Towards a resource theory of contextuality



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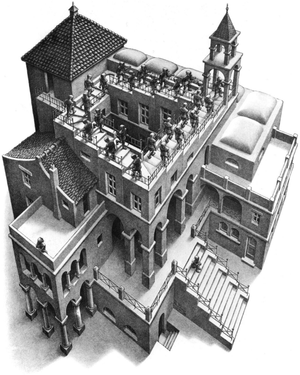
Introduction

- **Contextuality**: a fundamental non-classical phenomenon of QM



Introduction

- **Contextuality**: a fundamental non-classical phenomenon of QM

- Contextuality as a **resource** for QC:
  - Raussendorf (2013) – **MBQC**
    "Contextuality in measurement-based quantum computation"
  - Howard, Wallman, Veith, & Emerson (2014) – **MSD**
    "Contextuality supplies the 'magic' for quantum computation"

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- Abramsky–Brandenburger: unified framework for non-locality and contextuality in general measurement scenarios

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- in particular, “free” operations

- A–B: qualitative hierarchy of contextuality for empirical models
- quantitative grading – **measure of contextuality**
  (NB: there may be more than one useful measure)

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(generalising the idea of non-local fraction)

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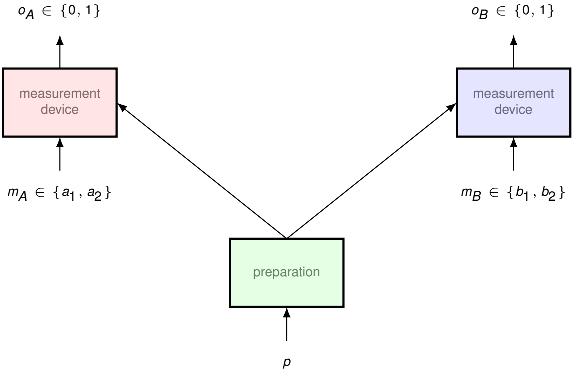
  $\rightsquigarrow$ **resource theory**
- Relates to quantifiable **advantages** in QC and QIP tasks

Contextuality

Empirical data

A	B	(0, 0)	(0, 1)	(1, 0)	(1, 1)
a_1	b_1	$1/2$	0	0	$1/2$
a_1	b_2	$3/8$	$1/8$	$1/8$	$3/8$
a_2	b_1	$3/8$	$1/8$	$1/8$	$3/8$
a_2	b_2	$1/8$	$3/8$	$3/8$	$1/8$



Abramsky–Brandenburger framework

Measurement scenario $\langle X, \mathcal{M}, O \rangle$:

- X is a finite set of measurements or variables
- O is a finite set of outcomes or values
- $\mathcal{M}$ is a cover of X, indicating **joint measurability** (contexts)

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Example: (2,2,2) Bell scenario

- The set of variables is $X = \{a_1, a_2, b_1, b_2\}$.
- The outcomes are $O = \{0, 1\}$.
- The measurement contexts are:

$$\{ \{a_1, b_1\}, \quad \{a_1, b_2\}, \quad \{a_2, b_1\}, \quad \{a_2, b_2\} \}.$$

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A joint outcome or **event** in a context C is $s \in O^C$, e.g.

$$s = [a_1 \mapsto 0, b_1 \mapsto 1] \,.$$

(These correspond to the cells of our probability tables.)

Another example: 18-vector Kochen–Specker

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- A set of 18 variables, $X = \{A, \ldots, O\}$
- A set of outcomes $O = \{0, 1\}$
- A measurement cover $\mathcal{M} = \{C_1, \ldots, C_9\}$, whose contexts C_i correspond to the columns in the following table:

U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9
A	A	H	H	B	I	P	P	Q
B	E	I	K	E	K	Q	R	R
C	F	C	G	M	N	D	F	M
D	G	J	L	N	O	J	L	O

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Compatibility condition: these distributions "agree on overlaps", i.e.

$$\forall_{C, C' \in \mathcal{M}}.\ e_C|_{C \cap C'} \ = \ e_{C'}|_{C \cap C'} \ .$$

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For multipartite scenarios, compatibility $=$ the **no-signalling** principle.

Contextuality

A (compatible) empirical model is **non-contextual** if there exists a **global distribution** $d \in \mathsf{Prob}(O^X)$ (on the joint assignments of outcomes to all measurements) that marginalises to all the e_C:

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The import of results such as Bell's and Bell–Kochen–Specker's theorems is that there are empirical models arising from quantum mechanics that are contextual.

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E.g. K–S models, GHZ, the PR box:

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a_1	b_1	✓	×	×	✓
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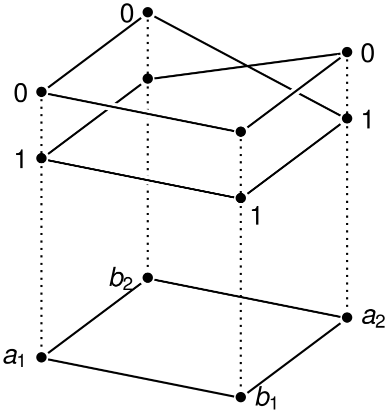

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Equivalently, maximum weight λ over all convex decompositions

$$e = \lambda e^{NC} + (1 - \lambda)e'$$

where e^{NC} is a non-contextual model.

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$$\text{NCF}(e) = \lambda \qquad \text{CF}(e) = 1 - \lambda$$

(Non-)contextual fraction via linear programming

Checking contextuality of e corresponds to solving

$$\begin{array}{ll} \texttt{Find} & \mathbf{d} \in \mathbb{R}^n \\ \texttt{such that} & \mathbf{M}\,\mathbf{d} = \mathbf{v}^e \\ \texttt{and} & \mathbf{d} \geq \mathbf{0} \end{array} .$$

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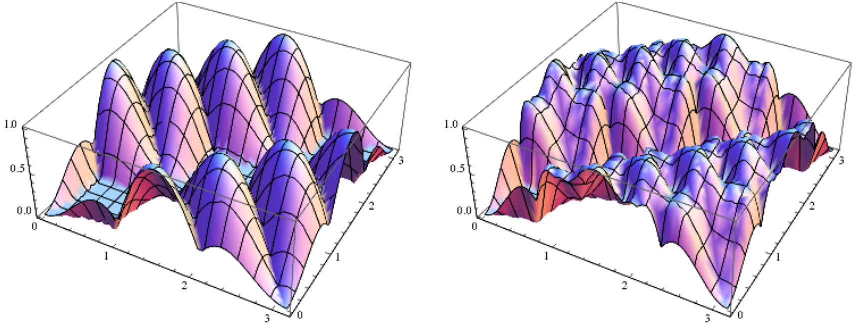
Computing the non-contextual fraction corresponds to solving the following linear program:

$$\begin{array}{ll} \texttt{Find} & \mathbf{c} \in \mathbb{R}^n \\ \texttt{maximising} & \mathbf{1} \cdot \mathbf{c} \\ \texttt{subject to} & \mathbf{M}\,\mathbf{c} \leq \mathbf{v}^e \\ \texttt{and} & \mathbf{c} \geq \mathbf{0} \end{array} \quad .$$

E.g. Equatorial measurements on GHZ(n)



(a) (b)

Figure: Non-contextual fraction of empirical models obtained with equatorial measurements at ϕ_1 and ϕ_2 on each qubit of $|\psi_{\mathrm{GHZ}(n)}\rangle$ with: (a) $n = 3$; (b) $n = 4$.

Violations of Bell inequalities

Generalised Bell inequalities

An **inequality** for a scenario $\langle X, \mathcal{M}, O \rangle$ is given by:

- a set of coefficients $\alpha = \{\alpha(C, s)\}_{C \in \mathcal{M}, s \in \mathcal{E}(C)}$
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For a model e, the inequality reads as

$$\mathcal{B}_\alpha(e) \;\leq\; R \,,$$

where

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Wlog we can take R non-negative (in fact, we can take $R = 0$).

It is called a **Bell inequality** if it is satisfied by every NC model. If it is saturated by some NC model, the Bell inequality is said to be **tight**.

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The **normalised violation** of a Bell inequality $\langle \alpha, R \rangle$ by an empirical model e is the value

$$\frac{\max\{0, \mathcal{B}_\alpha(e) - R\}}{\|\alpha\| - R} \; .$$

Bell inequality violation and the contextual fraction

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Let e be an empirical model.

- The normalised violation by e of any Bell inequality is at most $\mathsf{CF}(e)$.
- This is attained: there exists a Bell inequality whose normalised violation by e is exactly $\mathsf{CF}(e)$.
- Moreover, this Bell inequality is tight at “the” non-contextual model e^{NC} and maximally violated by “the” strongly contextual model e^{SC}:

$$e = \mathsf{NCF}(e)e^{NC} + \mathsf{CF}(e)e^{SC} .$$

Bell inequality violation and the contextual fraction

Quantifying Contextuality LP:

`Find`	$\mathbf{c} \in \mathbb{R}^n$
`maximising`	$\mathbf{1} \cdot \mathbf{c}$
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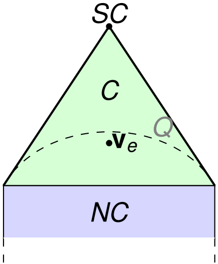

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Dual LP:

$$\begin{array}{ll} \texttt{Find} & \mathbf{y} \in \mathbb{R}^m \\ \texttt{minimising} & \mathbf{y} \cdot \mathbf{v}^e \\ \texttt{subject to} & \mathbf{M}^T\,\mathbf{y} \geq \mathbf{1} \\ \texttt{and} & \mathbf{y} \geq \mathbf{0} \quad . \end{array}$$

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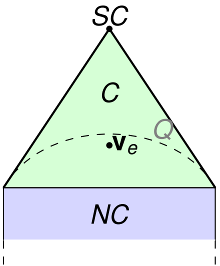

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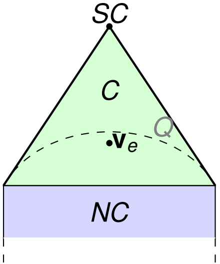

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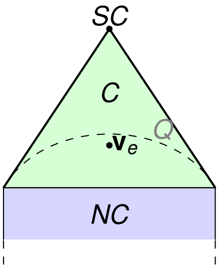

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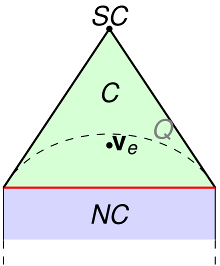

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and	$\mathbf{y} \geq \mathbf{0}$.

$\mathbf{a} := \mathbf{1} - |\mathcal{M}|\mathbf{y}$

Find	$\mathbf{a} \in \mathbb{R}^m$
maximising	$\mathbf{a} \cdot \mathbf{v}^e$
subject to	$\mathbf{M}^T \mathbf{a} \leq \mathbf{0}$
and	$\mathbf{a} \leq \mathbf{1}$.

computes tight Bell inequality (separating hyperplane)

Operations on empirical models

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- Towards a resource theory as for entanglement (e.g. LOCC), non-locality, . . .

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- The operations remind one of process algebras.

Operations

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  $$(e \otimes e')_{C \sqcup D} \langle s_1, s_2 \rangle := e_C(s_1)\, e'_D(s_2)$$

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Contextual fraction and quantum advantages

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- Measure of contextuality $\rightsquigarrow$ to quantify such advantages.

Contextual fraction and MBQC

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- Probabilistic version: non-linear function computed with sufficently large probability of success implies contextuality.

Contextual fraction and MBQC

- **average distance** between two Boolean functions $f, g : 2^m \longrightarrow 2^l$:
  $\tilde{d}(f, g) := 2^{-m} |\{\mathbf{i} \in 2^m \mid f(\mathbf{i}) \neq g(\mathbf{i})\}|$

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- Then, $1 - \bar{p}_S \ \geq \ \mathrm{NCF}(e)\tilde{\nu}(f)$.

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  - Connections with Contextuality-by-Default (Dzhafarov et al.)

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  - What (else) is this resource useful for?

Questions...

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