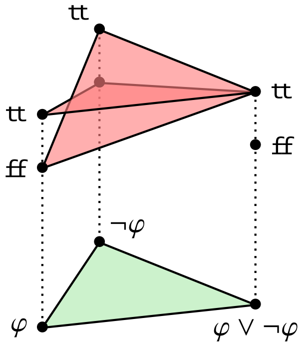


N.B. Constraints are functions (direction opposite to parsing).
So, normally,
assignments are globally defined by induction.

Contextuality and Logical Paradoxes

Topology is given along the composition of sentences:





N.B. Constraints are functions (direction opposite to parsing).
So, normally, i.e., when parsing is well-founded,
assignments are globally defined by induction.

Well-founded parsing can sometimes fail, since
sentences may refer to other sentences, or even to themselves:

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$$\begin{tikzcd}
\sigma_0 \arrow[d, equal] & \sigma_1 \arrow[d, equal] & \lambda \arrow[d, equal] \\
\mathrm{Tr}(\sigma_1) \arrow[d] & \neg\mathrm{Tr}(\sigma_0) \arrow[d] & \neg\mathrm{Tr}(\lambda) \arrow[d] \\
\sigma_1 & \mathrm{Tr}(\sigma_0) \arrow[d] & \mathrm{Tr}(\lambda) \arrow[d] \\
 & \sigma_0 & \lambda
\end{tikzcd}$$

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\sigma_1 \arrow[ur, blue] & \mathrm{Tr}(\sigma_0) \arrow[d] & \mathrm{Tr}(\lambda) \arrow[d] \\
 & \sigma_0 \arrow[uul, blue] & \lambda \arrow[uu, blue, bend left]
\end{tikzcd}
$$

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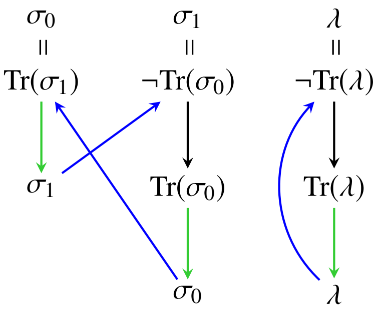

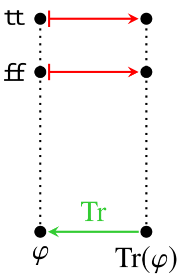

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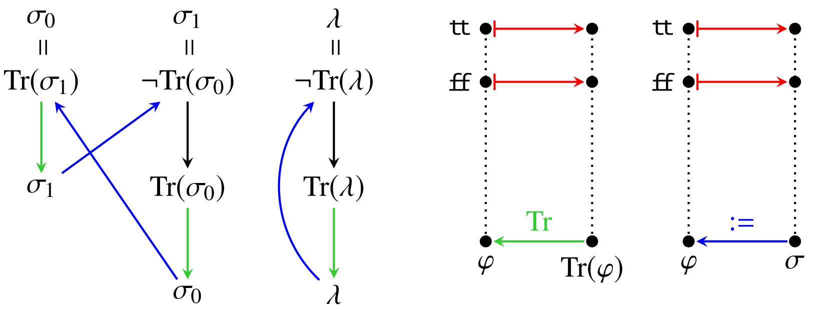

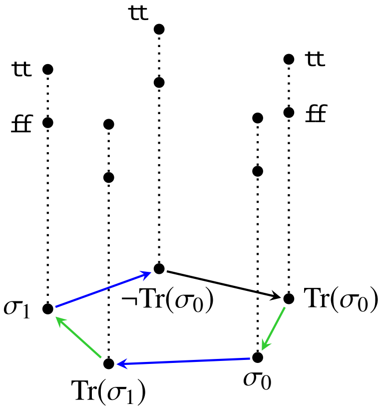
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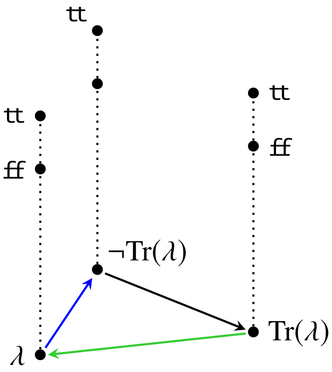
σ_1
$\neg\mathrm{Tr}(\sigma_0)$
$\mathrm{Tr}(\sigma_0)$
$\mathrm{Tr}(\sigma_1)$
σ_0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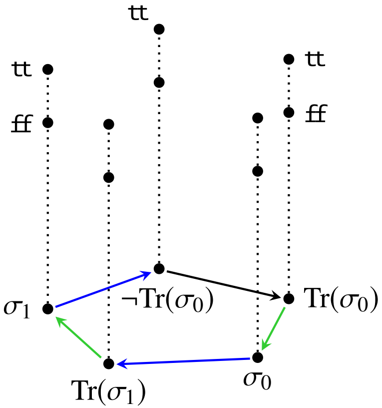
$\neg\mathrm{Tr}(\lambda)$
$\mathrm{Tr}(\lambda)$
λ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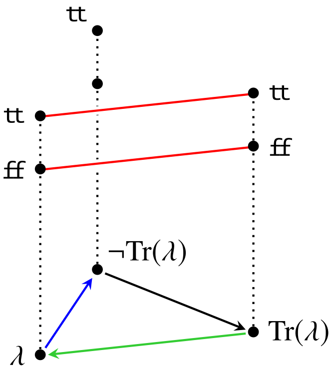
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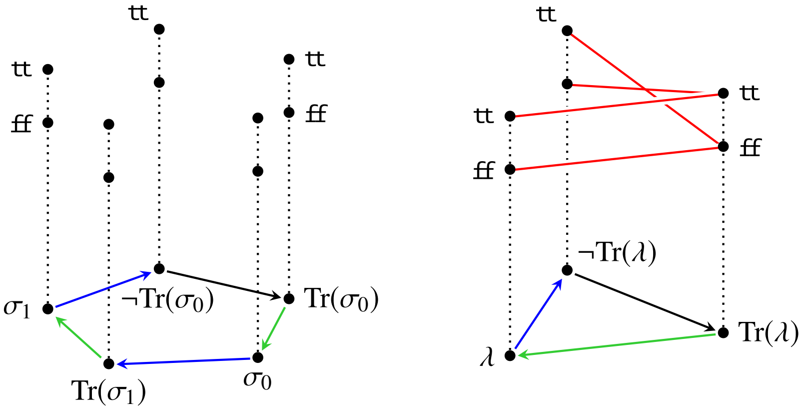
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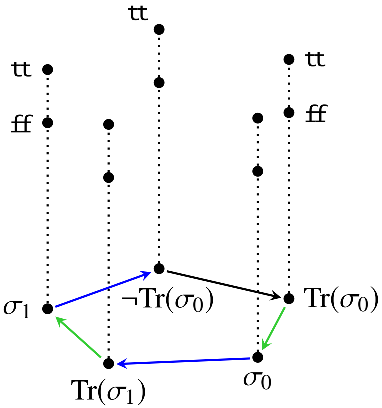
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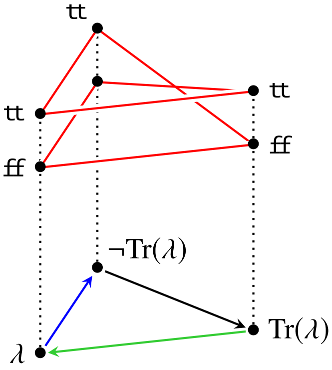
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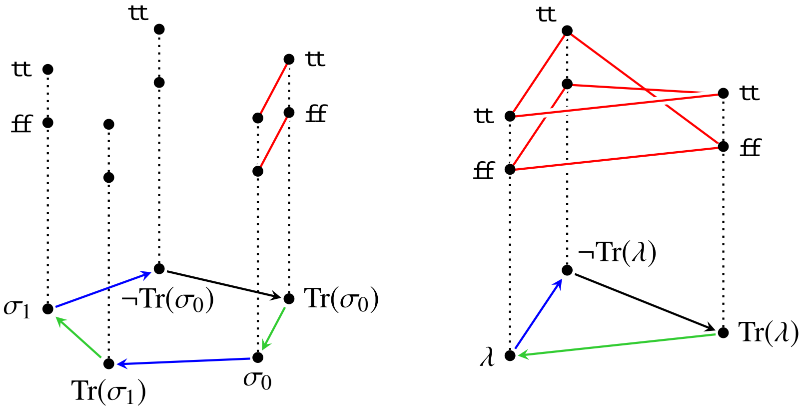
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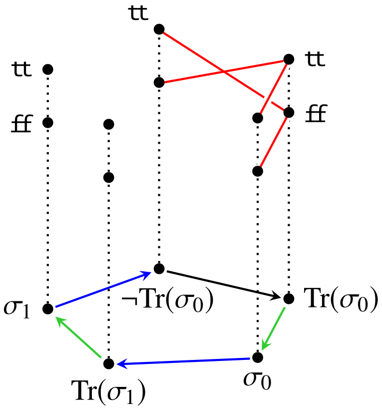

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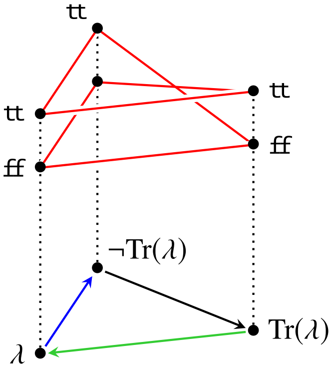

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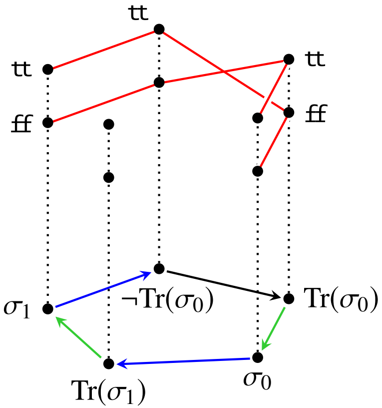
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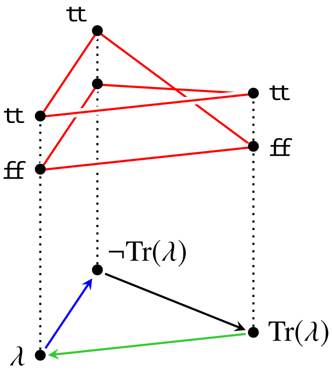
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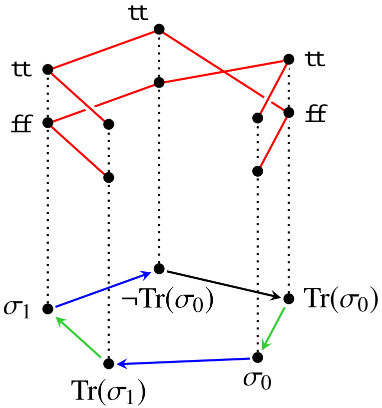
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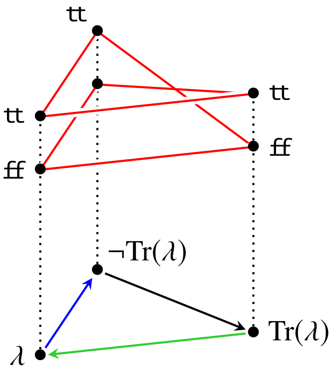
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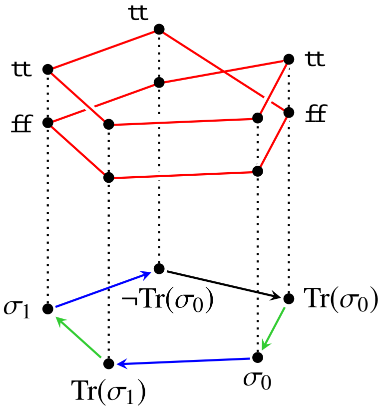
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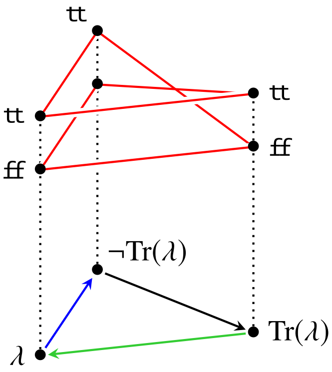
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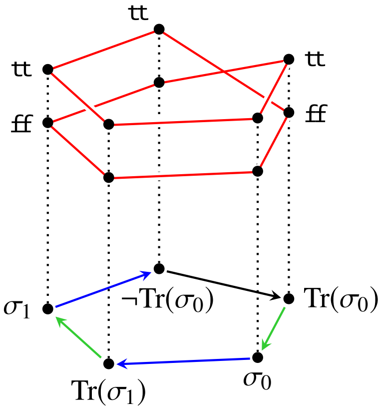
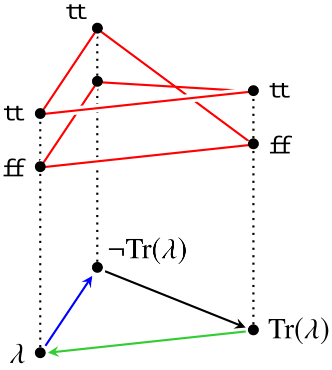
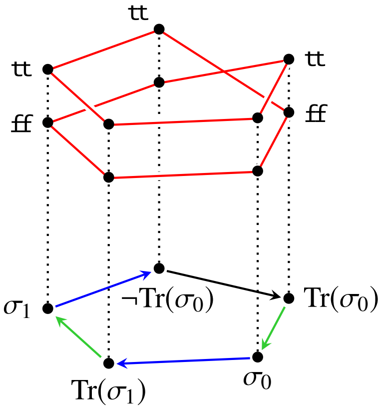
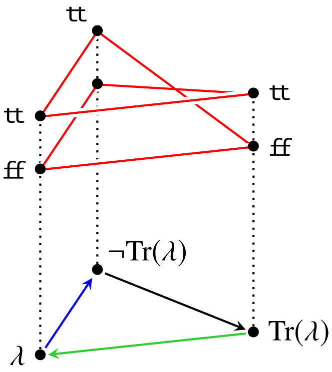




- The paradoxes have the same topology as the PR box!





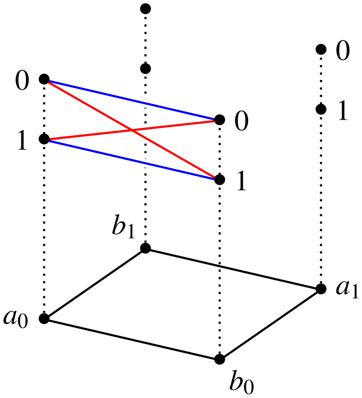
- The paradoxes have the same topology as the PR box!
- This leads to a new semantics for languages with non-well-founded parsing.

Contextuality Argument

Contextuality Argument

Joint outcomes may / may not satisfy certain properties, e.g.:

$(0,0) \models x \oplus y = 0$
$(0,1) \models x \oplus y = 1$
$(1,0) \models x \oplus y = 1$
$(1,1) \models x \oplus y = 0$



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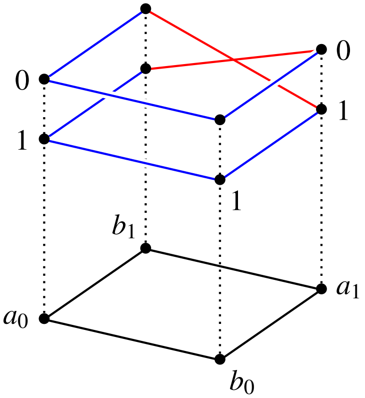

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$a_0 \oplus b_0 = 0$
$a_0 \oplus b_1 = 0$
$a_1 \oplus b_0 = 0$
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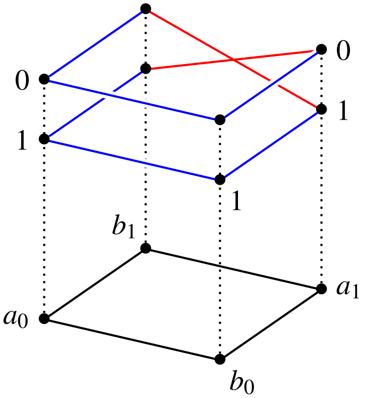
$\bigoplus$ LHS's = $\bigoplus$ RHS's



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$a_0 \oplus b_0 = 0$

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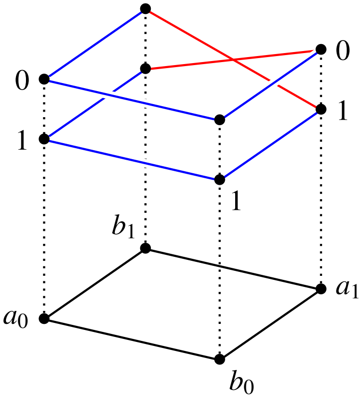
$\bigoplus$ LHS's $\neq \bigoplus$ RHS's

The equations are inconsistent,



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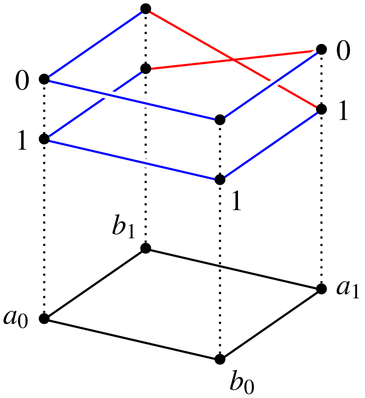


The equations are inconsistent,
i.e. no global assignment consistent with the constraints,
i.e. strongly contextual!

This method subsumes
“all vs nothing” arguments in the QM literature:

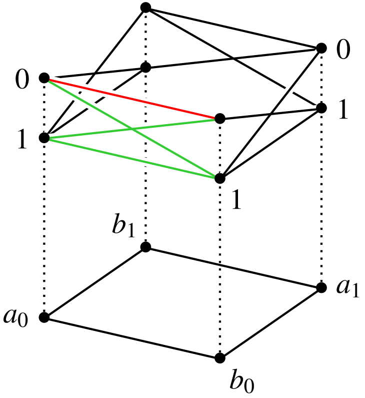
- GHZ state:
$$\begin{aligned} a_0 \oplus b_0 \oplus c_0 &= 0 \\ a_0 \oplus b_1 \oplus c_1 &= 1 \\ a_1 \oplus b_0 \oplus c_1 &= 1 \\ a_1 \oplus b_1 \oplus c_0 &= 1 \end{aligned}$$
$$\bigoplus \text{LHS's} = 0 \neq 1 = \bigoplus \text{RHS's}$$

- Kochen-Specker-type:

  18 variables, each occurs twice, so $\bigoplus$ LHS’s = 0;
  9 equations, all of parity 1, so $\bigoplus$ RHS’s = 1.

- etc., etc. . . .

- Can use other vocabulary,
- Works for
  logical contextuality, too



- Can use other vocabulary,
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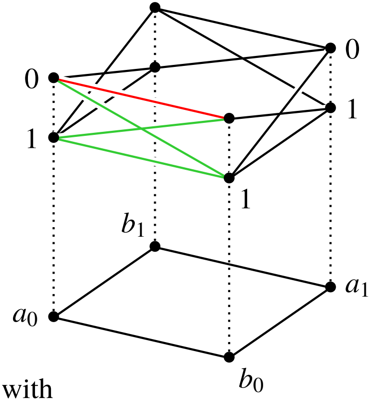
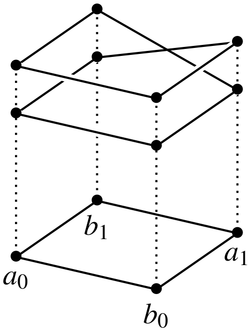
  logical contextuality, too

$$
\begin{array}{rc}
 & a_0 \vee b_1 \\
 & a_1 \vee b_0 \\
 & \neg(a_1 \vee b_1) \\
 & \neg a_0 \wedge \neg b_0 \\
\therefore & \bot
\end{array}
\qquad\qquad
\begin{array}{rc}
 & a_0 \vee b_1 \\
 & a_1 \vee b_0 \\
 & \neg(a_1 \vee b_1) \\
\therefore & a_0 \vee b_0
\end{array}
$$



No global assignment (consistent with the other constrants) satisfies $\neg a_0 \wedge \neg b_0$, i.e. logically contextual!

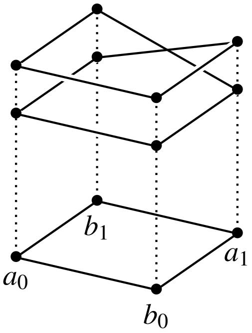
Logic of Contextualiy?



$$
\begin{array}{lrcl}
 & a_0 \oplus b_0 & = & 0 \\
 & a_0 \oplus b_1 & = & 0 \\
 & a_1 \oplus b_0 & = & 0 \\
 & a_1 \oplus b_1 & = & 1 \\
\therefore & & \bot &
\end{array}
$$

$$\Gamma \vdash \bot$$

Logic of Contextualiy?

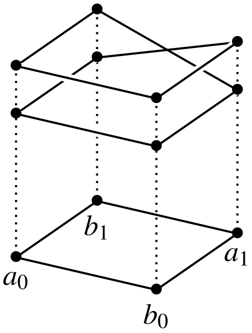


$$
\begin{array}{rl}
 & a_0 \oplus b_0 = 0 \\
 & a_0 \oplus b_1 = 0 \\
 & a_1 \oplus b_0 = 0 \\
 & a_1 \oplus b_1 = 1 \\
\therefore & \quad \bot
\end{array}
$$

$\Gamma \vdash \bot$ does NOT mean

"no model satisfies Γ",

Logic of Contextualiy?



$\Gamma \vdash \varphi$

$$
\begin{aligned}
& a_0 \oplus b_0 = 0 \\
& a_0 \oplus b_1 = 0 \\
& a_1 \oplus b_0 = 0 \\
\therefore\ & a_1 \oplus b_1 = 0
\end{aligned}
$$

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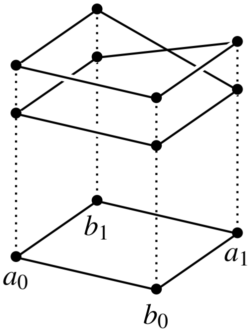

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& a_0 \oplus b_0 = 0 \\
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\therefore\ & a_1 \oplus b_1 = 0
\end{aligned}
$$

$\Gamma \vdash \bot$ does NOT mean
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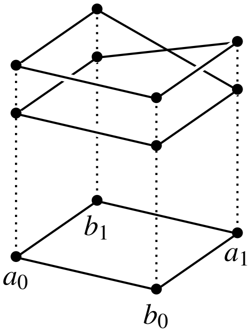
$\Gamma \vdash \varphi$ does NOT mean "models satisfying Γ also satisfy φ",

Logic of Contextualiy?



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\begin{aligned}
a_0 \oplus b_0 &= 0 \\
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a_1 \oplus b_0 &= 0 \\
\therefore\ a_1 \oplus b_1 &= 0
\end{aligned}
$$

$\Gamma \vdash \perp$ does NOT mean
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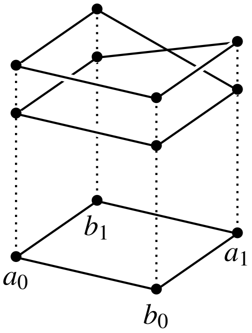
All-vs-nothing argument is NOT sound w.r.t. contextual models.

Logic of Contextualiy?



$$
\begin{aligned}
a_0 \oplus b_0 &= 0 \\
a_0 \oplus b_1 &= 0 \\
a_1 \oplus b_0 &= 0 \\
\therefore a_1 \oplus b_1 &= 0
\end{aligned}
$$

$\Gamma \vdash \perp$ does NOT mean
"no model satisfies Γ",
but "no global section satisfies Γ".

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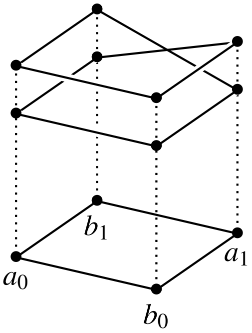

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a_0 \oplus b_0 &= 0 \\
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a_1 \oplus b_0 &= 0 \\
\therefore a_1 \oplus b_1 &= 0
\end{aligned}
$$

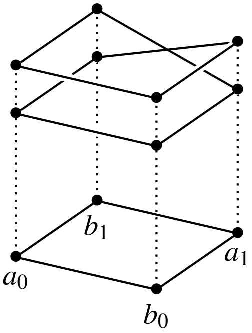
$\Gamma \vdash \bot$ does NOT mean "no model satisfies Γ", but "no global section satisfies Γ".

$\Gamma \vdash \varphi$ does NOT mean "models satisfying Γ also satisfy φ", but "global sections satisfying Γ also satisfy φ".

All-vs-nothing argument is NOT sound w.r.t. contextual models.

—"Global logic" of global sections

Logic of Contextualiy?



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a_1 \oplus b_0 &= 0 \\
\therefore a_1 \oplus b_1 &= 0
\end{aligned}
$$

$\Gamma \vdash \perp$ does NOT mean
"no model satisfies Γ",
but "no global section satisfies Γ".

$\Gamma \vdash \varphi$ does NOT mean "models satisfying Γ also satisfy φ",
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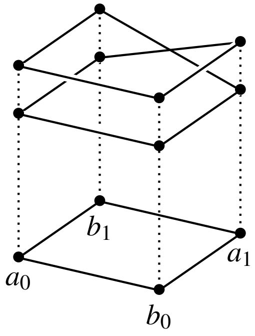
All-vs-nothing argument is NOT sound w.r.t. contextual models.

—"Global logic" of global sections vs local (in)consistency.

Logic of Contextualiy?



$$\begin{array}{rcl} a_0 \oplus b_0 &=& 0 \\ a_0 \oplus b_1 &=& 0 \\ a_1 \oplus b_0 &=& 0 \\ a_1 \oplus b_1 &=& 1 \\ \therefore \quad & \bot & \end{array}$$

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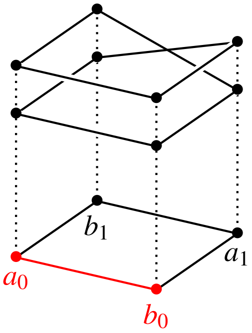

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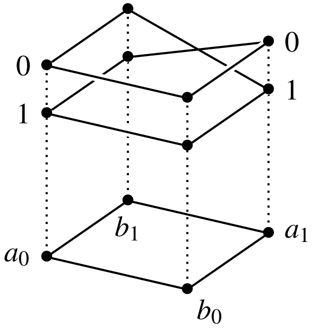

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$$
\begin{array}{l}
a_0 = b_0 \\
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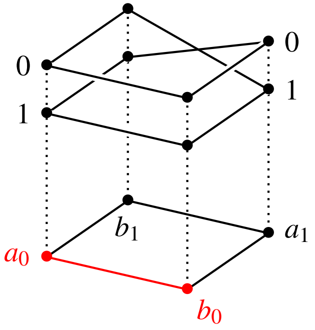

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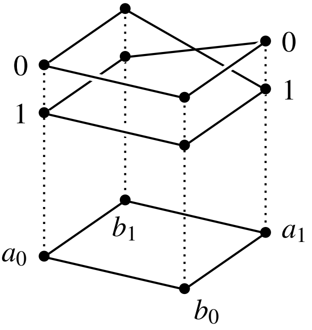

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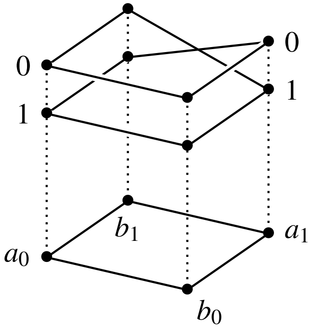

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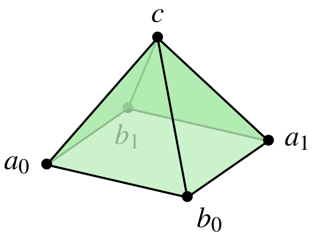

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\qquad\qquad
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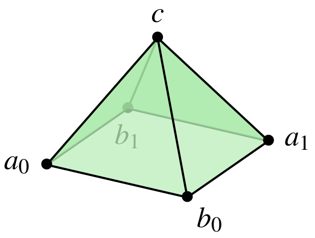

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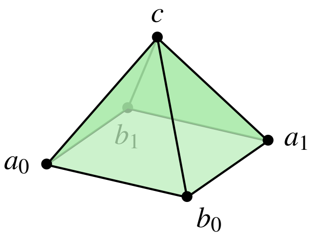

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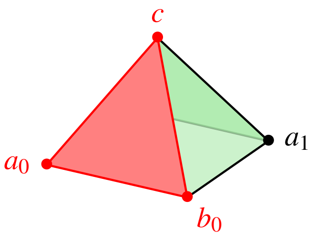

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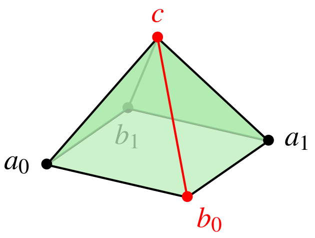

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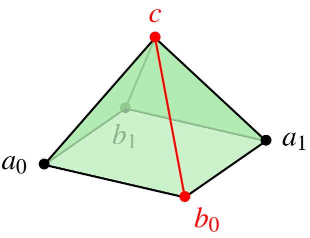
(This step needs no-signalling. . . .)



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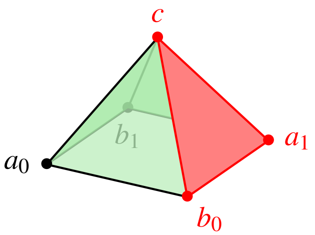

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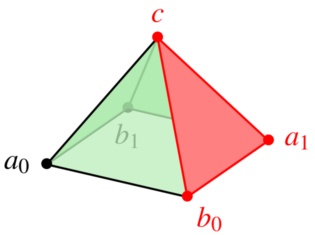

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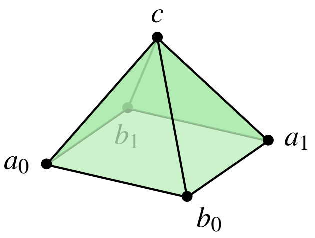

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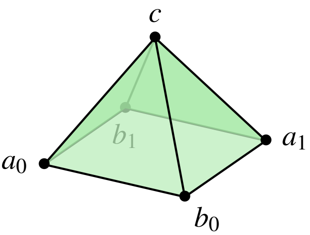
Passing on info locally:



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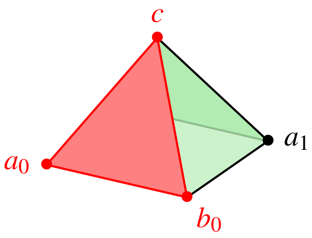

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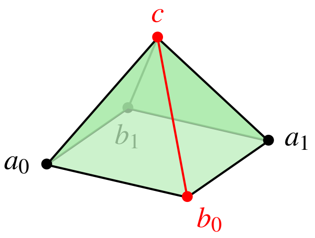

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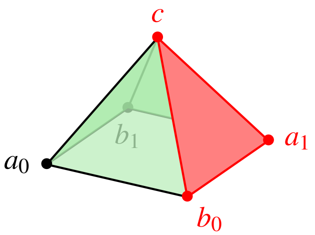

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Inchworm Logic!

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Replace "onto" (used in no-signalling, "$\cdots \neq \varnothing$", etc.) with "regular epi", because:

Fact. In regular **S**, any $D \xrightarrow{f} C$ has

$$\mathrm{Sub}_{\mathbf{S}}(D) \underset{f^{-1}}{\overset{\exists_f}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{Sub}_{\mathbf{S}}(C)$$

Moreover, if f is a regular epi then $\exists_f \circ f^{-1} = 1$.

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Def. Let $\vdash$ be a "global" theory in $\mathcal{L}$.
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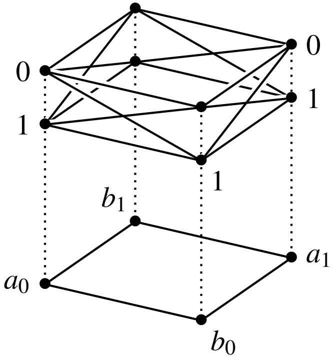

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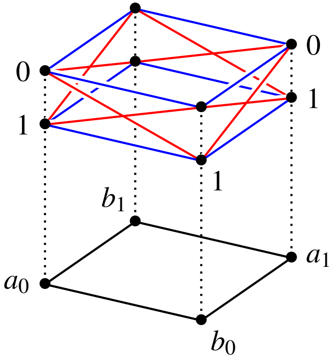

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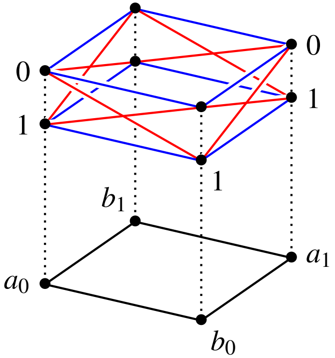

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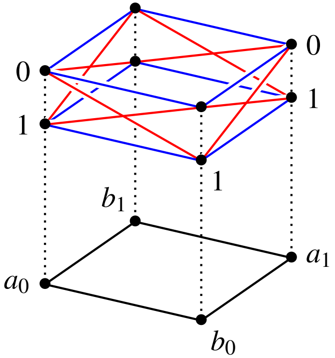

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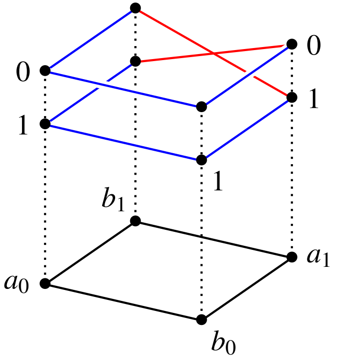

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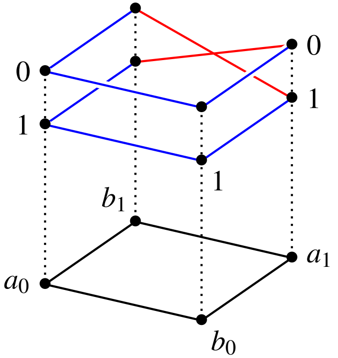

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"Parasite lemma" for transferring completeness.

Suppose $\vdash$ is C-finite, and that $[\![-]\!]$ is a classifying model of $\vdash$
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(A slightly more general semantics gives
the analogous results without C-finiteness assumed.)

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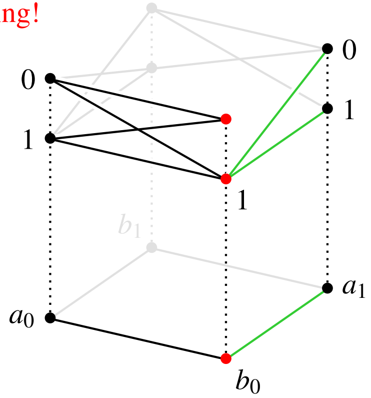

2. No-signalling = no more job for the inchworm.

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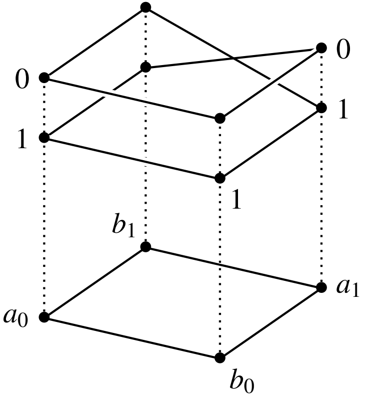

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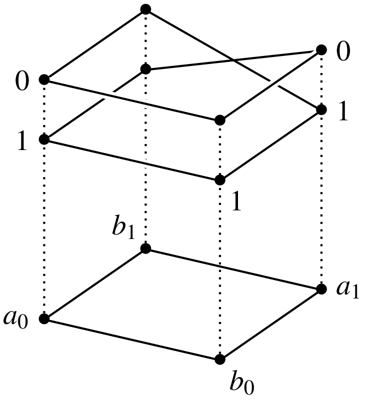

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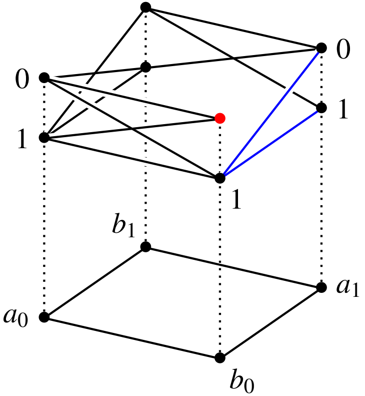

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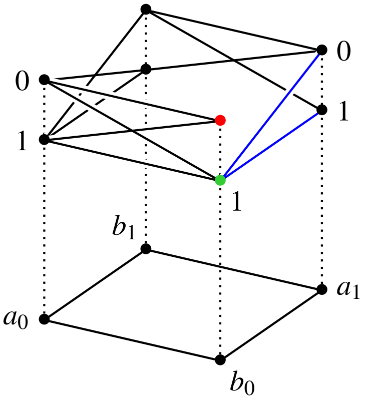

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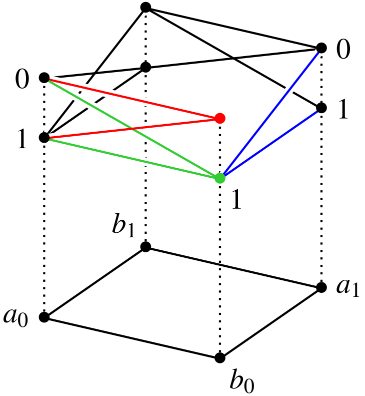

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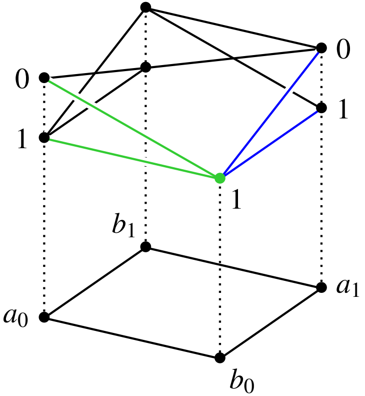

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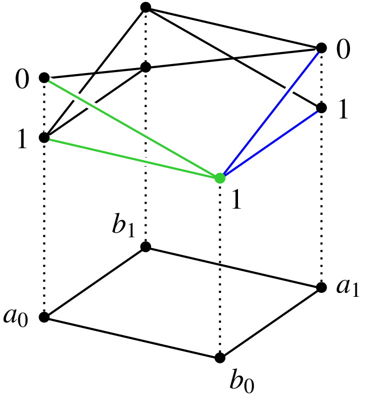


❷ No-signalling = no more job for the inchworm.

Def. Call $\Gamma \subseteq \Phi_C$ "inchworm-saturated" if $\Gamma_V \vdash \varphi$ implies $\Gamma_U \vdash \exists_{V \setminus U}.\, \varphi$.

Fact. If Γ is inchworm-saturated (and if $\Gamma_U = \Gamma \cap \Phi_U$ is finite for every $U \in C$), then the family $(\bigwedge_{\varphi \in \Gamma_U} [\![\varphi]\!]_U)_U$ forms a no-signalling model.

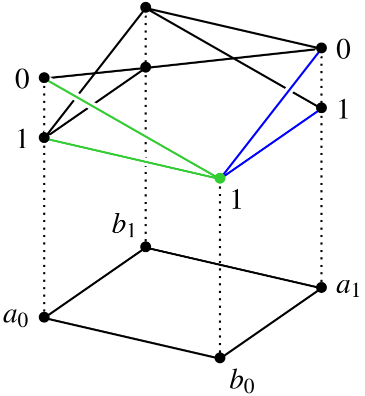
E.g. $a_0 \vee b_1$

$\neg(a_1 \wedge b_1)$

$(a_1 \wedge b_0) \vee (\neg a_1 \wedge b_0)$

$\therefore b_0$

the inchworm carves out a no-signalling model!



This fact is used in the completeness proof.

Summary

1. Topological approach expresses contextuality as "global inconsistency" + "local consistency".
2. It shows contextuality to be isomorphic to phenomena in many other subjects, e.g. relational databases.
3. It gives a powerful logical method of contextuality proof; but this method needs to address the "local consistency" part.
4. Hence the inchworm logic, sensitive to the topology of contexts; (categorical) semantics with soundness and completeness.

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Future Work and Directions

- Complexity and algorithms for inchworm satisfiability.
- Apply inchworm logic to other subjects.
- Import methods from other subjects to contextuality.
- Take advantage of the generality of regular categories.