A topological approach to compositionality in complex systems

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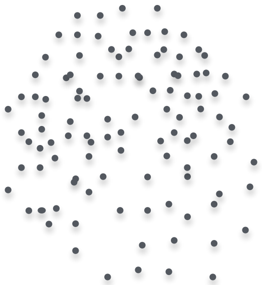
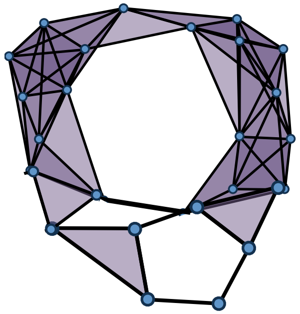
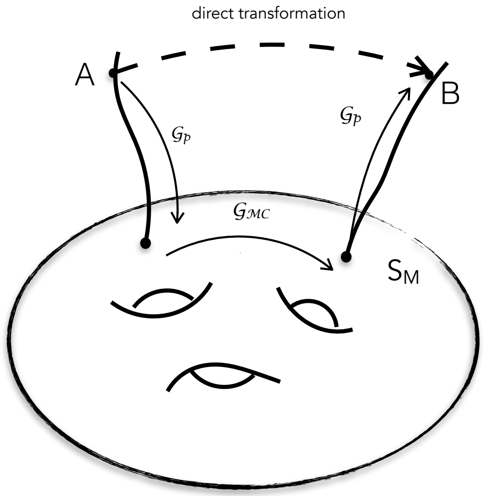
Workshop on Compositionality at Simons Institute, 8th December 2016



data space



simplicial complex
base space



fiber bundle + field action
relation patterns
topological data field

COMPLEX SYSTEMS

Complex systems are composed of many non-identical elements, **entangled in loops of nonlinear interactions**, and characterized by the characteristic 'emergence' behaviours.

M. Rasetti, E. Merelli Topological field theory of data: mining data beyond complex networks, Cambridge University Press, 2016

topological data field theory

The TFTD is based on

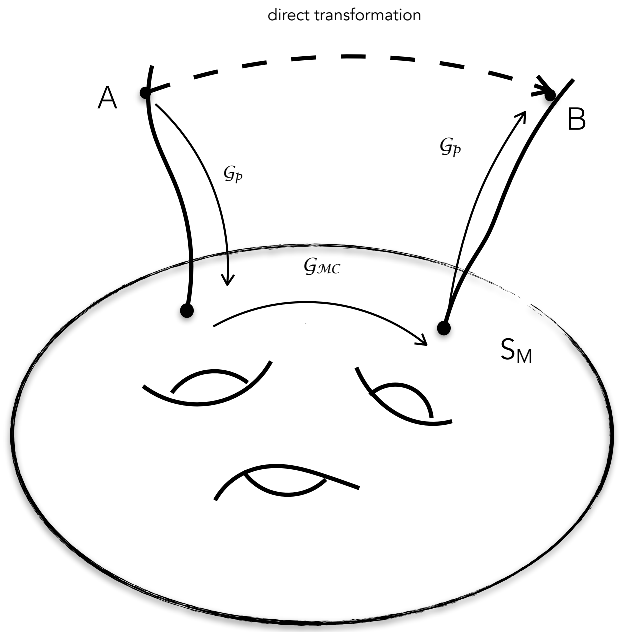
1. embedding data space into a combinatorial topological object, a **simplicial complex**;

2. considering the **complex as base space** of a **(block) fiber bundle**

3. assuming **a field action** (which has a free part, the combinatorial Laplacian over the simplicial complex, and an interaction part depending on the process algebra)

4. constructing the **gauge group** (semi-direct product of the group generated by the algebra of processes (the fibers) and the group of (simplecio-morphisms modulo isotopy) of the data space.

Emergent features of data-represented complex systems were shown to be expressed by the correlation functions of the field theory."



fiber bundle + field action

relation patterns

topological data field

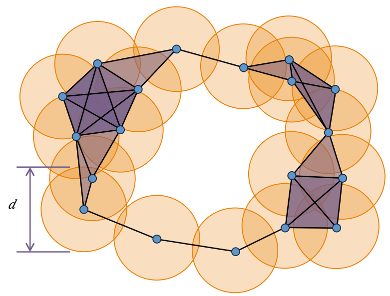
M. Rasetti, E. Merelli Topological field theory of data: mining data beyond complex networks, Cambridge University Press, 2016

Persistent Homology

Persistent homology is an algebraic method for discerning topological features of data

e.g. components, graph structure holes

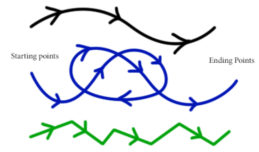
set of discreet points



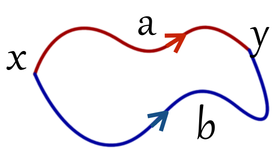
movie by Matthew L. Wright

A graph captures connectivity, but ignores higher-order features, such as holes.

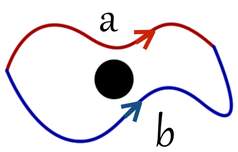
Topological Invariants



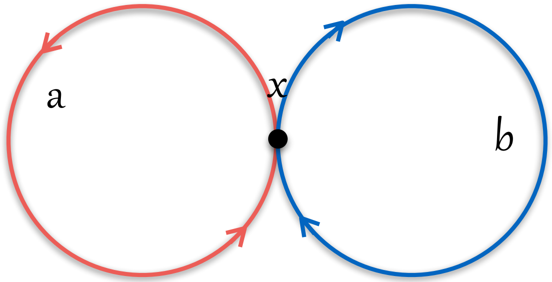
the language of paths



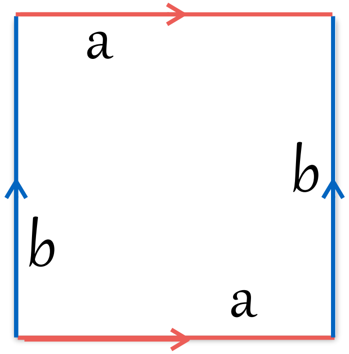
$a \simeq b$



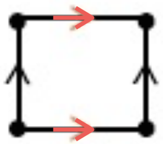
$a \neq b$

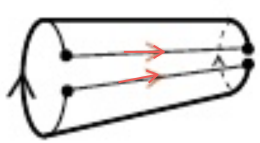


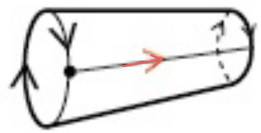
$ab \neq ba$



1. a path in a space S is a continuos map
2. **homotopy** is an equivalence relation on paths in space S

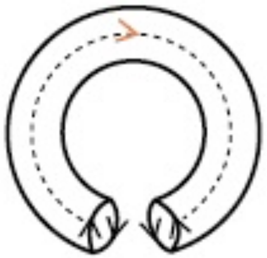


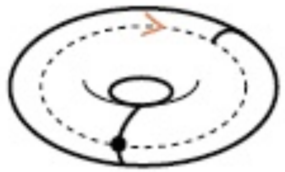




$G = \langle a, b \mid aba^{-1}b^{-1} \rangle$

$ab \simeq ba$

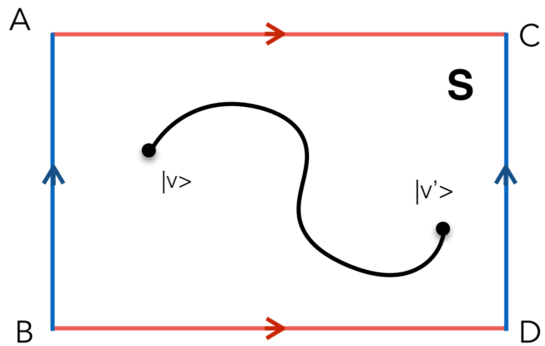


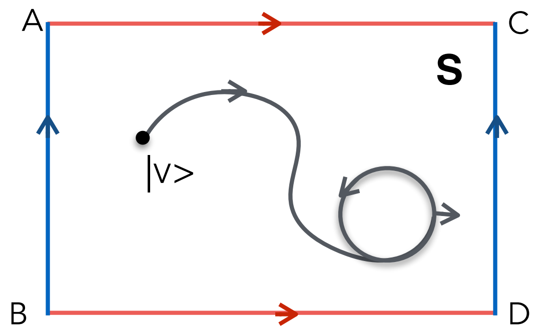


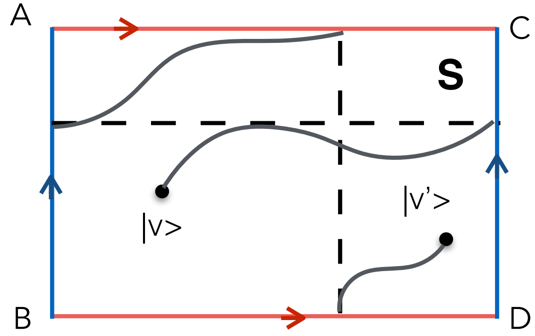
Each equivalence class of paths might be called *behavior* of its members

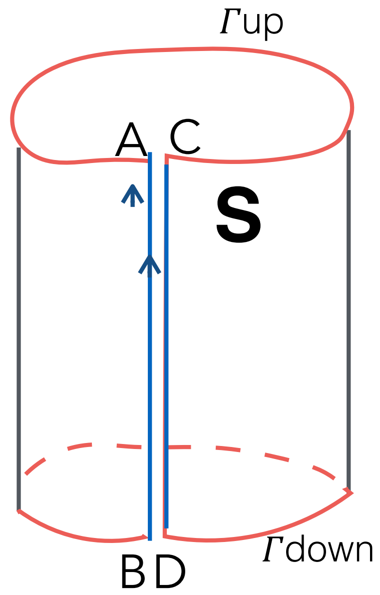
Topological Interpretation of Dynamics of a System

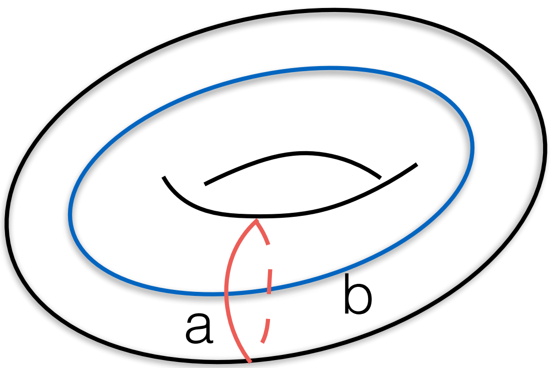
- **S** is the space of states
- Each state is defined by a vector that moves over S driven by a dynamical system
- If the dynamics moves the vector towards a boundary, we can say that there is a deadlock.
- This happens because S has not been defined globally. In fact the boundary breaks the translational symmetry.
- If we allow the boundary to disappear by adding an extra-relation, global in nature, we obtain a global topology that is not trivial
- For example we can add two relations among the generators of the manifold

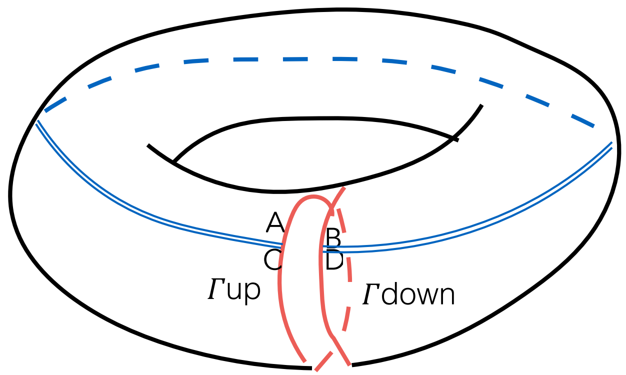








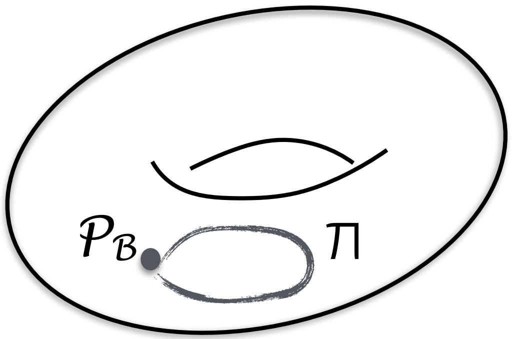




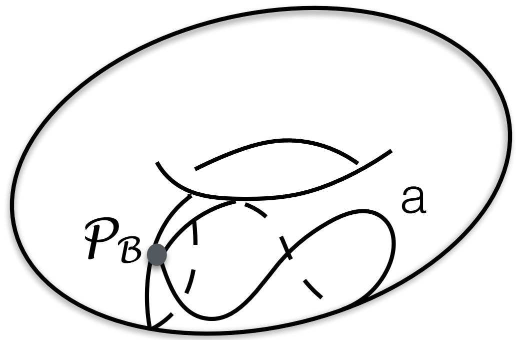
The trivial case of torus:

for any g ≠ 0, (e.g. g=1 torus) there are three irreducible classes behaviors

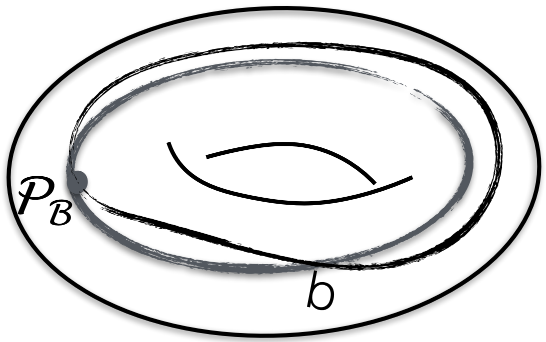
i) the set of closed paths homotopy to 0. In this case, we are given a local interpretation and we are not aware that at global level the genus can be different that 0.

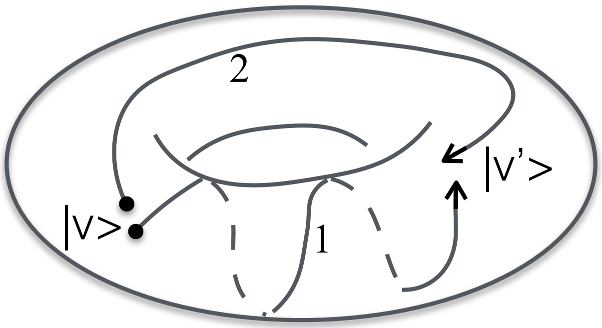


ii) the set of closed paths homotopy to the first generator a of the topological space (the homology group) whose basic cycle fixed on point P_B can be used to reduce any path going around the neck of the torus **a** by a continuos deformation;



iii) similar to the previous set, but the paths are **homotopy to the second generator b**





to each point | v> of the states space
is associated a path, represented by the arrow
|v> → |v'>
that is nothing but an operator $\mathcal{G}$ that determines
the dynamic of the system moving
|v> to $\mathcal{G}$(|v>)=|v'>

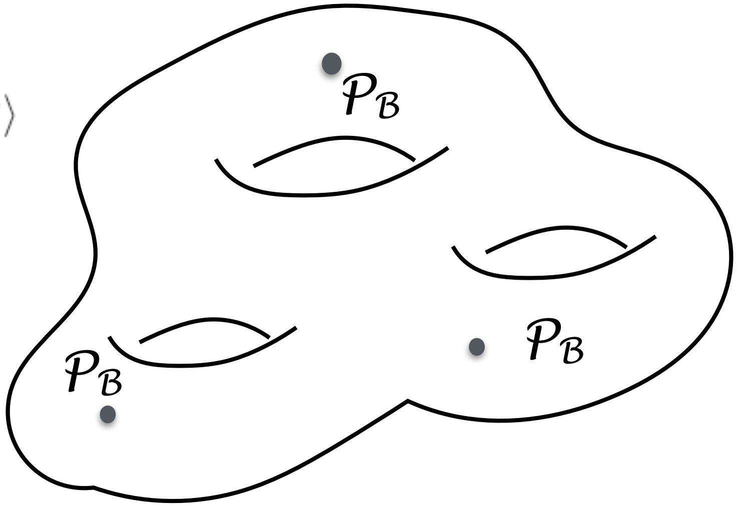
the set of all possible $\mathcal{G}$ represents a group, i.e the group of transformations

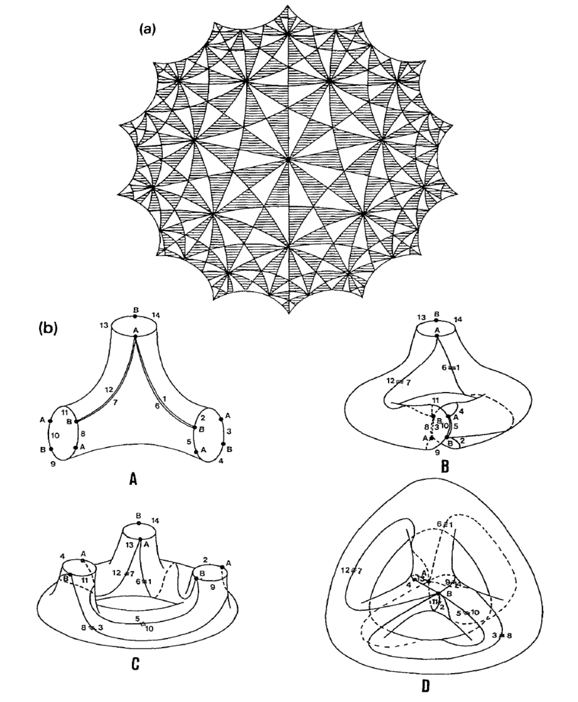
More complex case:

$$\mathcal{G}_{168} \sim \langle U, V \mid V^2, (UV)^3, U^7, (VU^4)^4 \rangle$$

homology generators

$$\mathbf{a}_1 = (VU^3)^4$$
$$\mathbf{a}_2 = U^{-1}(VU^4)^3VU(VU^4)^3VU$$
$$\mathbf{a}_3 = (VU^3)^3U^3(VU^3)^3U^3$$
$$\mathbf{b}_1 = U^{-1}(VU^3)^4U$$
$$\mathbf{b}_2 = (VU^3)^3VU(VU^3)^3U^3(VU^3)^2VU\ ,$$
$$\mathbf{b}_3 = U^{-1}(VU^3)^3VU(VU^3)^3U^3(VU^3)^2VU^2$$



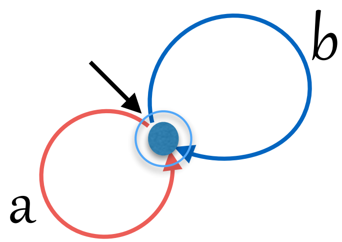


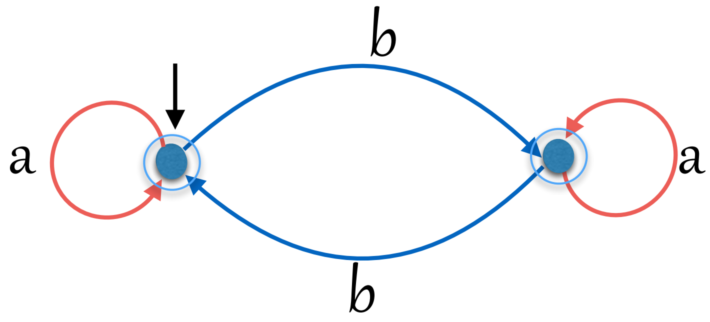
(beyond) Frege's principle of compositionality

The principle of **non-linear** composition states that the emergent behaviour of a complex systems can not be fully determined by the behaviour of its constituents and the rules to combine them, because is the global context that induces the non-linearity interactions among them.

Compositionality over a Paths

Frege's principle: in mathematics, semantics, and philosophy of language, the principle of compositionality is the principle for which the meaning of a complex expression is determined by the meanings of its constituent expressions and the rules used to combine them;

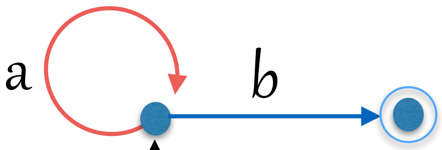




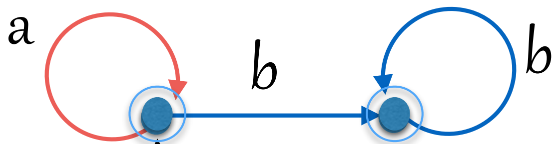
Theorem [Milner 1984]: not every X-behaviour is a star behaviour.

$$(a+b)^* \quad \simeq \quad a^*(b+a^*)^*$$

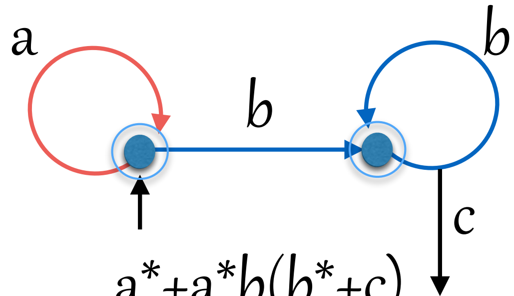
under **bisimulation** relation, but cannot be proven in any axioms system



a^*b



$a^*+a^*bb^*$



$a^*+a^*b(b^*+c)$

Topological Interpretation of Processes

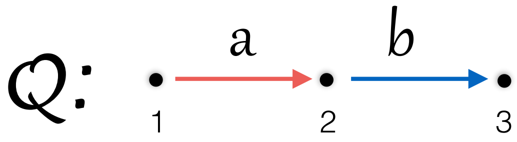
The 'process interpretation' scheme of **P** in $\mathcal{P}$ is indeed nothing but a quiver $\mathcal{Q}$ (or, more generally, a set of quivers, over some arbitrary ring κ). Associate to quiver $\mathcal{Q}$ its 'natural' path algebra $\mathcal{A} \equiv \mathcal{P}_{k\mathcal{Q}}$, i.e., the path algebra of which $\mathcal{Q}$ is the basis.

The structure is simpler and elegant because space $\mathcal{P}$ has an underlying natural formal language (that generates in general a subgroup of the much wilder group of all possible homeomorphisms of $\mathcal{P}(\mathbf{P})$)

1. define a processes as Quivers Q

The *process* quiver $\mathcal{Q}$ represents a system behavior

Behavioural equivalences: the paths in process quivers are distinguished by some homotopic equivalence.

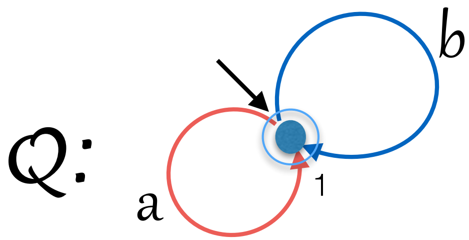


paths: v_1, v_2, v_3, a, b, a b

A quiver $\mathcal{Q}$ is a direct graph. $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1, s, e)$, where $\mathcal{Q}_0$ is a set of vertices (states) and $\mathcal{Q}_1$ a set of arrows (transitions) and $s, e: \mathcal{Q}_1 \to \mathcal{Q}_0$, are maps.

Given an arrow $a \in \mathcal{Q}_1$ with $a: i \to j$ for $i, j \in \mathcal{Q}_0$

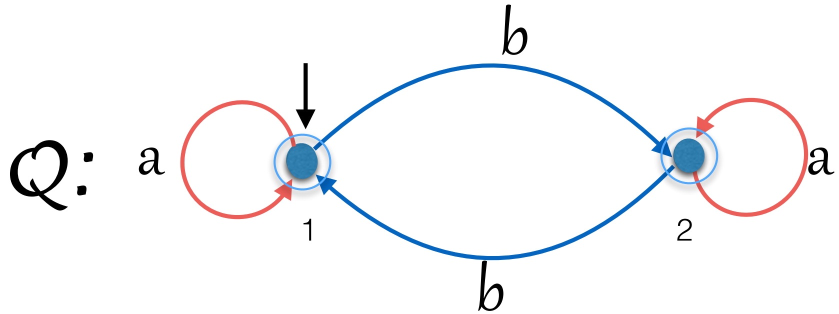
When $s(a) = e(a)$, arrow a is said to be a *loop*.



paths: v_1, a^*, b^*, $(a+b)^*$

An path in a quiver $\mathcal{Q}$ is either an ordered composition of arrows $p = a_1 a_2 \ldots a_n$ with $e(a_t) = s(a_{t+1})$ for $1 \leq t < n$ or the symbol v_i for $i \in \mathcal{Q}_0$

A path p that starts and ends at the same vertex is a *cycle*. Loops are cycles.



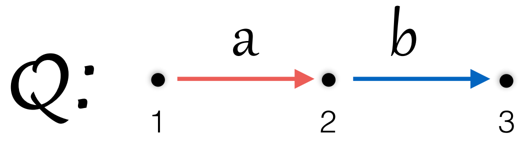
paths: v_1, v_2, a^*, b^*, $(a+b)^*$

A quiver with relations is a pair $(\mathcal{Q}, \mathcal{R})$,

2. associate a *natural* path algebra $\mathcal{P}_{kQ}$ to given Q

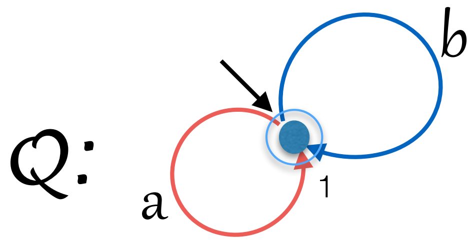
- Let k be a field. The path algebra $\mathcal{P}_{kQ}$ of the quiver Q is defined to be the k-vector space generated by all paths in Q. The composition (product) of two paths is induced by simple concatenation of paths if it exists, and zero otherwise. Q is the basis of $\mathcal{P}_{kQ}$.

- $\mathcal{P}_{kQ}$ is unitary if Q_0 is finite



$Q=(\{1,2,3\},\{a,b\},s,e)$

$\{v_1,v_2,v_3, a, b, ab\}$ is the basis for the $\mathcal{P}_{kQ}$



$\{v_1, a^*, b^*, a^*b^*, (a+b)^*\}$ is the basis for the $\mathcal{P}_{kQ}$

$\mathcal{P}_{QR}$ is a path co-algebra of quivers with relations

3. identify the Lie algebra $\mathcal{L}$ given by $\mathcal{P}$

Identify the Lie algebra $\mathcal{L}$ arising from $\mathcal{P}_{kQ}$

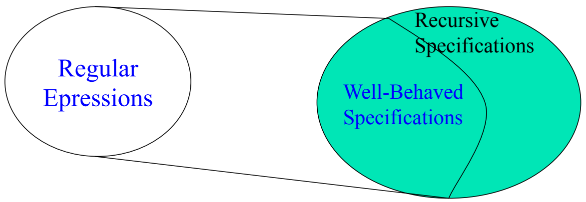
4. define $\mathcal{L}$ in Hopf algebra $\mathcal{H}$

Turn $\mathcal{L}$ into a Hopf algebra $\mathcal{H}$, equipping it in particular with a a coalgebra

5. redefine the theorem G in $\mathcal{H}$ and prove that hold in $\mathcal{H}$ and not in $\mathcal{P}$

Theorem G:
The SHC [Star Height Conjecture] is a ***topological application*** to the space $\mathcal{P}$, generated by the formal representation of a (any) process **P**.

Corradini's Star Heigh conjecture:
the set of regular expressions (without 0) with *hnewp* is the largest language for which bisimulation admits a finite equational axiomatization.



Def. *nhewp* structural property:

1. each *-behaviour must avoid to enter in a pure cycle,
2. each cycle must be of the form
   E*=1+EE*
   E*F → X=EX + F
3. in *-behaviour a*a≠ aa*

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Sala degli Stemmi

Sala della Muta

thanks!