Compositionality in Categorical Quantum Mechanics

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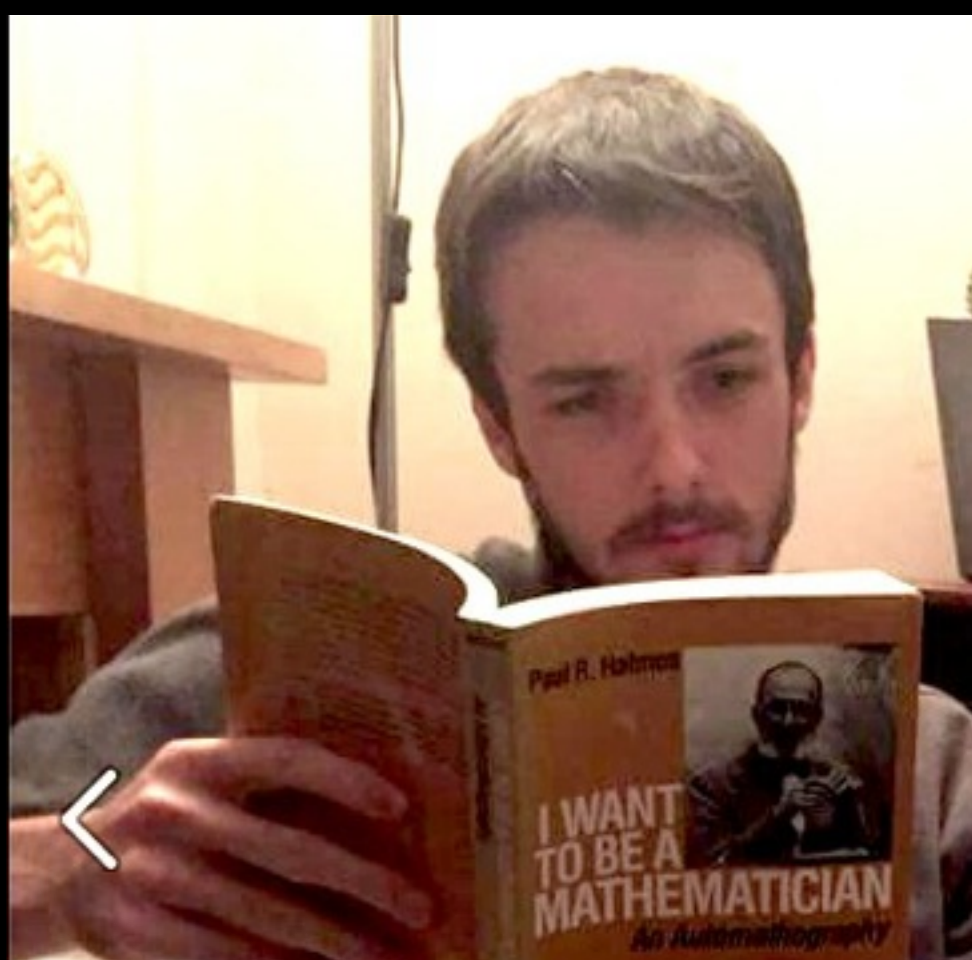
Simon Perdrix



Kevin Dunne

Niel de Beaudrap

Compositionality x 3

- Plain old monoidal category theory:
  —— quantum computing in string diagrams

- Rewriting and substitution:
  —— taking the syntax seriously

- “Quantum theory” as a composite theory
  —— Lack’s composing PROPS

An application: compiling for quantum architecture

1. Quantum theory as string diagrams

How much quantum theory can be expressed as an internal language in some monoidal category?


PAST / HEAVEN
FUTURE / HELL

F.D. Pure state QM

- States : Hilbert spaces
- Compound systems : Tensor product
- Dynamics : Unitary maps
- Non-degenerate measurements : O.N. Bases

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Ambient mathematical framework: †-symmetric monoidal categories

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Ambient mathematical framework: †-symmetric monoidal categories

Choose some good generators and relations to capture this stuff

Frobenius Algebras

Theorem: in $\mathbf{fdHilb}$ orthonormal bases are in bijection with †-special commutative Frobenius algebras.

$\delta : A \to A \otimes A$

$\mu : A \otimes A \to A$

$\epsilon : A \to I$

$\eta : I \to A$

Via:

$\delta :: |a_i\rangle \to |a_i\rangle \otimes |a_i\rangle$

$\mu = \delta^\dagger$

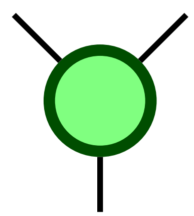
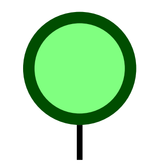
$\epsilon :: |a_i\rangle \to 1$

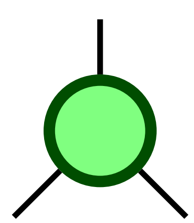
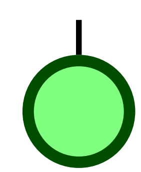
$\eta = \eta^\dagger$

Coecke, Pavlovic, and Vicary, “A new description of orthogonal bases”, MSCS 23(3), 2013. arxiv:0810.0812

Frobenius Algebras

Represent observables by †-special commutative Frobenius algebras:

$\mu =$  , $\eta =$ 

$\mu^\dagger =$  , $\eta^\dagger =$ 

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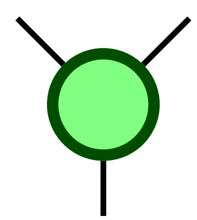
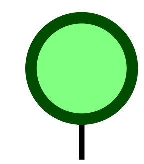
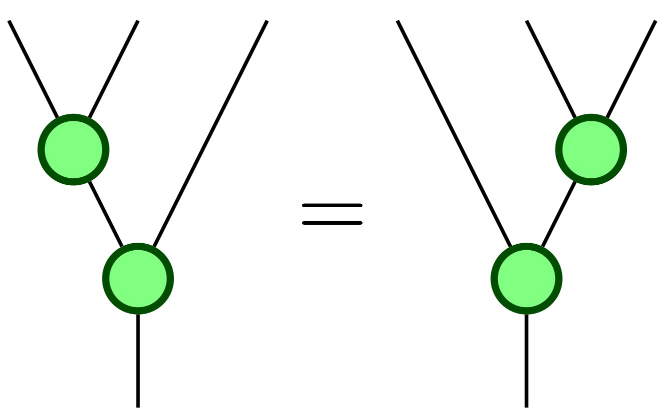
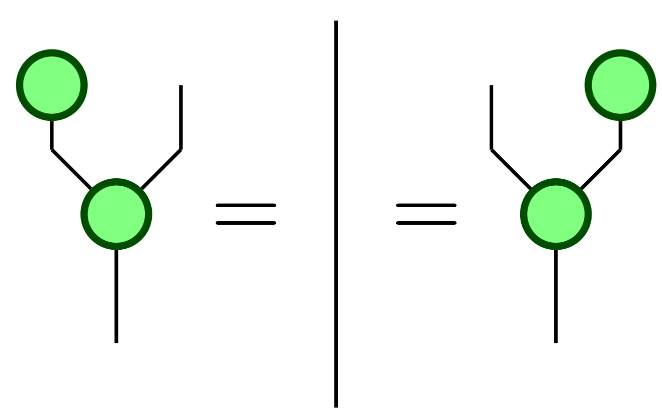
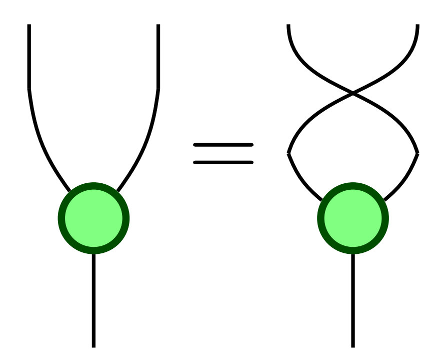
$\mu =$  , $\eta =$ 







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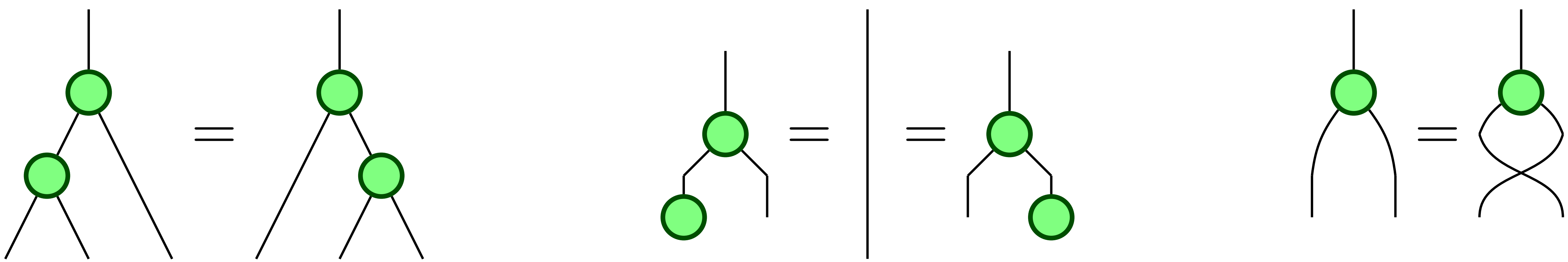
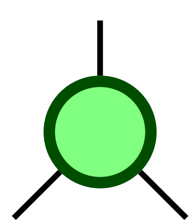
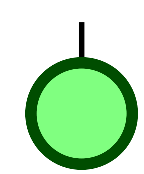


$\mu^\dagger =$  , $\eta^\dagger =$ 

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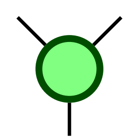
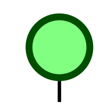
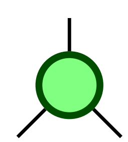
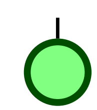
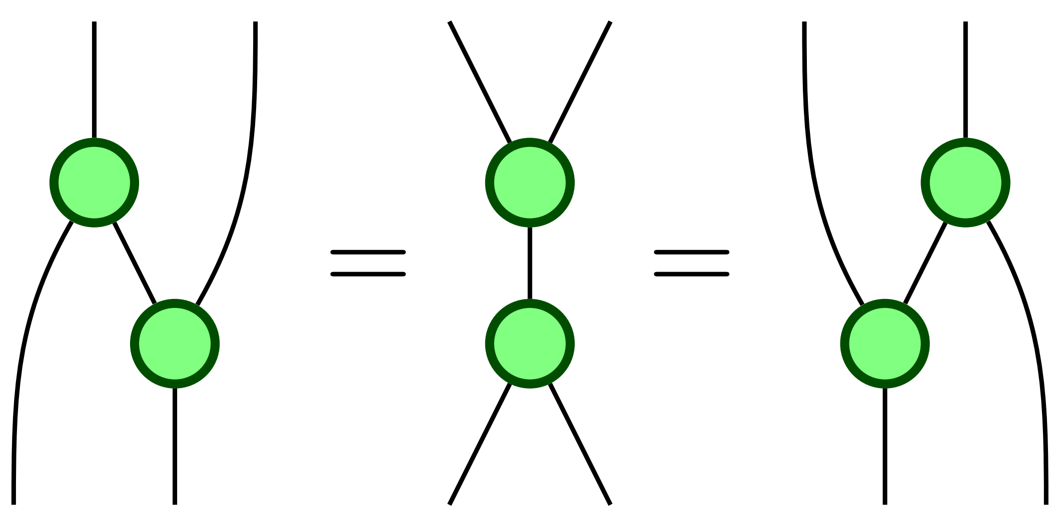
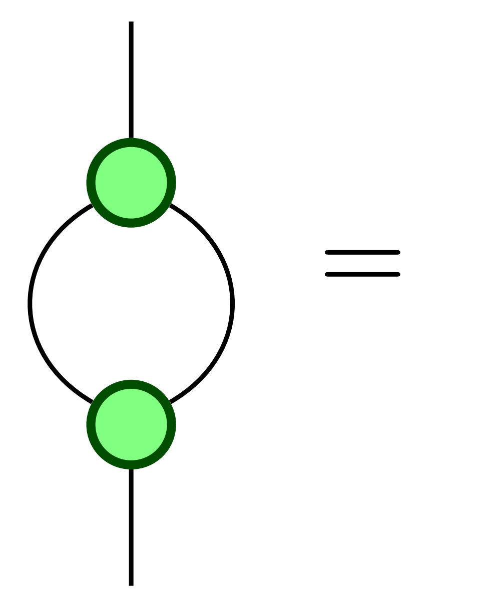
$\mu =$ , $\eta =$ 

$\mu^\dagger =$ , $\eta^\dagger =$ 





Phases

- **Defn**: a *phase* is unitary map that commutes with the Frobenius algebra like this:

- **Thm**: the phases form an abelian group

Example: Z-spin

- The following define a Frobenius algebra on the qubit:

$$\delta : \begin{array}{l} |0\rangle \mapsto |00\rangle \\ |1\rangle \mapsto |11\rangle \end{array} \qquad \epsilon : \begin{array}{l} |0\rangle \mapsto 1 \\ |1\rangle \mapsto 1 \end{array}$$

- Its group of phases is:

$$Z_\alpha : \begin{array}{l} |0\rangle \mapsto |0\rangle \\ |1\rangle \mapsto e^{i\alpha} |1\rangle \end{array}$$

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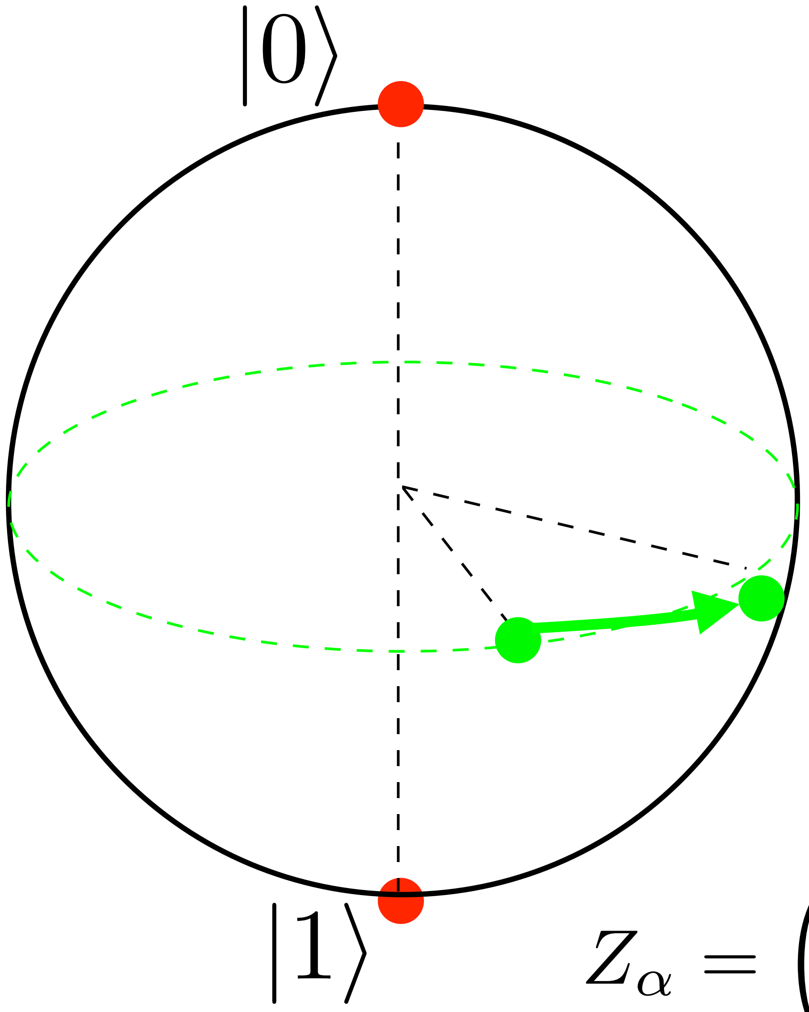
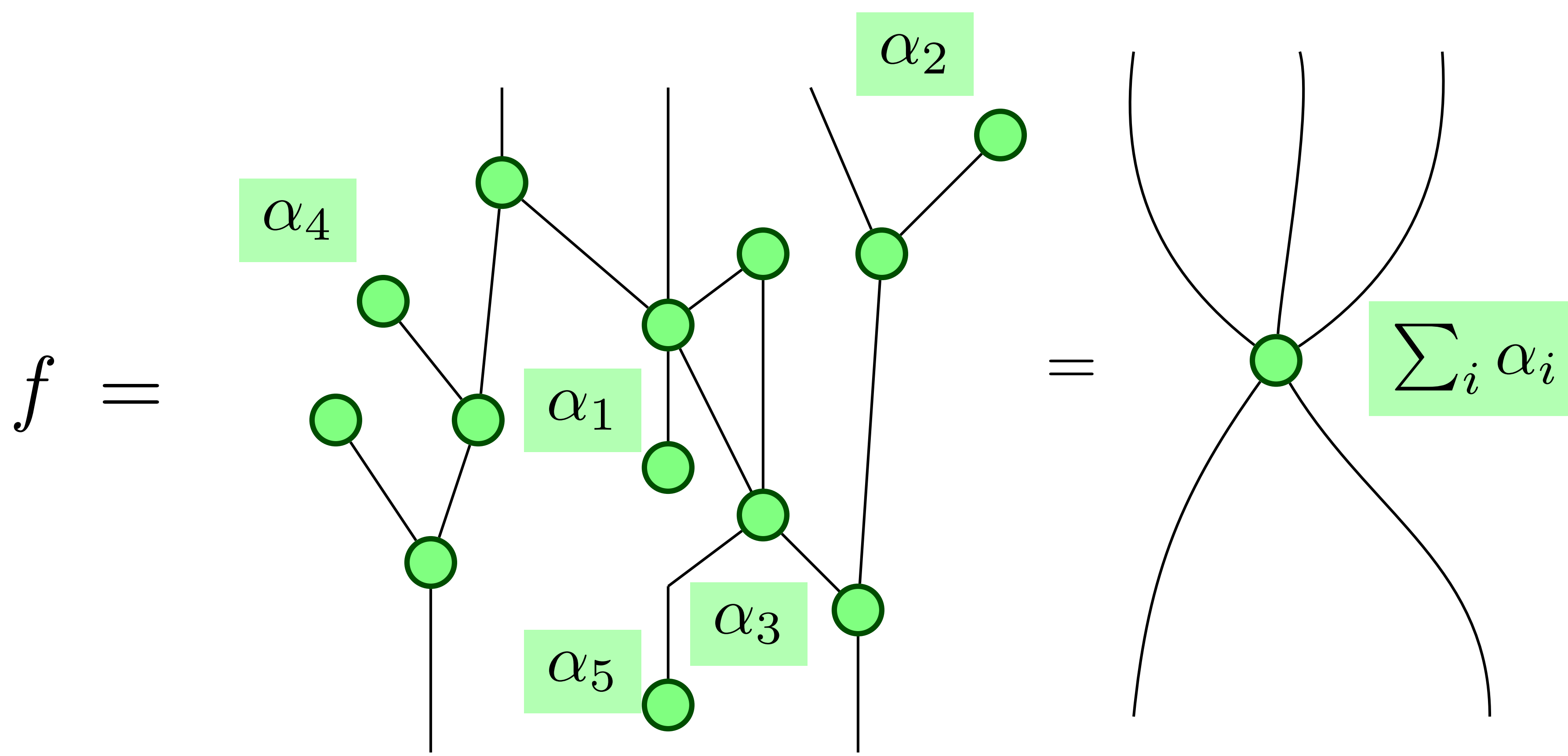


$$Z_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

Frob. algebras + phases

Theorem: let $f : n \longrightarrow m$ be connected.



Example: X-spin

- The following define a Frobenius algebra on the qubit:

$$\delta : \begin{array}{l} |+\rangle \mapsto |++\rangle \\ |-\rangle \mapsto |--\rangle \end{array} \qquad \epsilon : \begin{array}{l} |+\rangle \mapsto 1 \\ |-\rangle \mapsto 1 \end{array}$$

- Its group of phases is:

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X and Z spins

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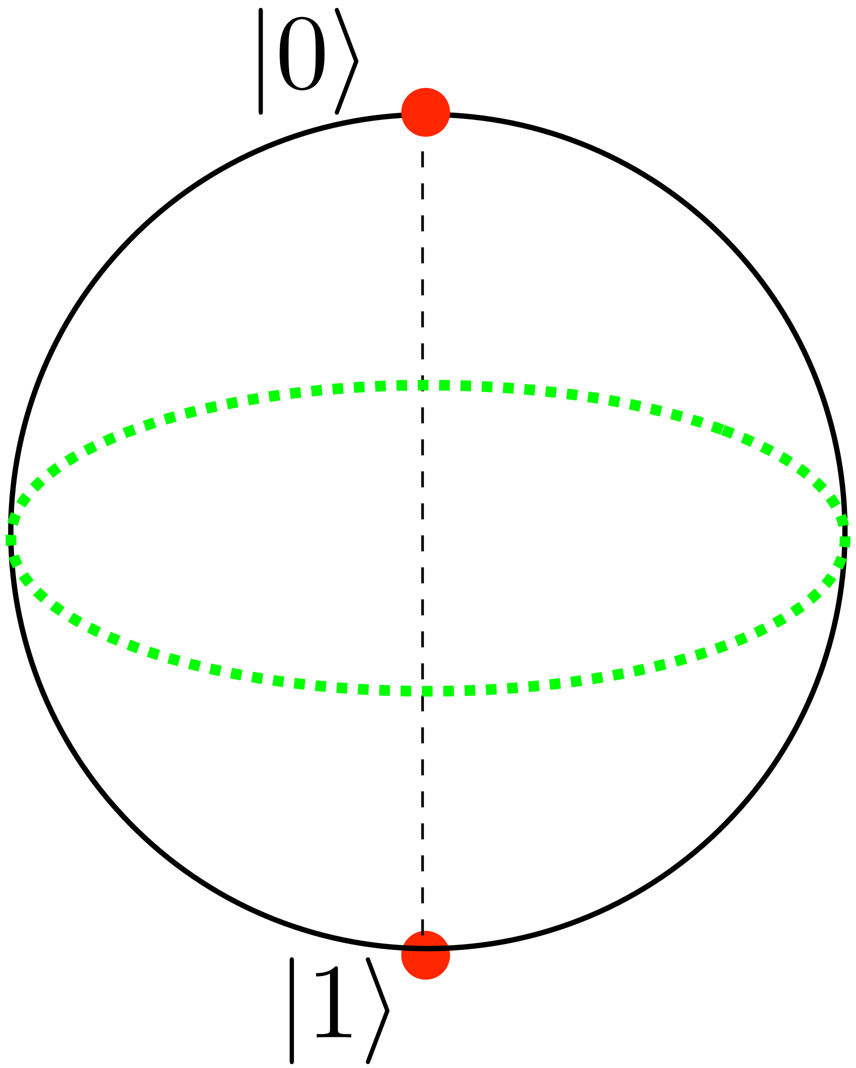

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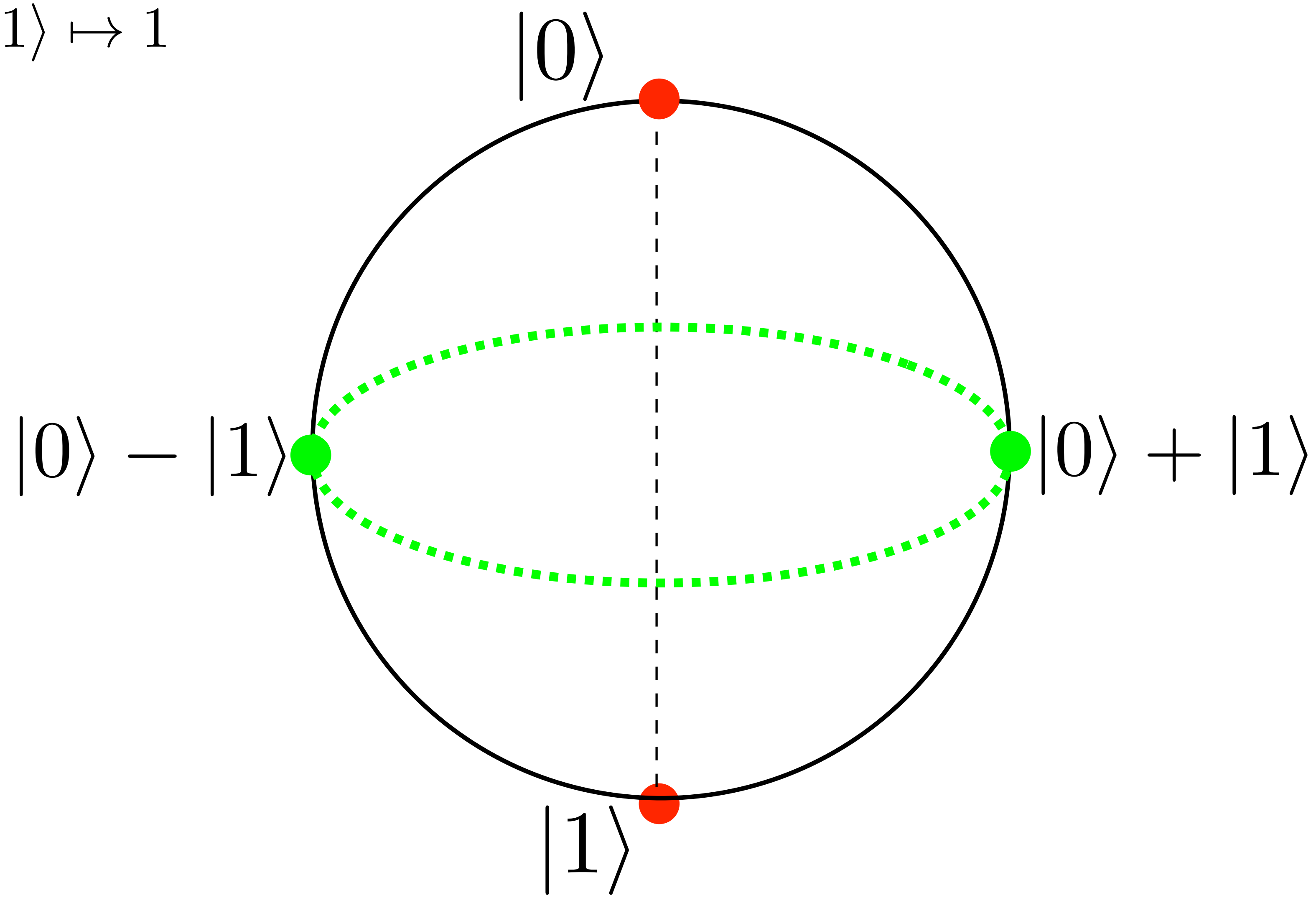

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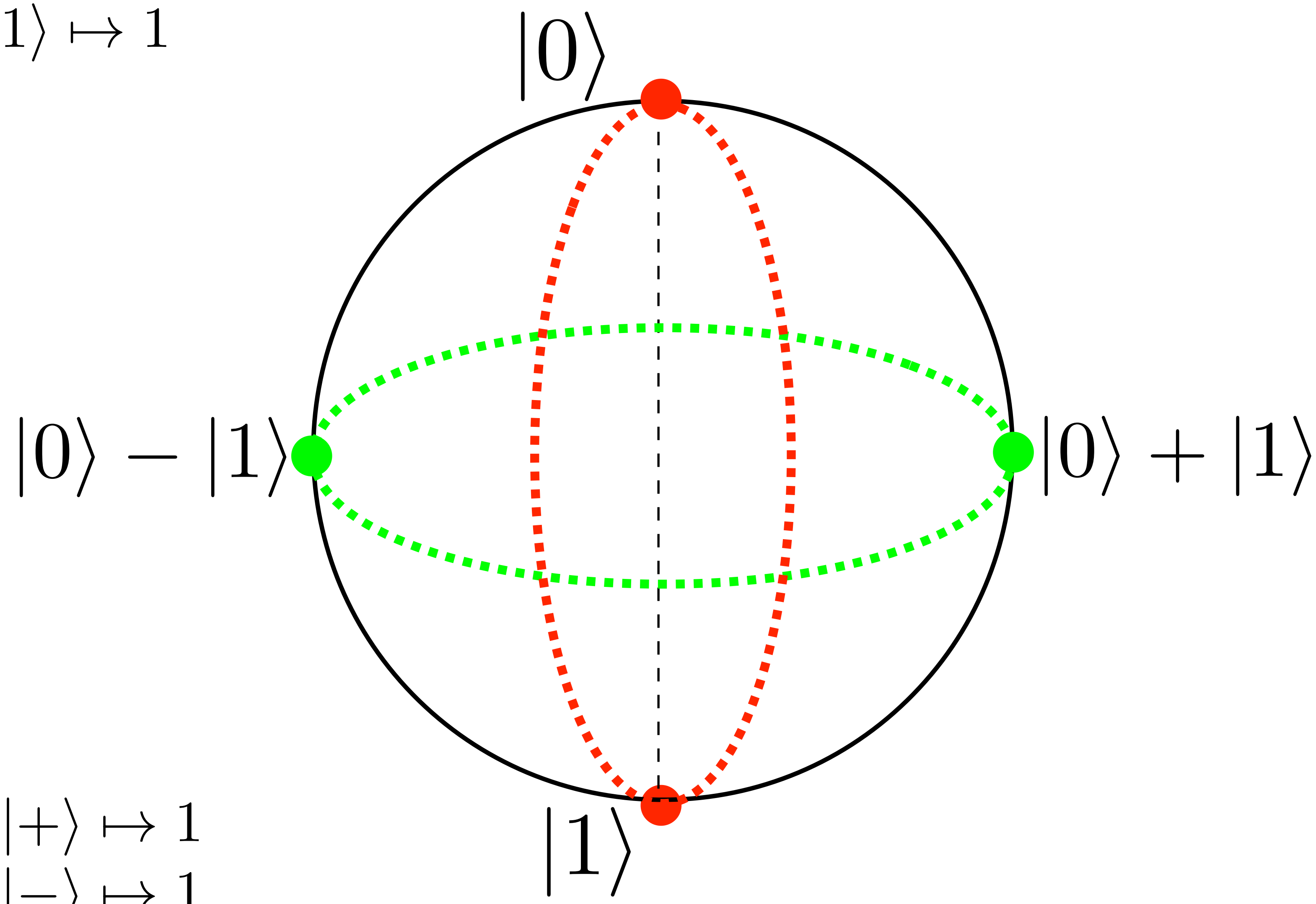

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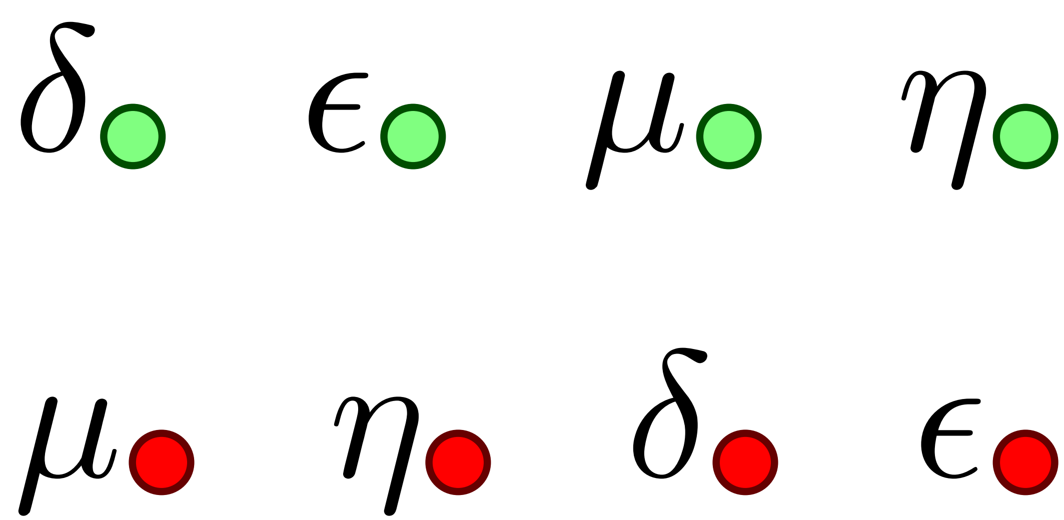
Strongly Complementary Observables are Hopf algebras

Theorem 3: Two observables are strongly complementary iff they form a Hopf algebra



Coecke and Duncan, "Interacting Quantum Observables: categorical algebra and diagrammatics", NJP 13(043016), 2011, arXiv:0906.4725.

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Frobenius

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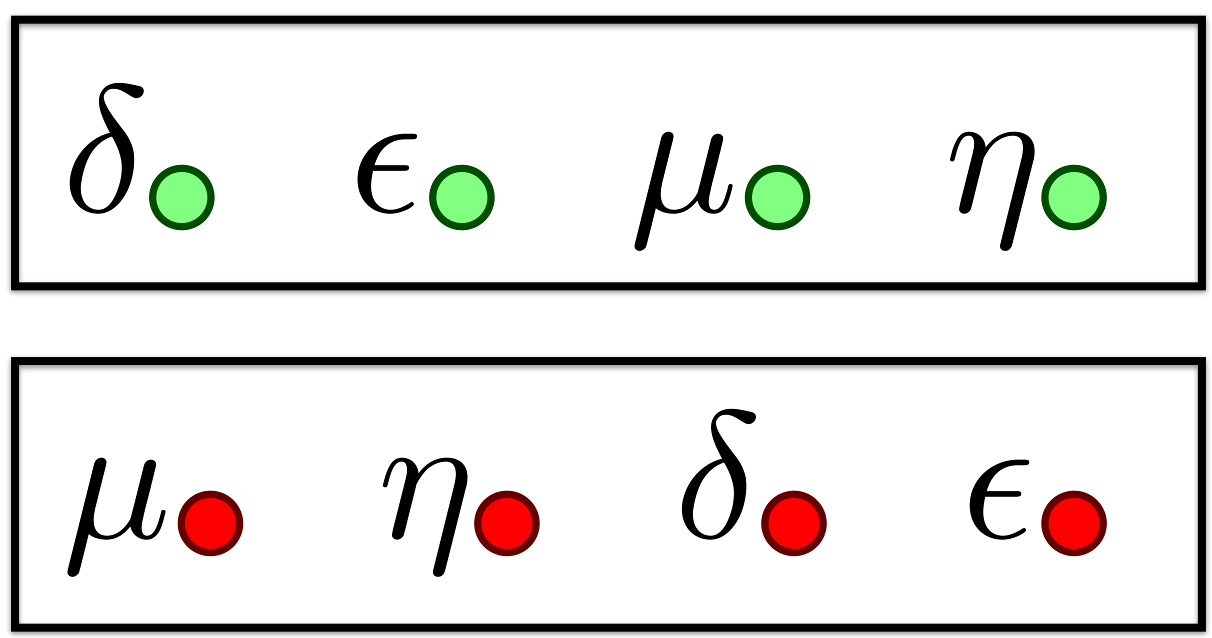

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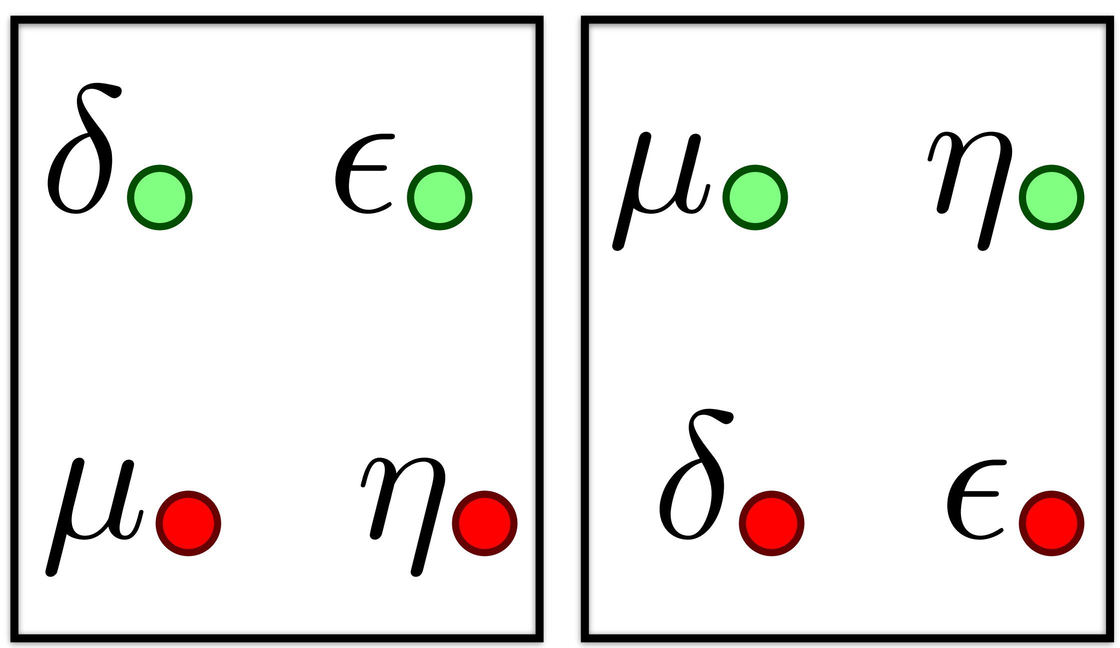

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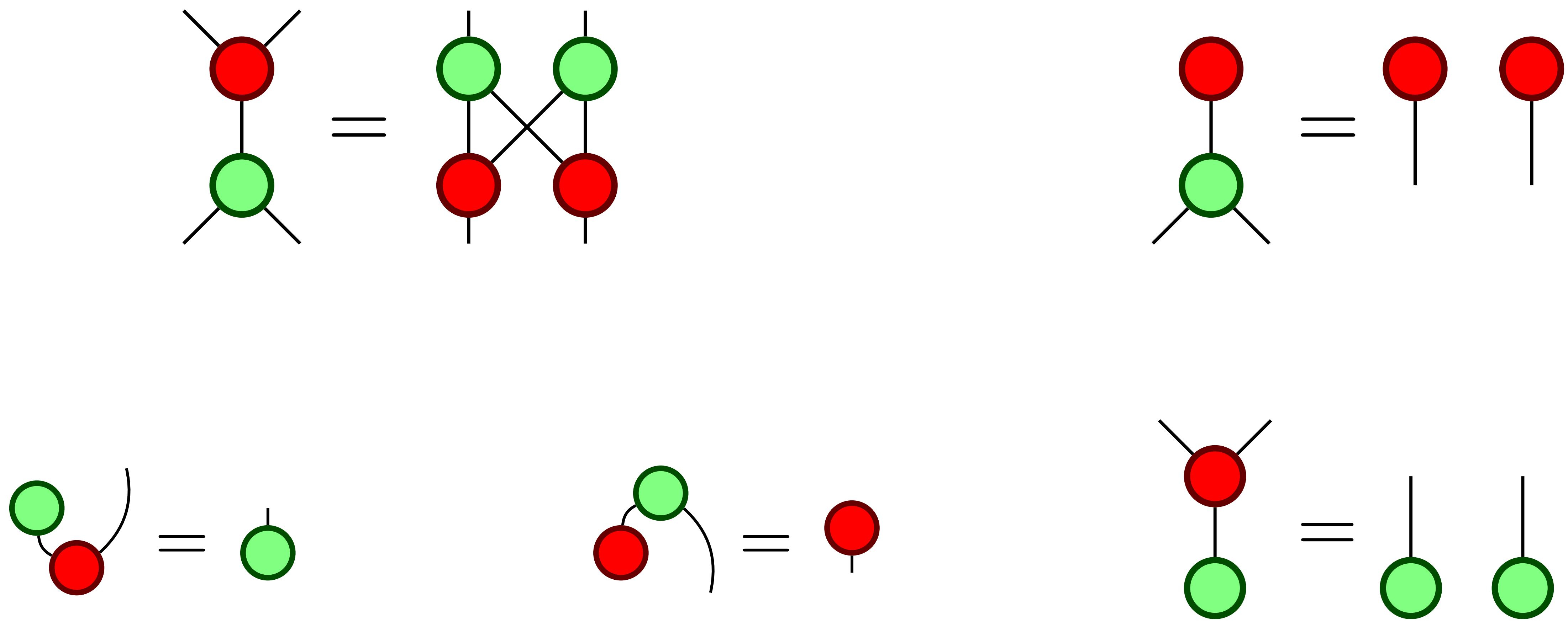

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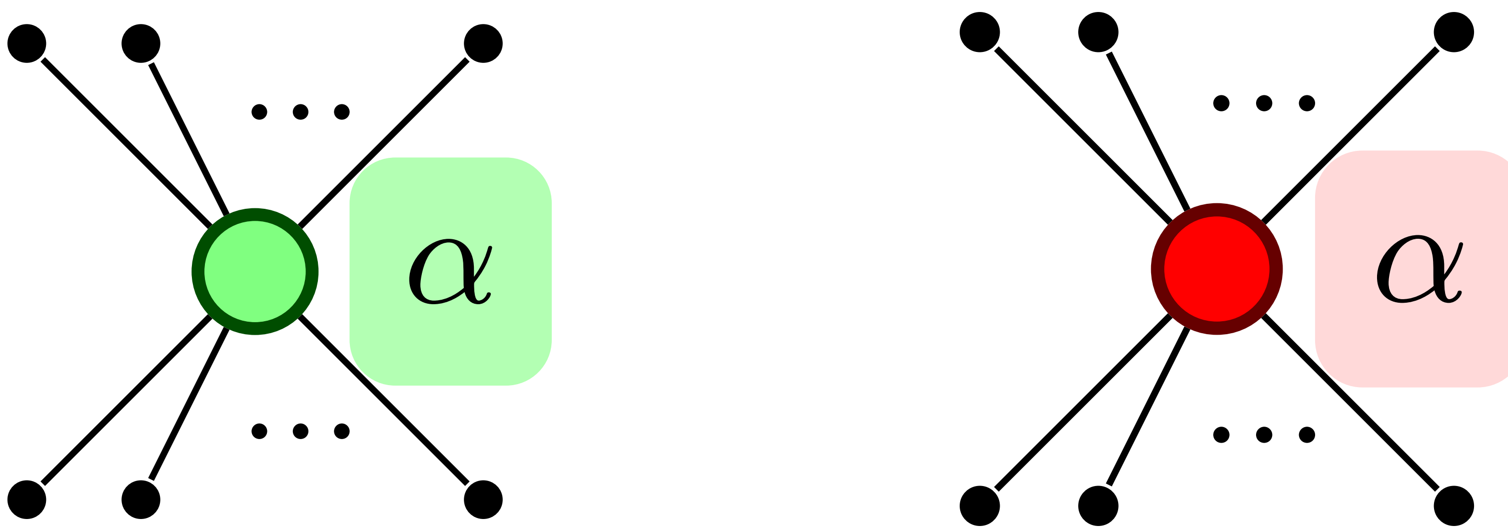
ZX-calculus

- Since we are interested in quantum computing we'll focus on the X and Z observables.
- This is called the **ZX-calculus**

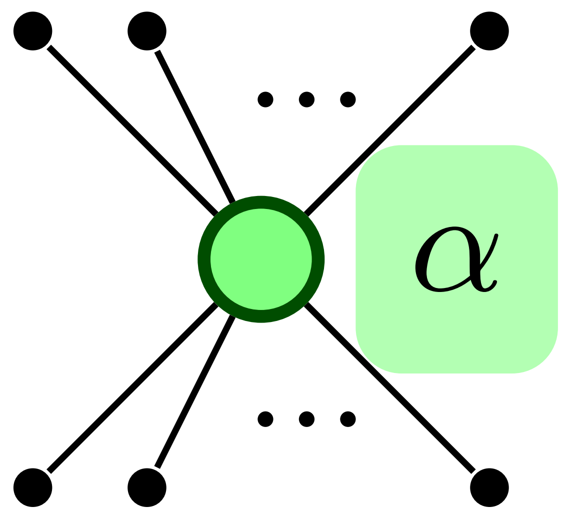
ZX-calculus syntax



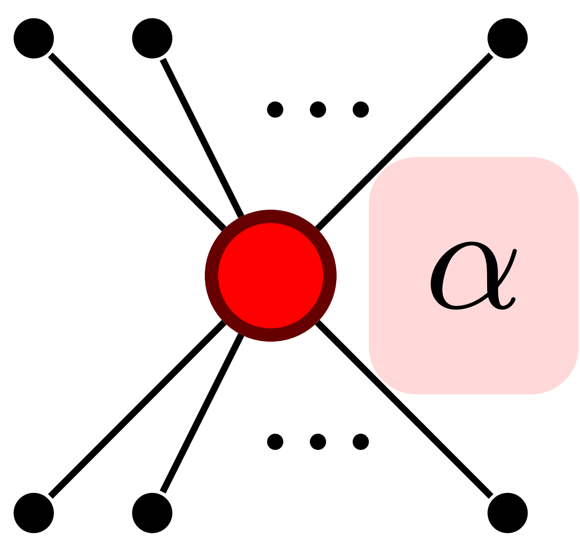
$$\alpha \in [0, 2\pi)$$

Defn: A *diagram* is an undirected open graph generated by the above vertices.

ZX-calculus semantics



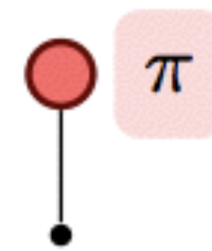
$$|0\rangle^{\otimes n} \mapsto |0\rangle^{\otimes m}$$
$$|1\rangle^{\otimes n} \mapsto e^{i\alpha} |1\rangle^{\otimes m}$$



$$|+\rangle^{\otimes n} \mapsto |+\rangle^{\otimes m}$$
$$|-\rangle^{\otimes n} \mapsto e^{i\alpha} |-\rangle^{\otimes m}$$

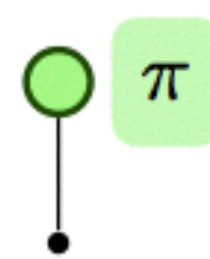
Representing Qubits

$[\![$  $]\!] = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$

$[\![$ 

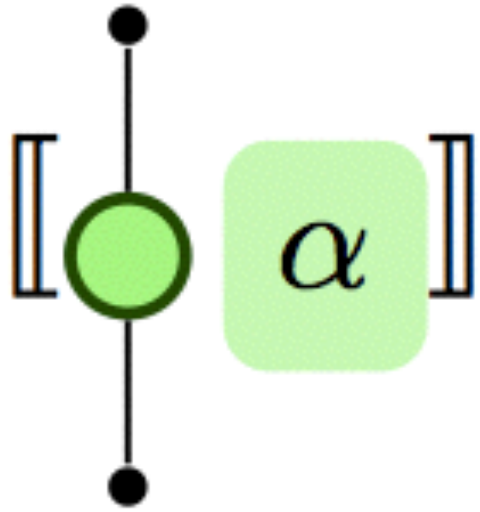
$]\!] = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$

$[\![$  $]\!] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = |+\rangle$

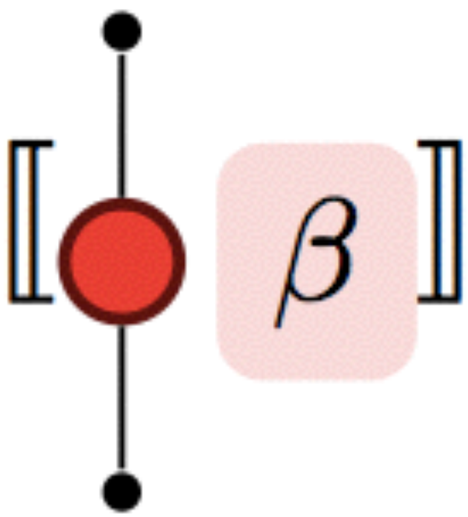
$[\![$ 

$]\!] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = |-\rangle$

Representing Phase shifts

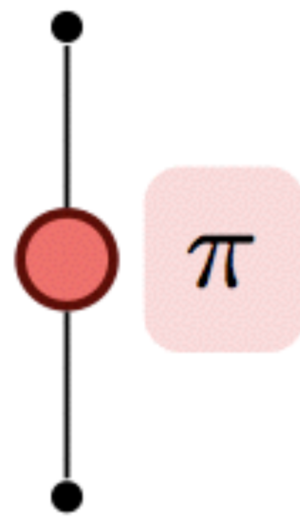


$$= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

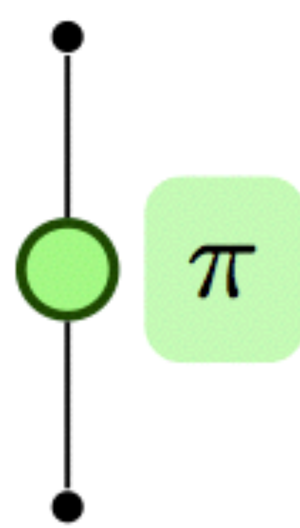


$$= \begin{pmatrix} \cos\frac{\beta}{2} & -i\sin\frac{\beta}{2} \\ -i\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}$$

Representing Paulis

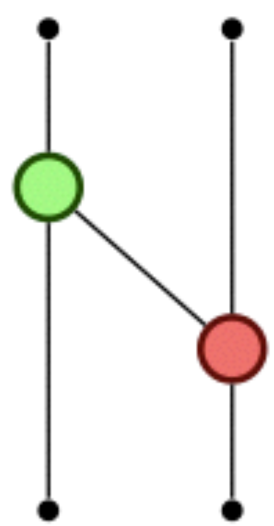
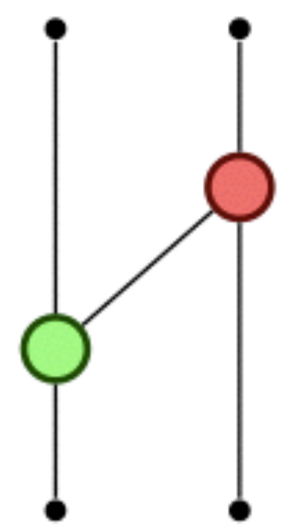
$[\![$ 

$]\!] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$[\![$ 

$]\!] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Representing CNot

$$\wedge X = [\![$$

$$]\!] = [\![$$

$$]\!] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The ZX-calculus is universal

Theorem: Let U be a unitary map on n qubits; then there exists a ZX-calculus term D such that:

$$[\![D]\!] = U$$

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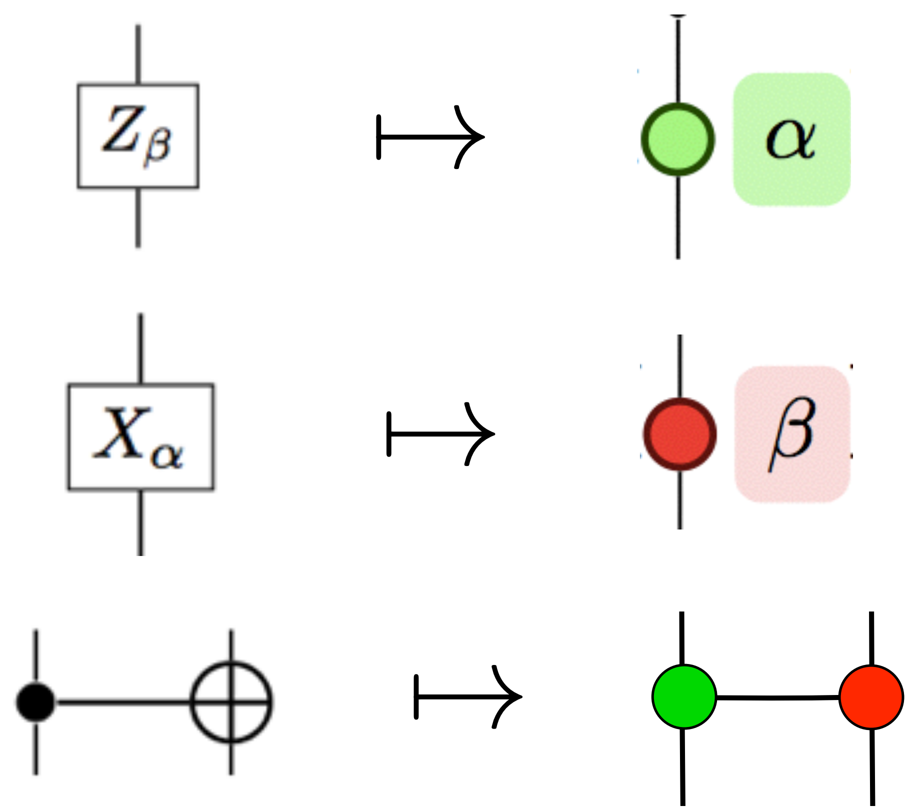
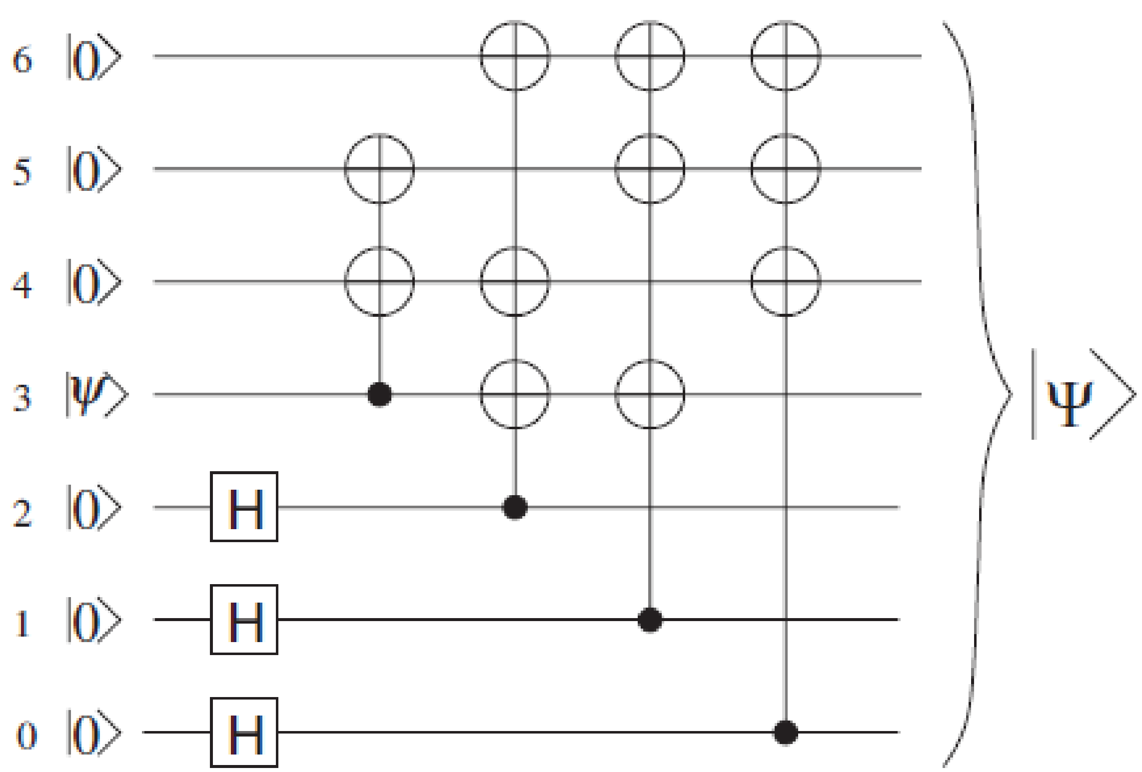
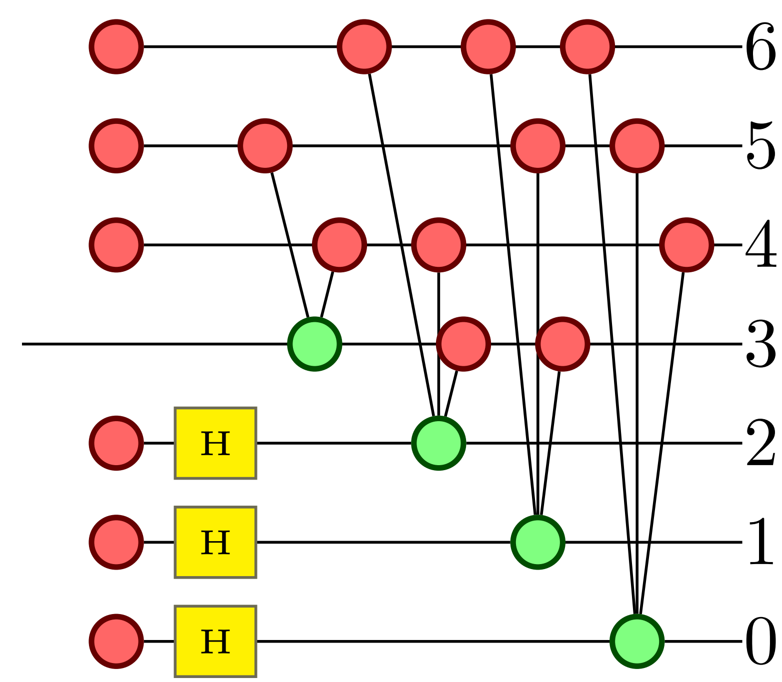


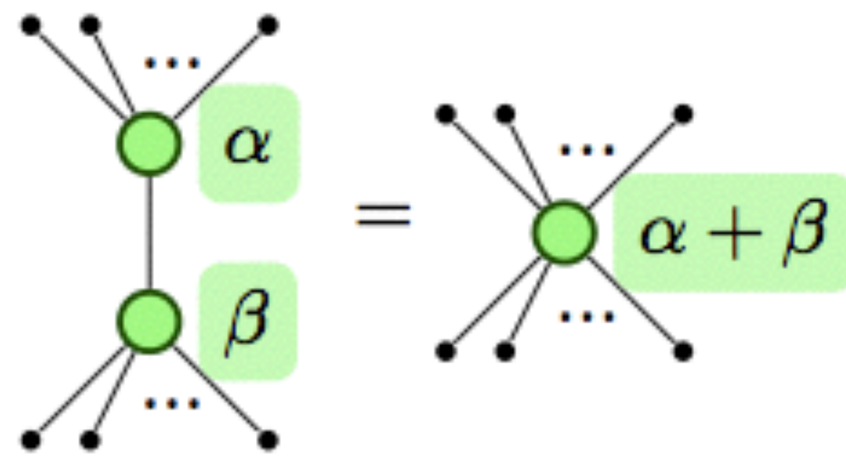
Translating circuits

Steane code encoder:

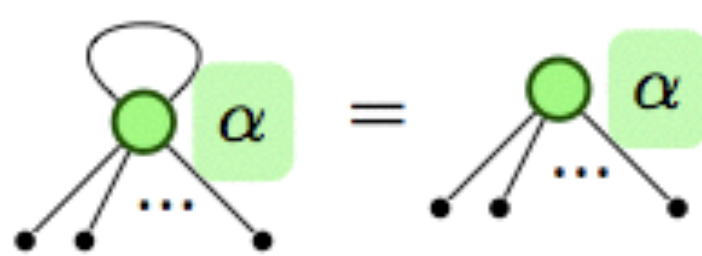




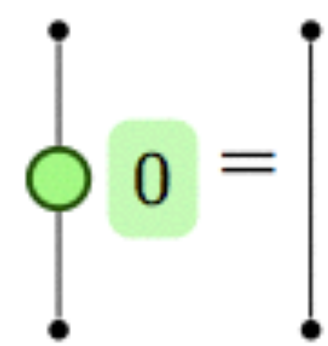
Equations



(spider)



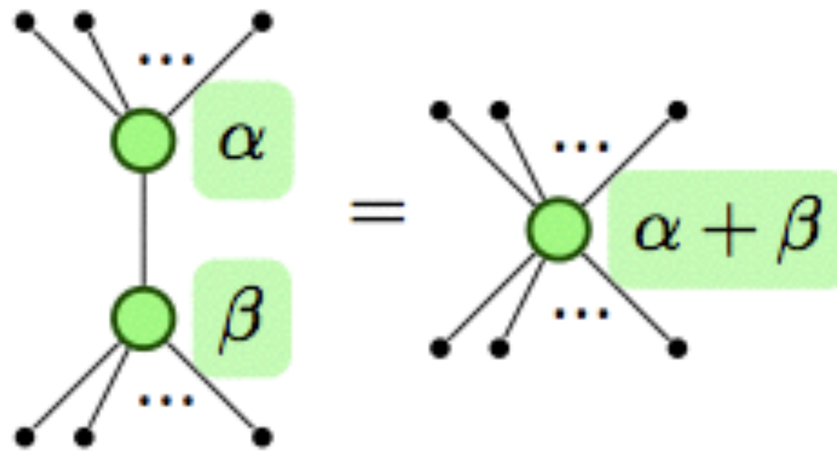
(anti-loop)



(identity)

Equations

Generalised Spider



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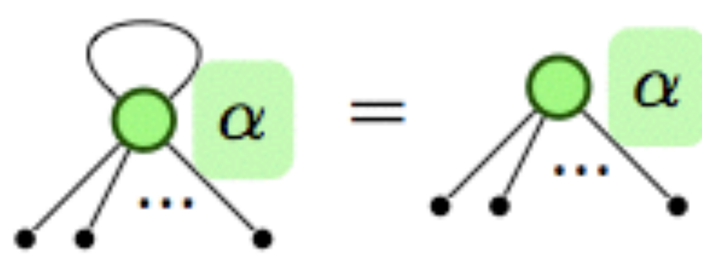

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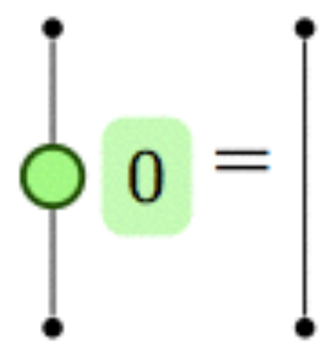

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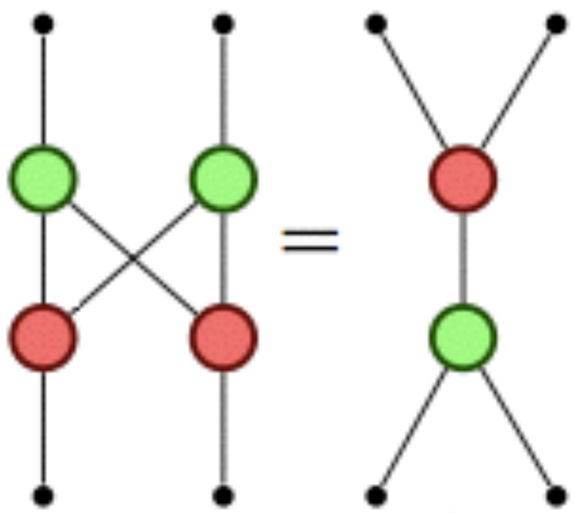
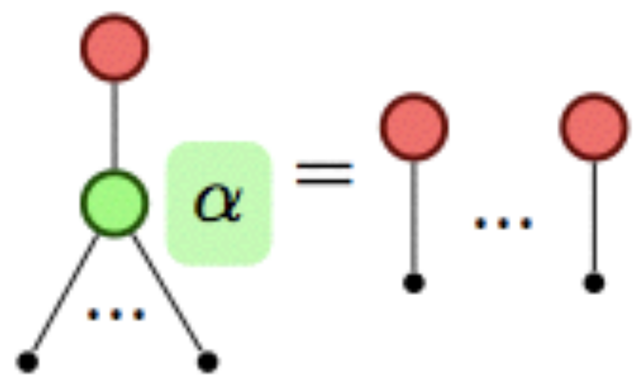


(bialgebra)



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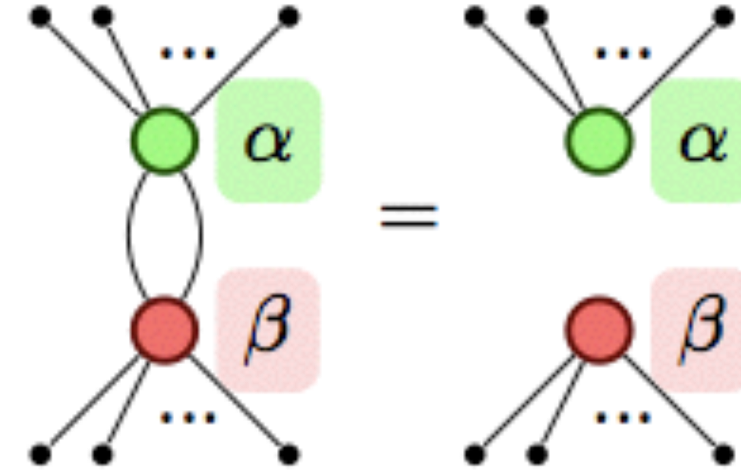
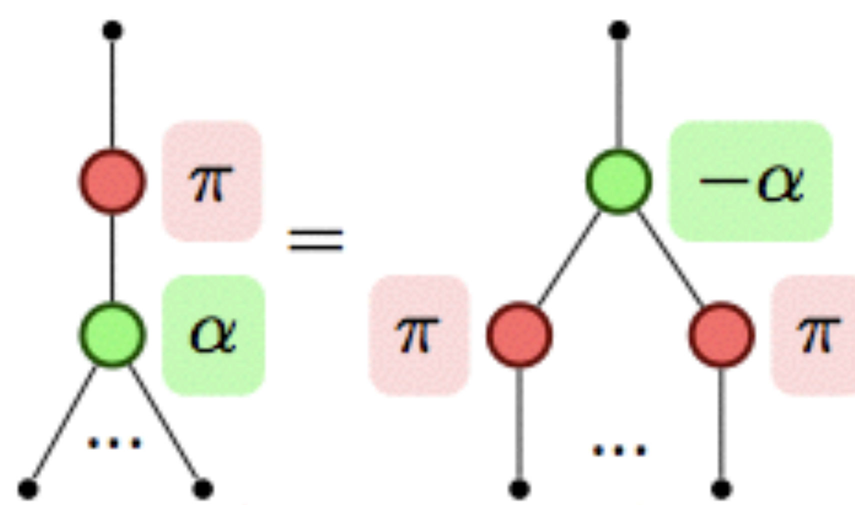


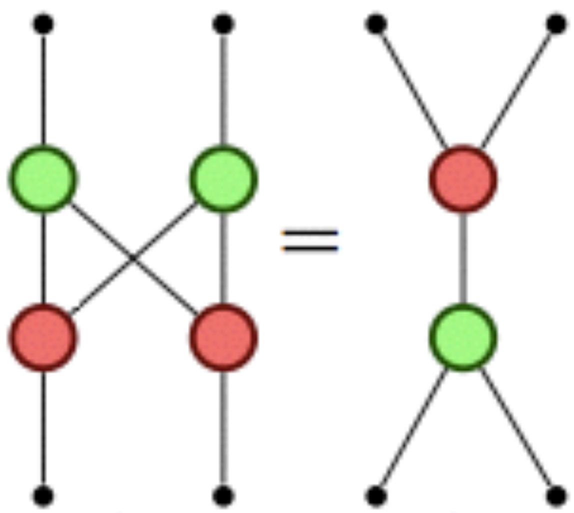
(hopf)



(π-commute)

Equations

“Strong Complementarity”



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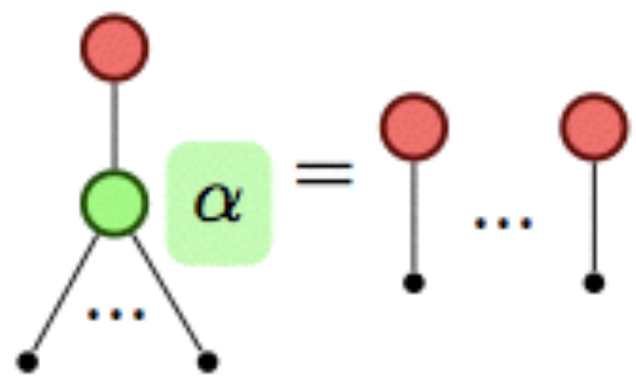

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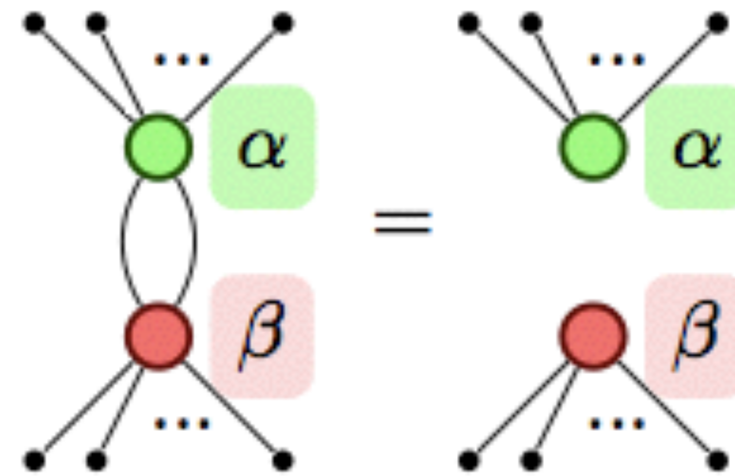
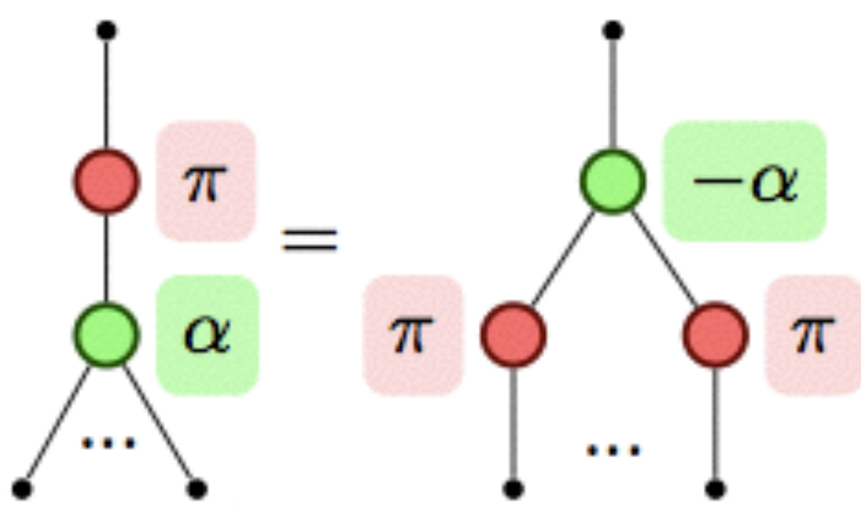
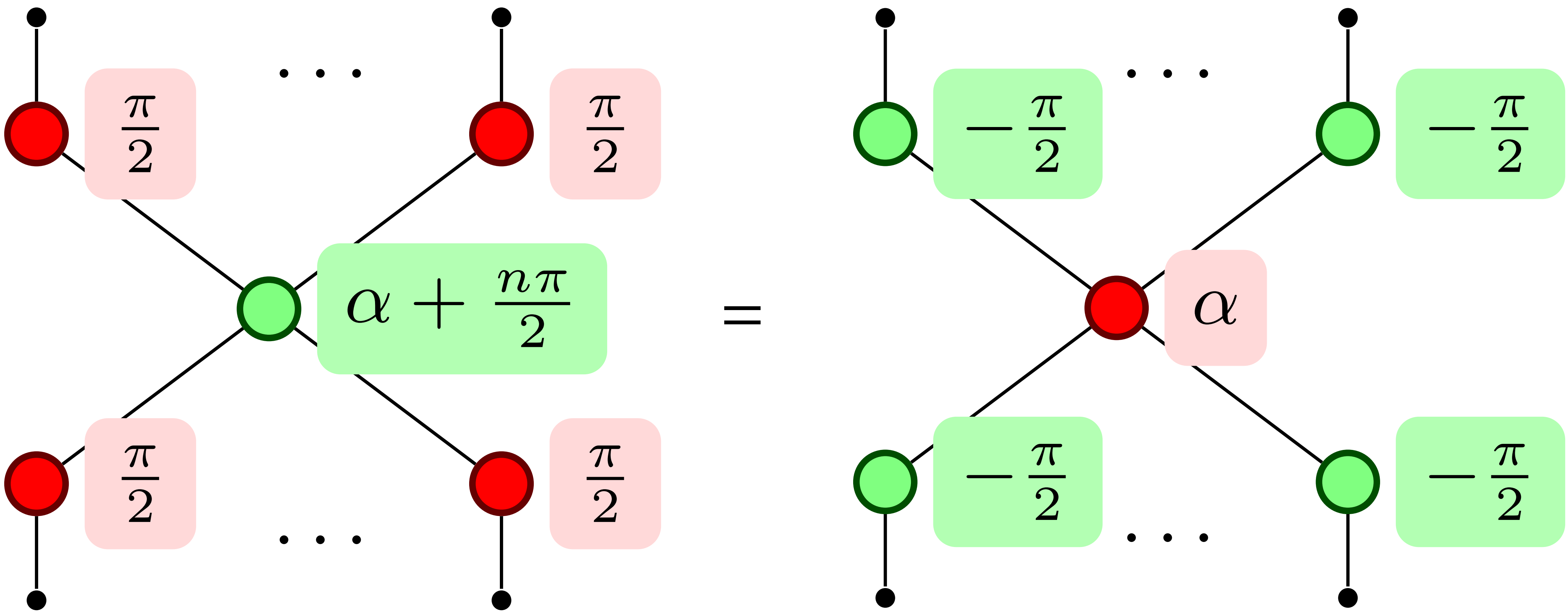


(hopf)



(π-commute)

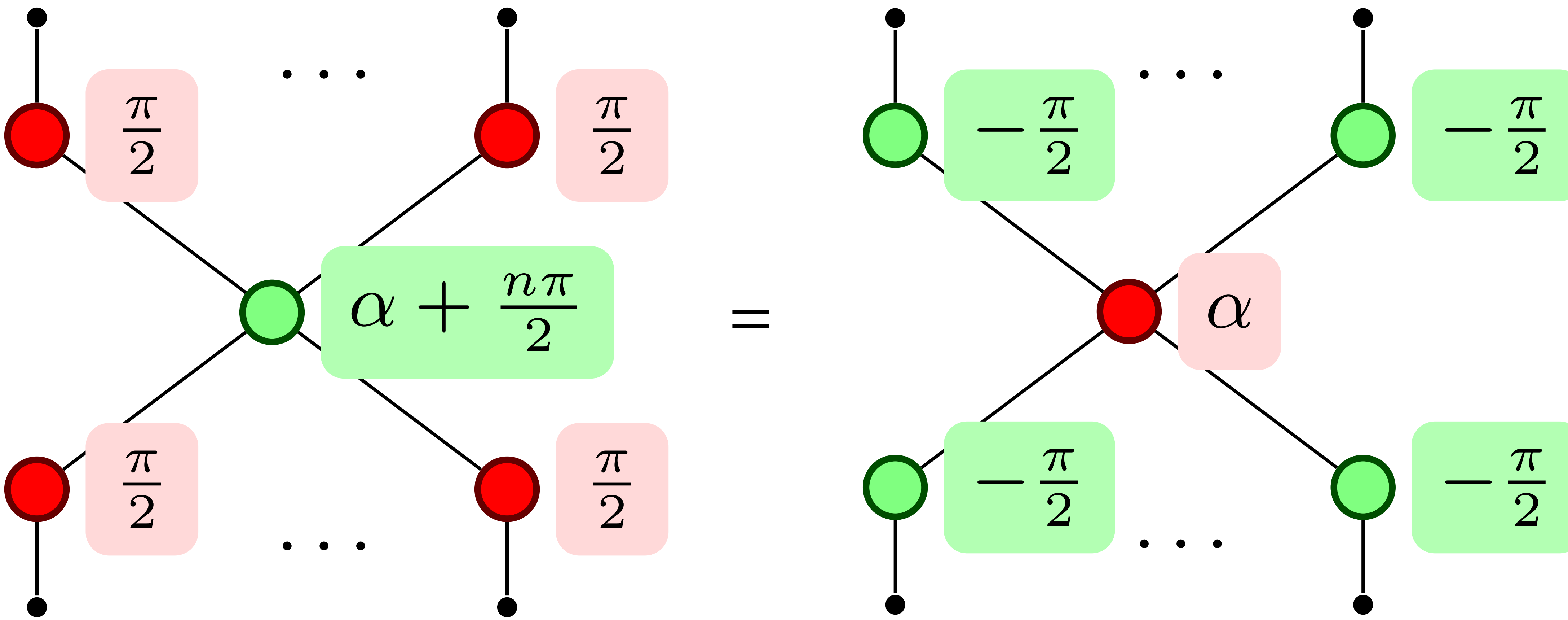
Equations



(colour change)

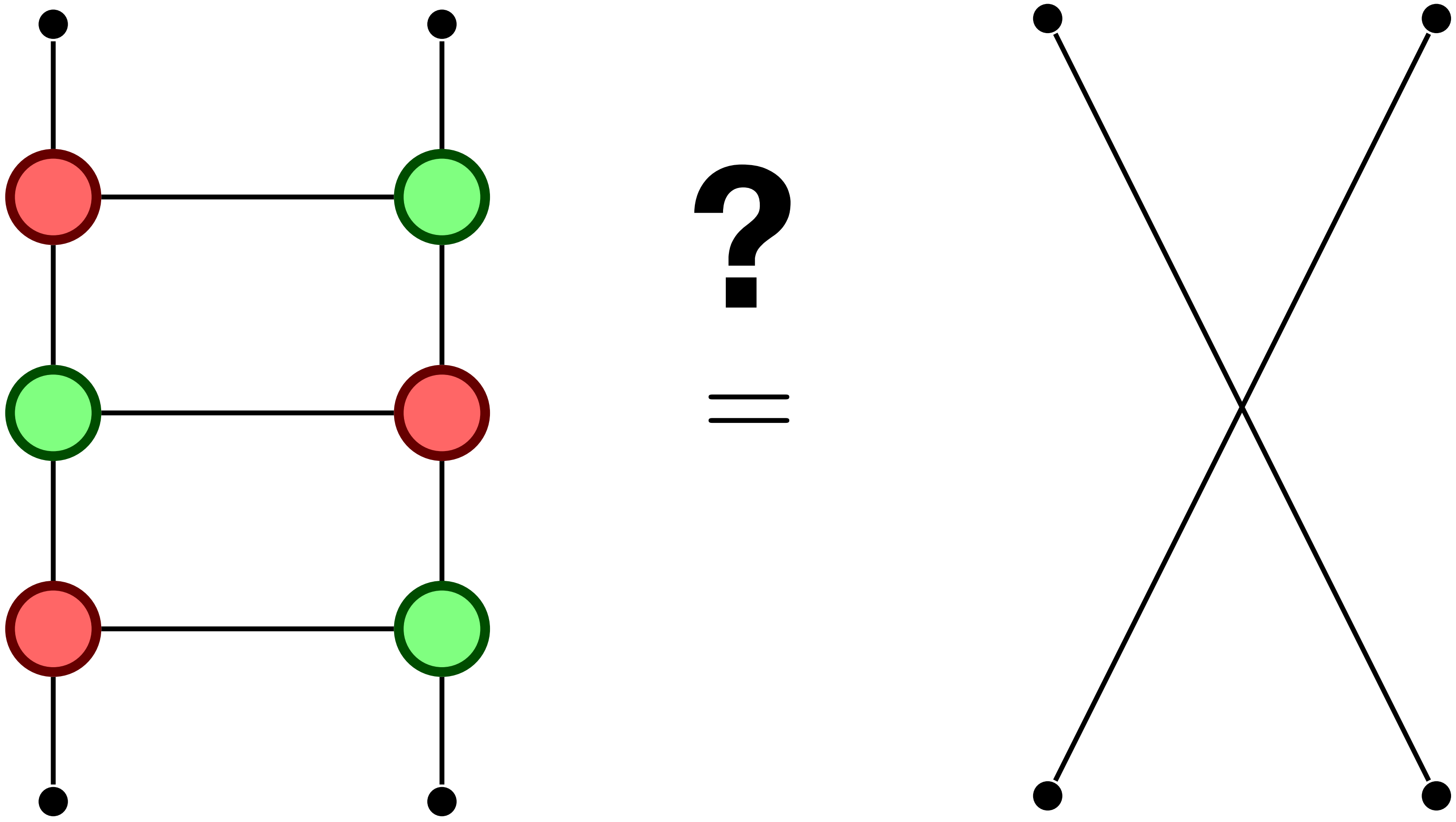
Equations

A weird one specific to ZX

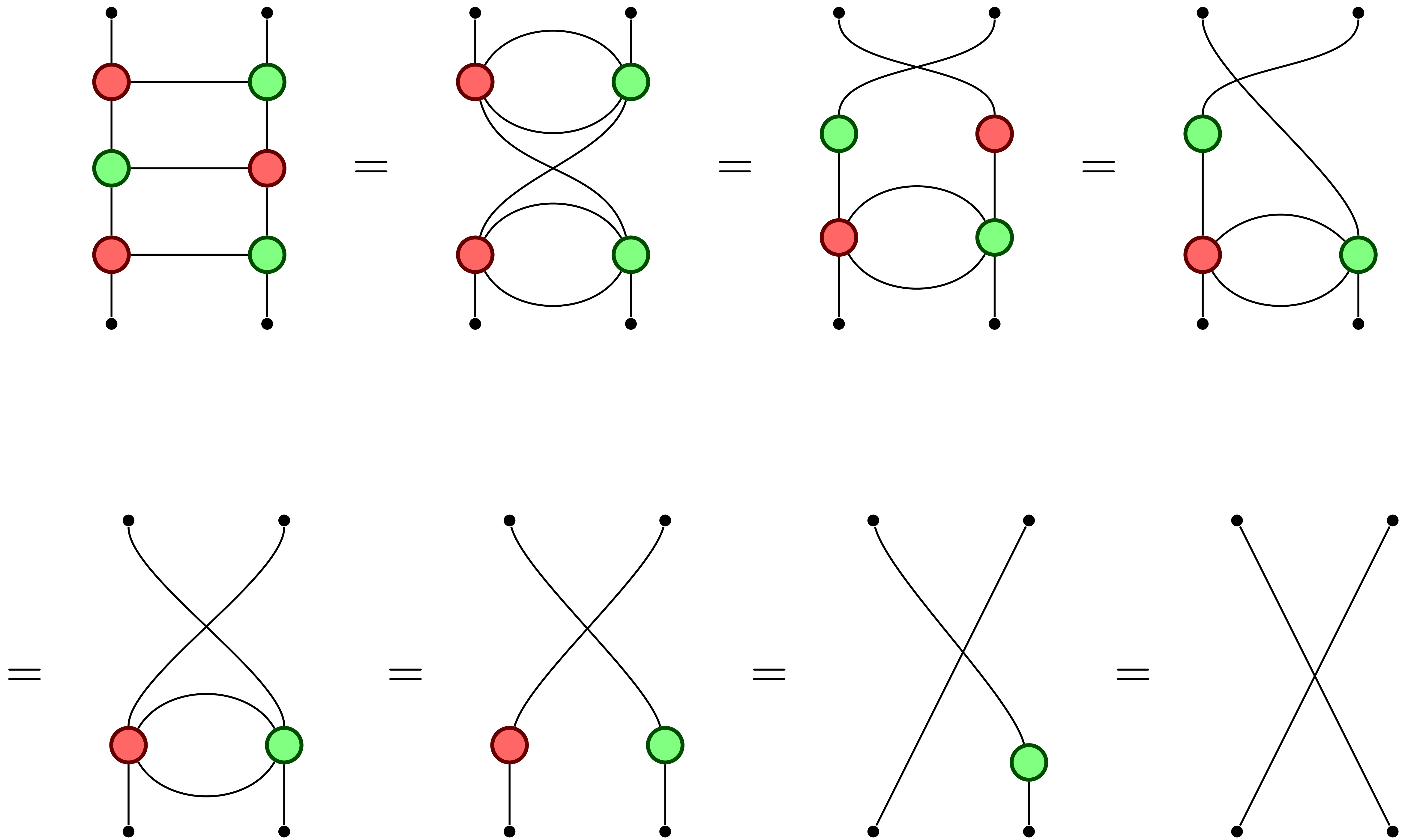


(colour change)

Example: CNOTS



Example: CNOTS

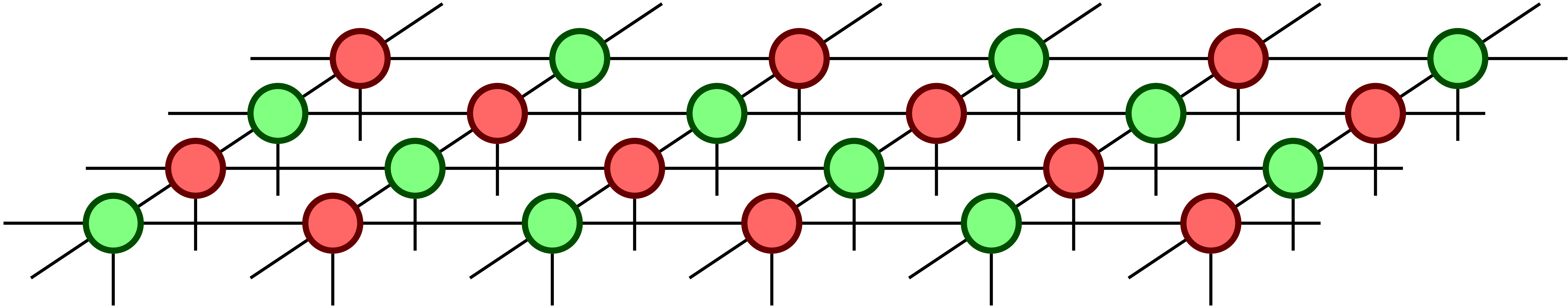


Graph States

Let G = (V,E) be a simple, undirected graph. Then define:

$$|G\rangle = \bigotimes_{(v,u)\in E} CZ_{vu} \bigotimes_{v\in V} |+\rangle$$

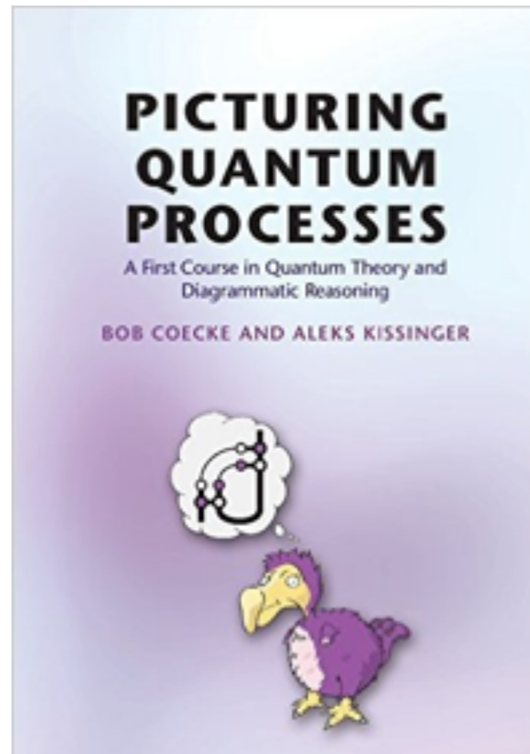
Or in 2D:



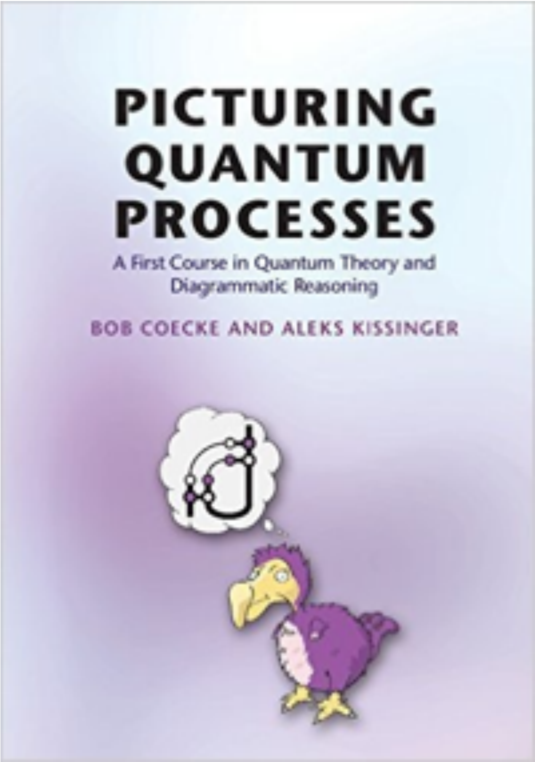
STOP!

QUANTO-TIME!

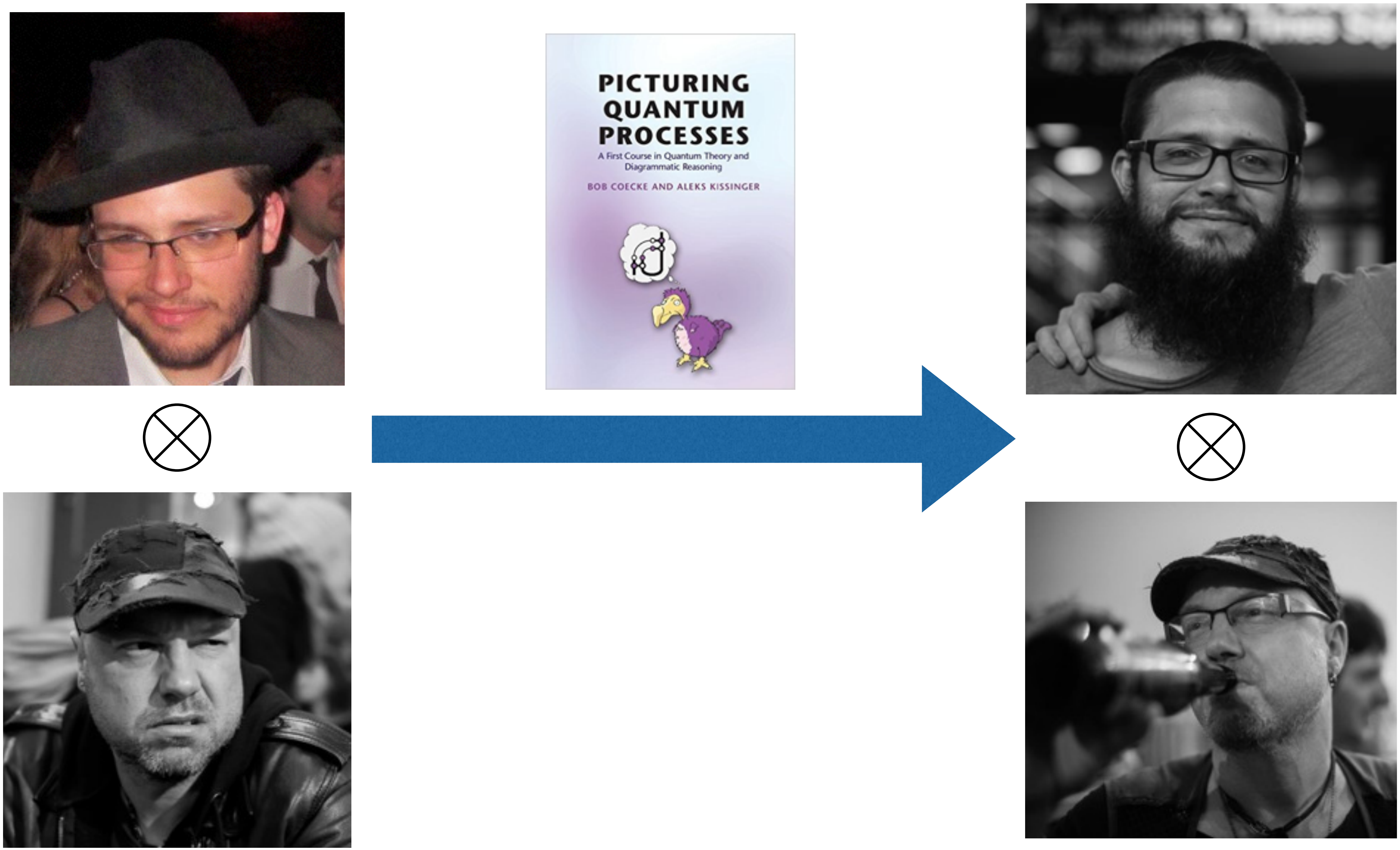
A good reference



A good reference



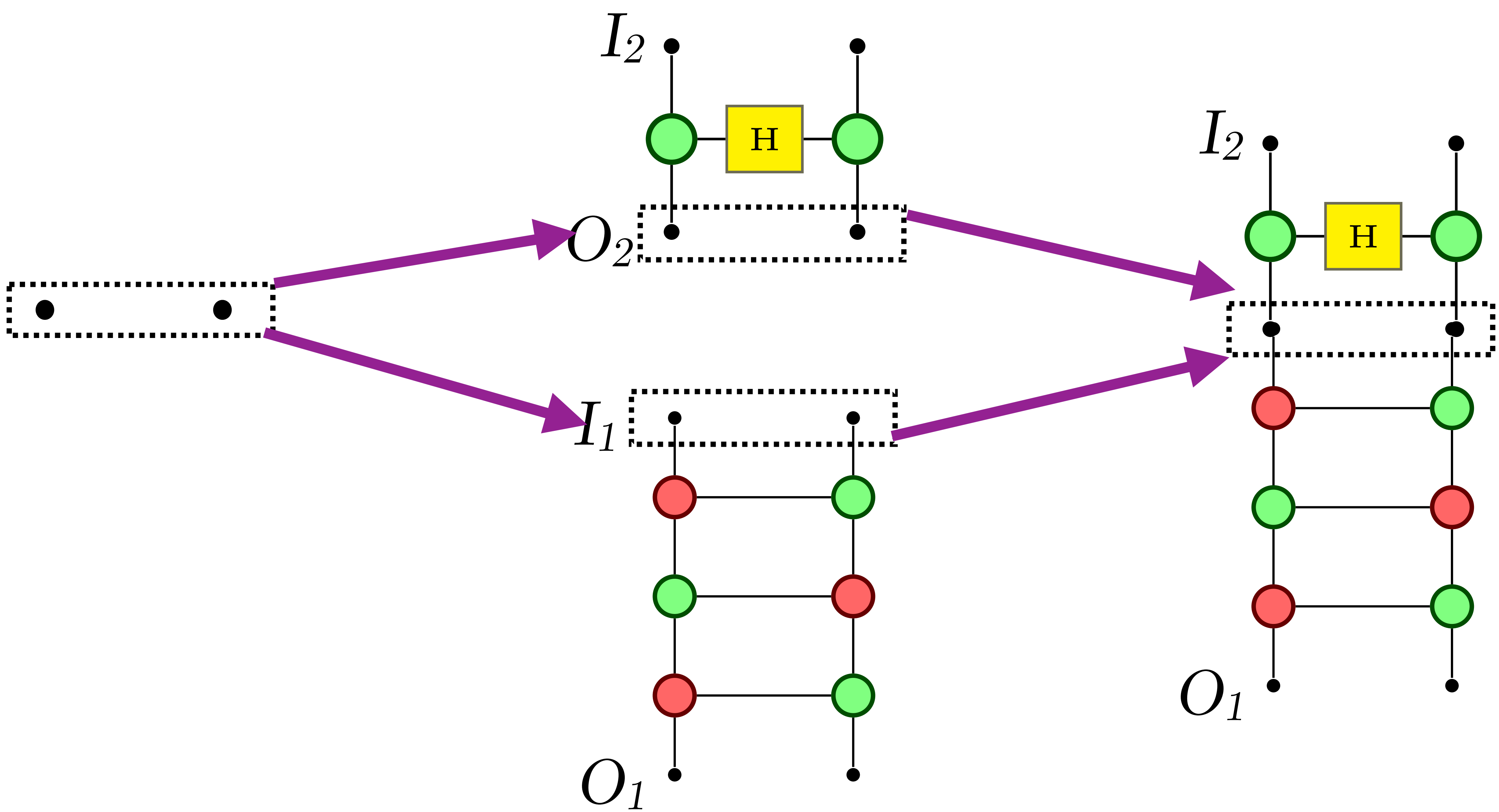
A good reference



2. Composition in graphical syntax

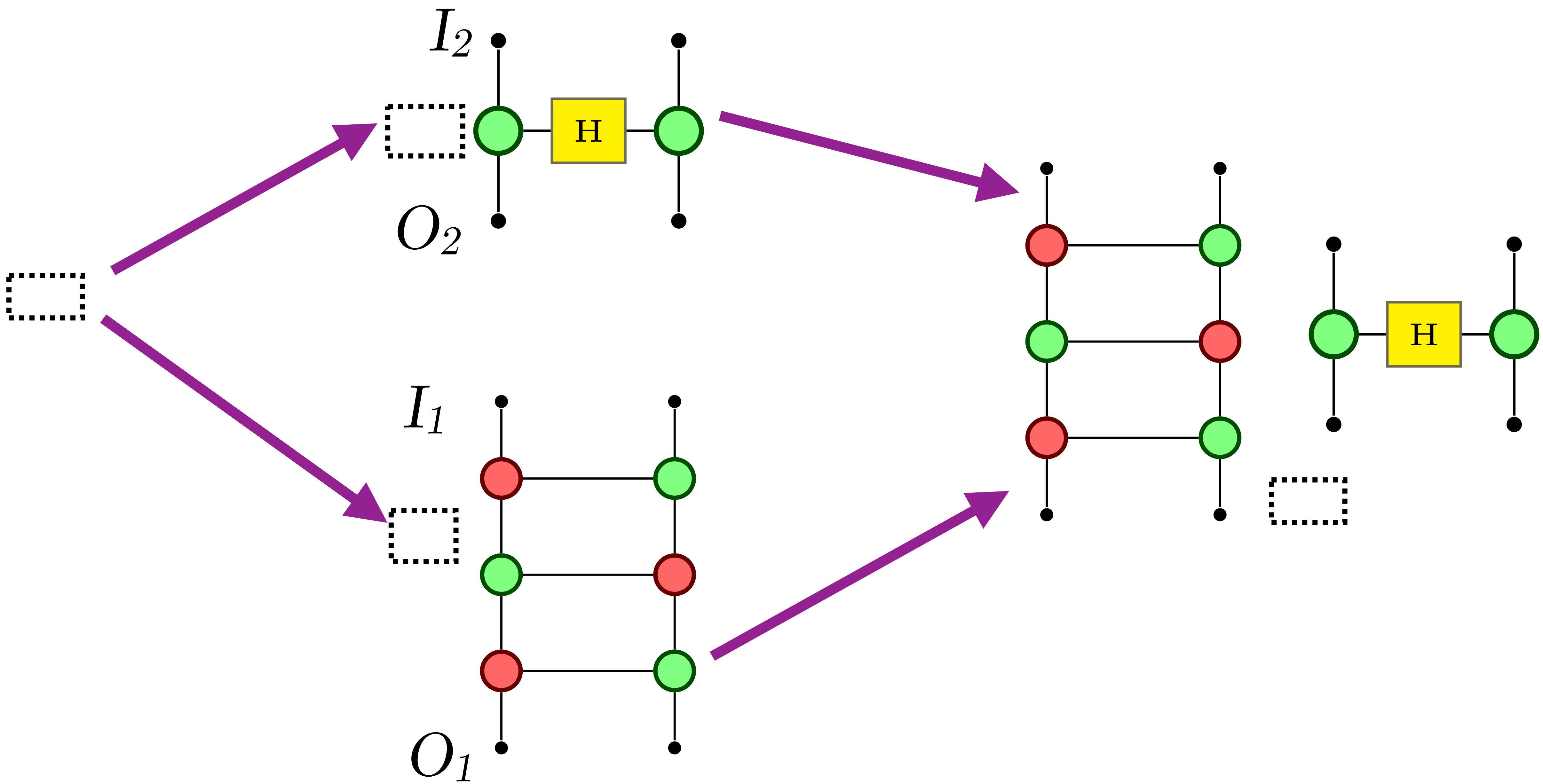
Composing diagrams

- ZX-calculus terms are arrows in PROP
  — Compose them push-out style

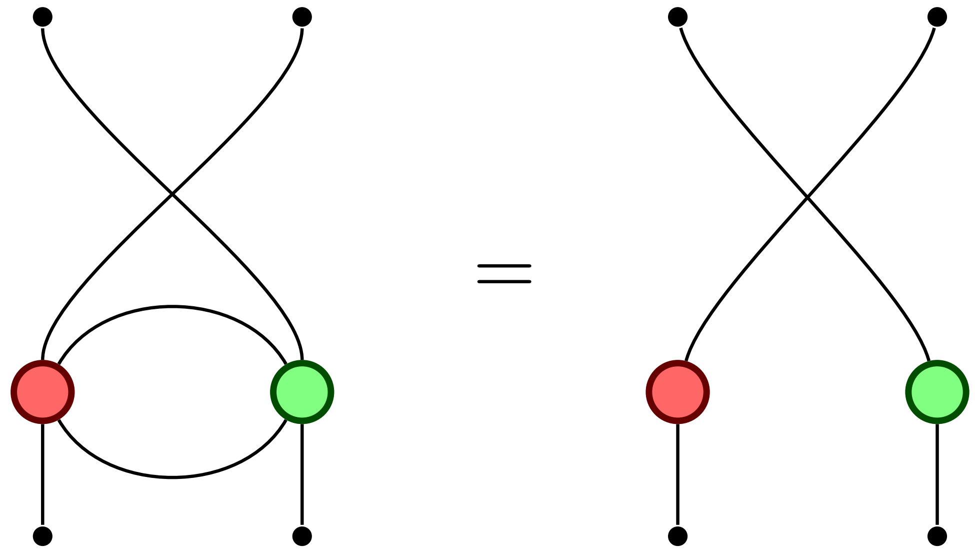


Composing diagrams

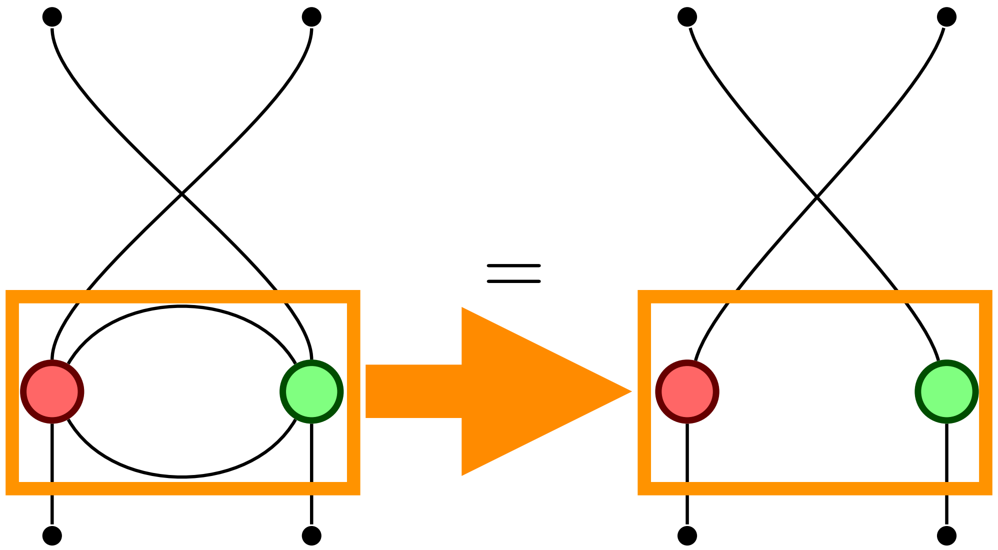
- ZX-calculus terms are arrows in PROP
  — Tensor them push-out style



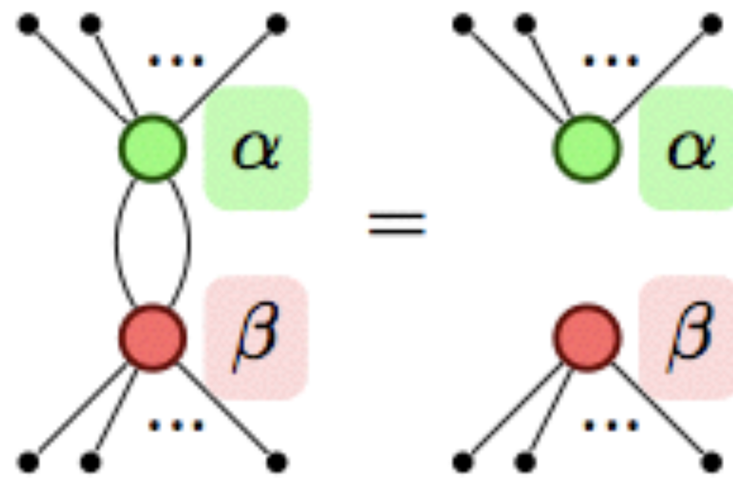
Equational Reasoning



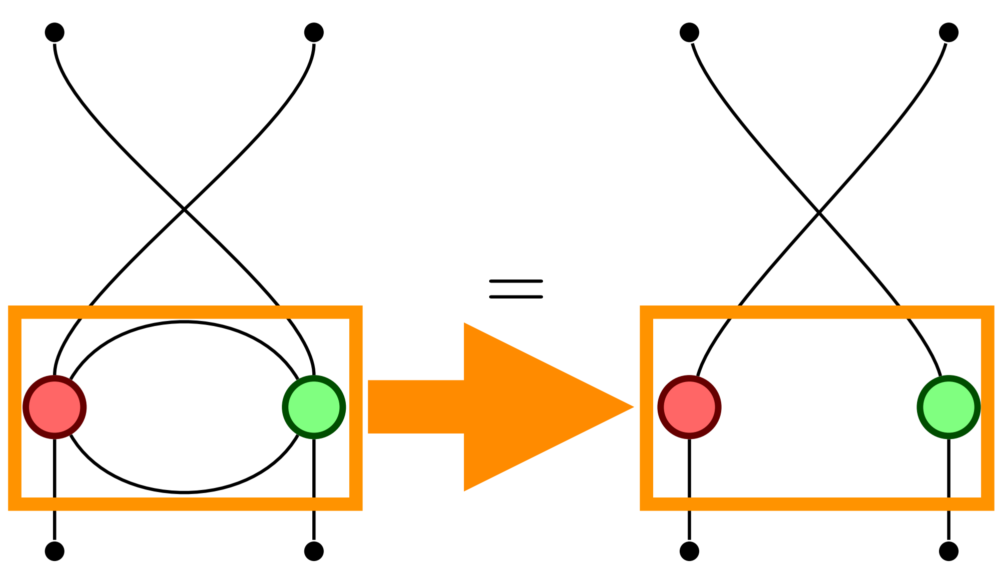
Equational Reasoning



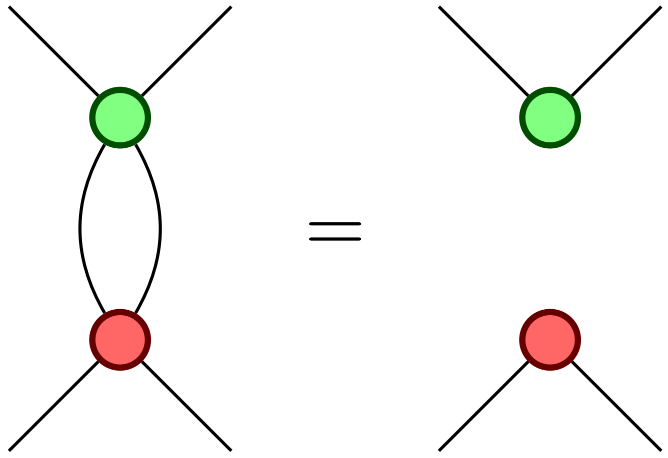
Equational Reasoning

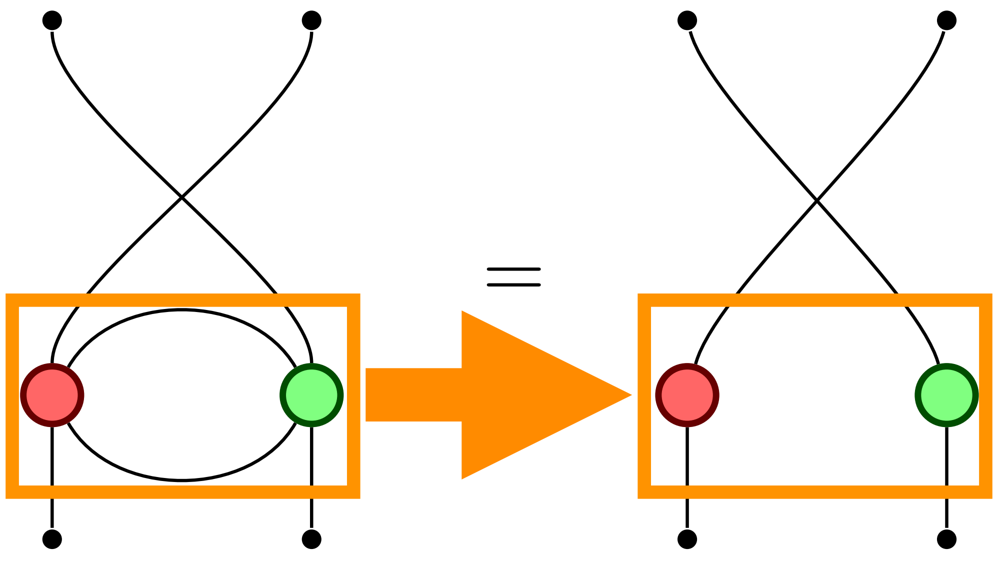


(hopf)

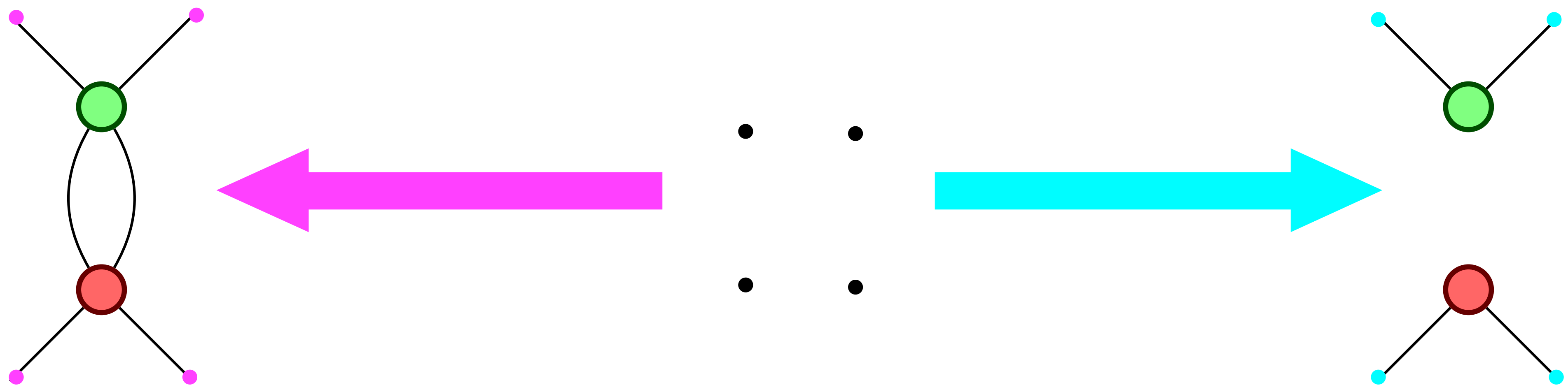


Equational Reasoning

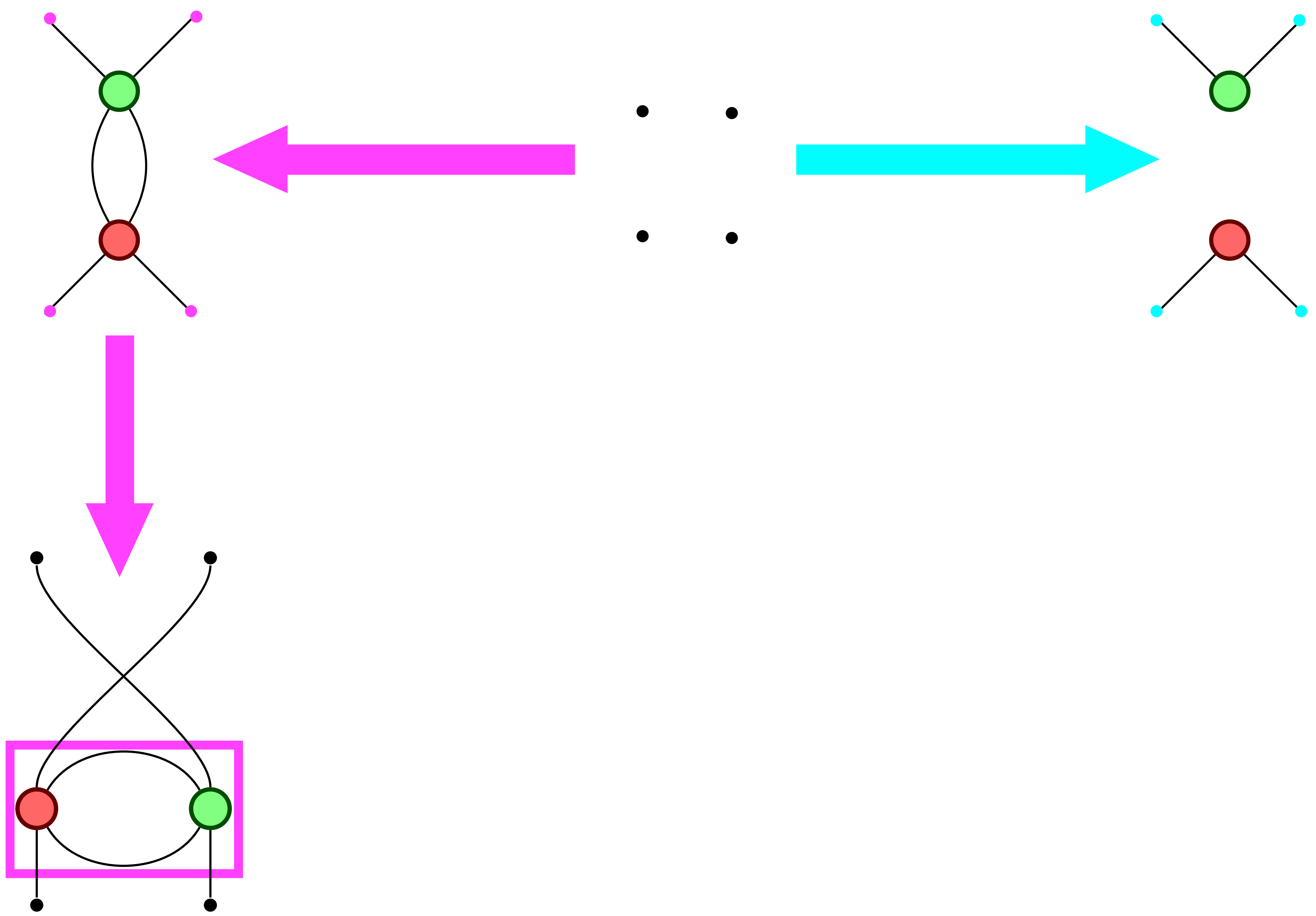




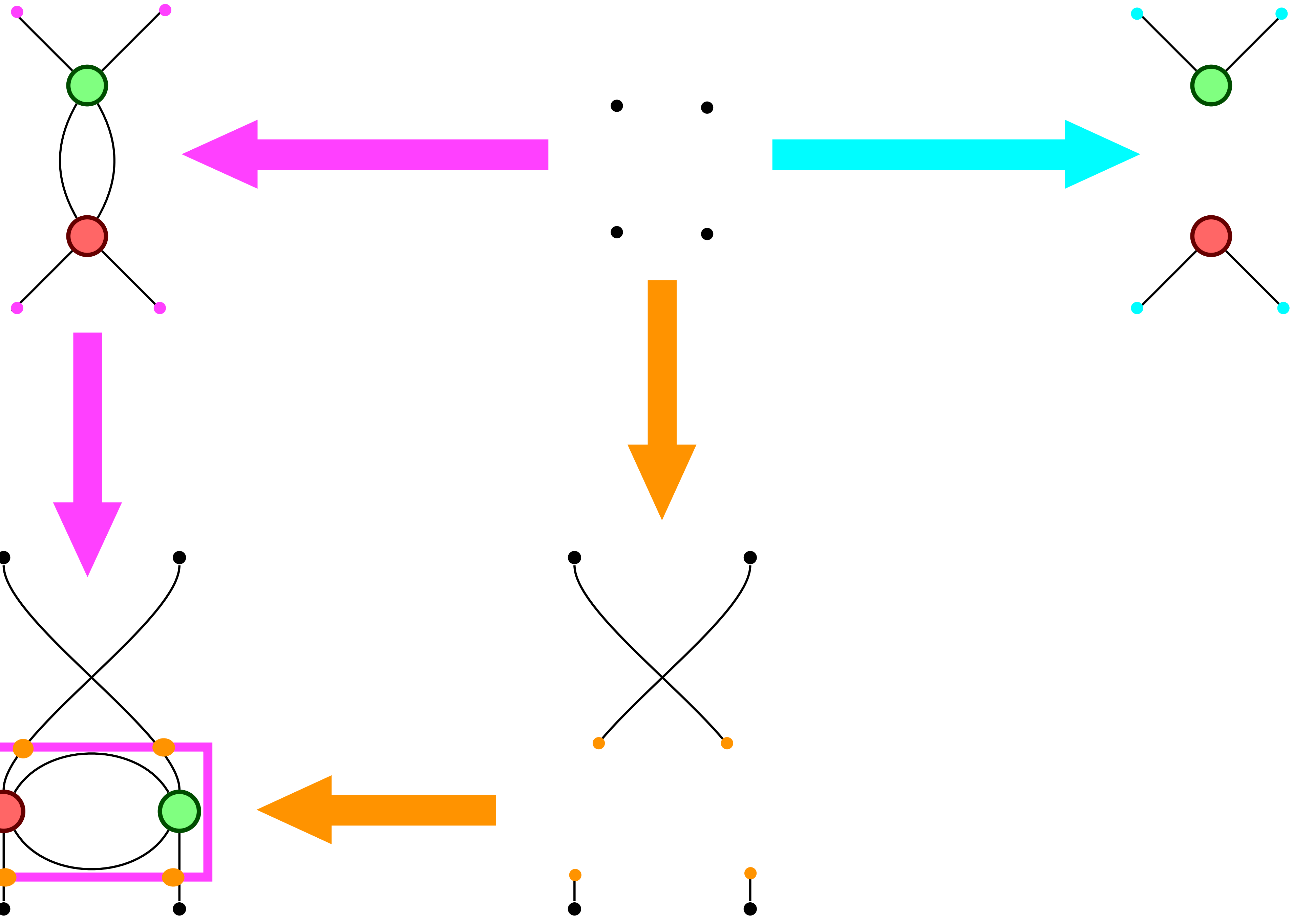
Equational Reasoning



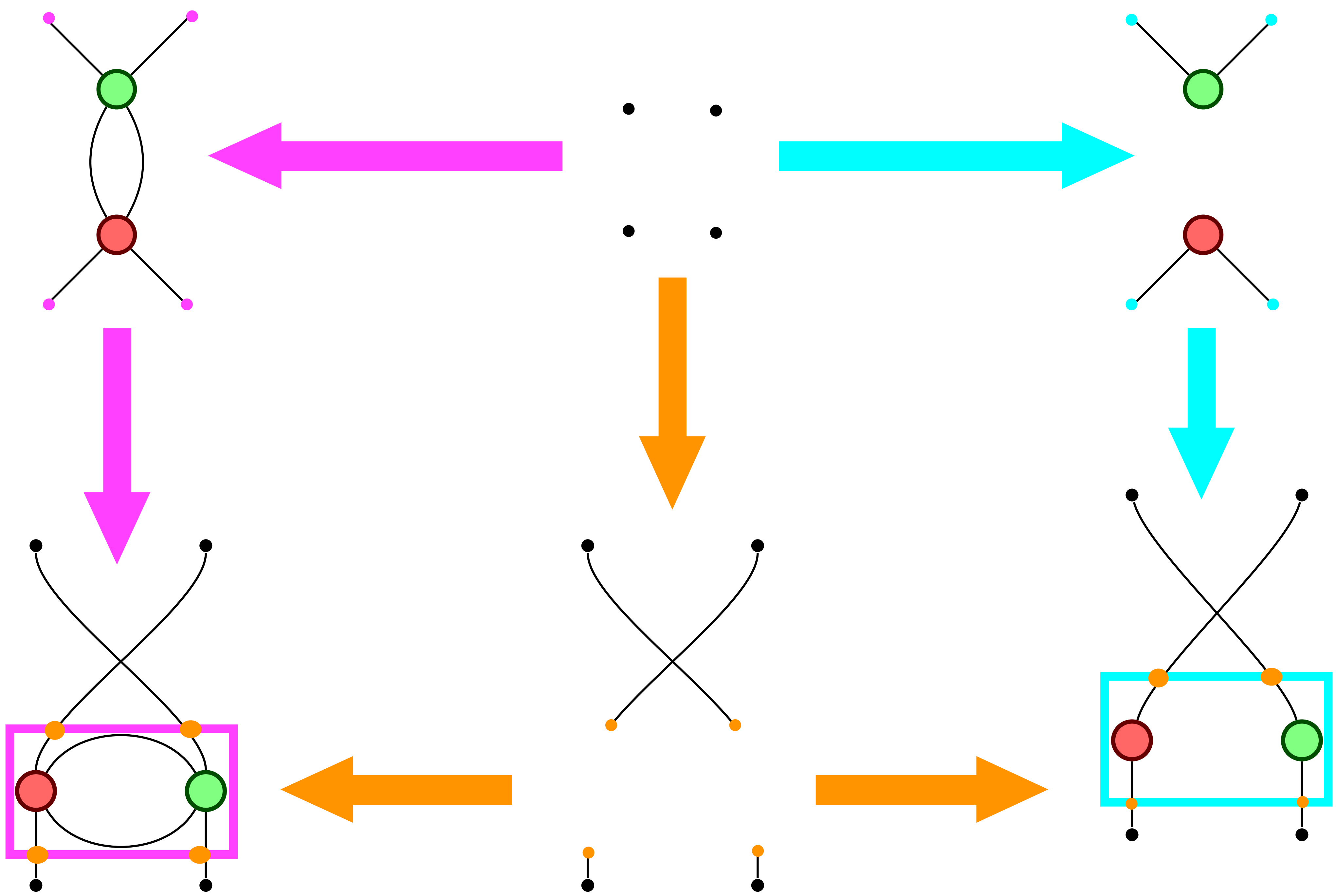
Equational Reasoning



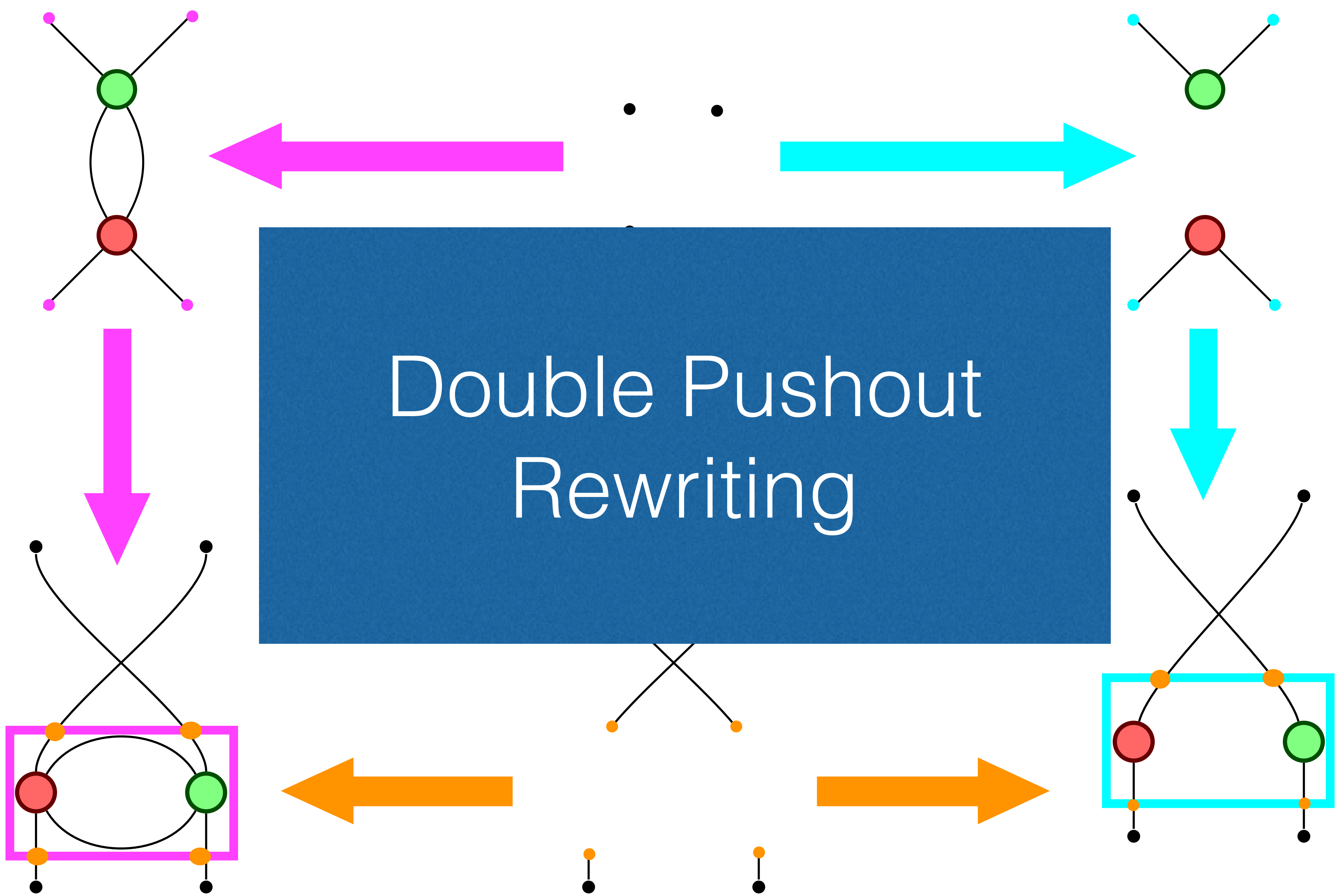
Equational Reasoning



Equational Reasoning



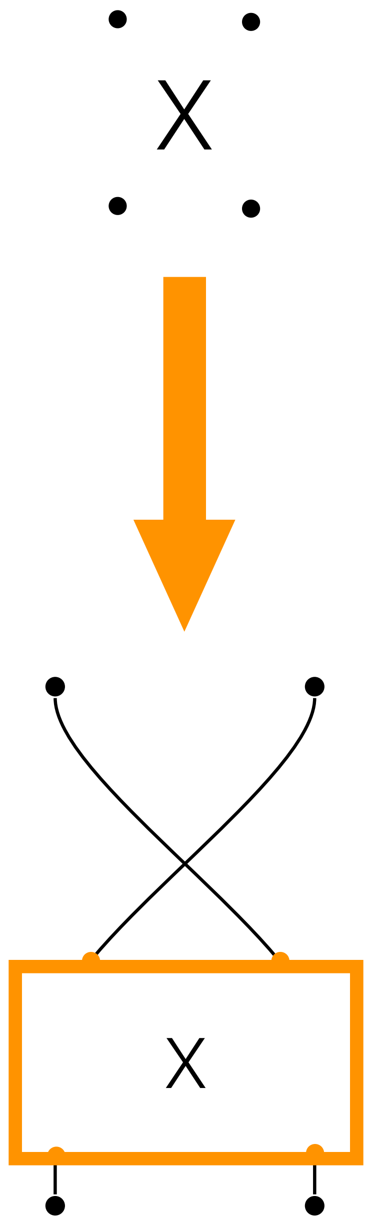
Equational Reasoning



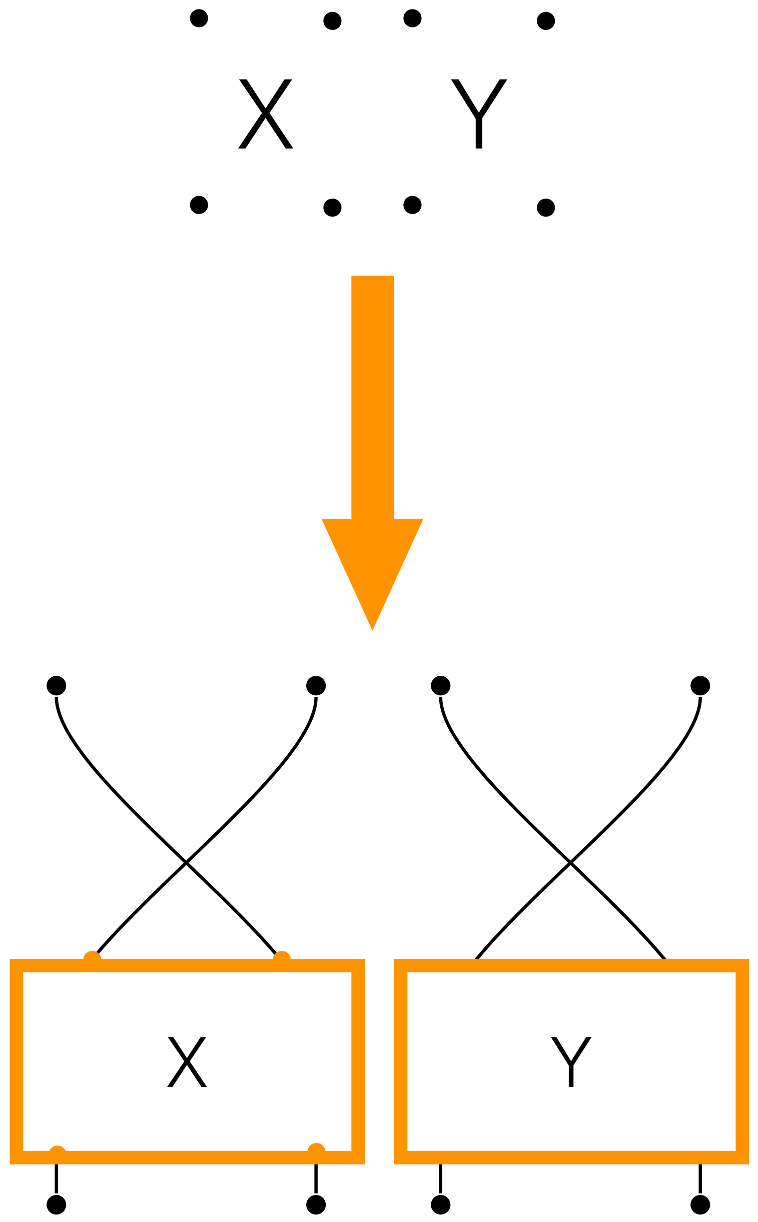
Equational Reasoning

X

Equational Reasoning



Equational Reasoning



3. Composite Theories

I learned all this from Pawel: thanks mate!

PROPs

Defn. A *PROP* is a strict symmetric monoidal category whose objects are the natural numbers.

Defn. A †-*PROP* is a PROP which has a dagger.

Let $\mathbb{T}$ be a PROP and let $\mathbf{C}$ be strict monoidal category.

Defn: a $\mathbb{T}$-algebra in $\mathbf{C}$ is a strict monoidal functor from $\mathbb{T}$ to $\mathbf{C}$.

PROPs

Syntactic presentation of a PROP:

$$(\Sigma, E)$$

Generators
symbols with arity and coarity

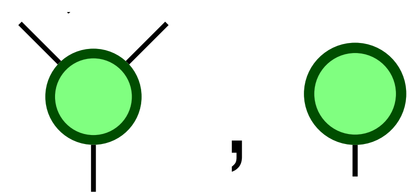
Relations
equations between terms of same type

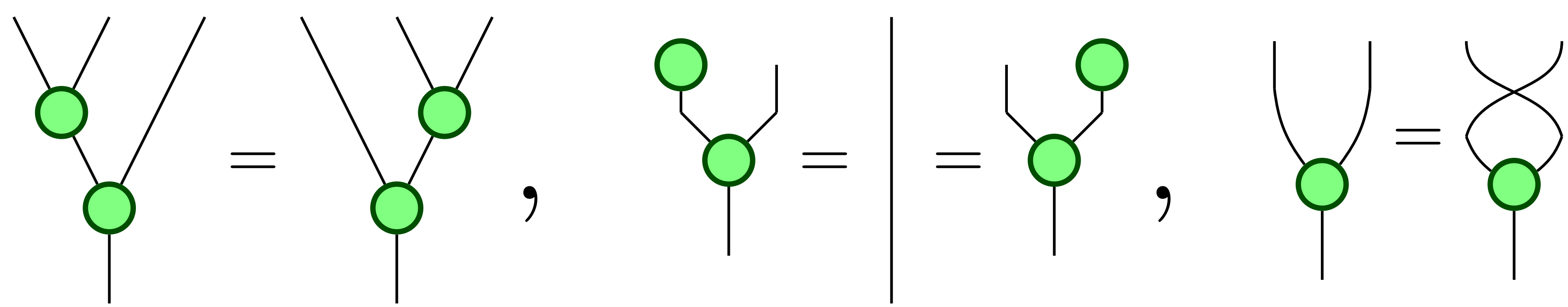
The coproduct of PROPs is very simple:

$$(\Sigma_1, E_1) + (\Sigma_2, E_2) = (\Sigma_1 + \Sigma_2, E_1 + E_2)$$

Example

The PROP of commutative monoids $\mathbb{M}$

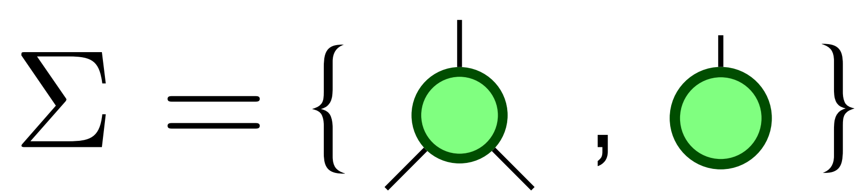
$\Sigma = \{$  $\}$

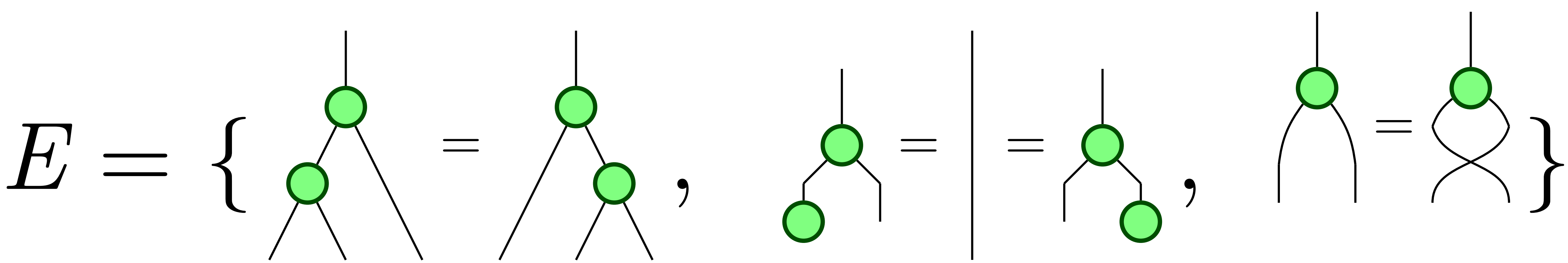
$E = \{$  $\}$

The $\mathbb{M}$-algebras in $\mathbf{C}$ are exactly the monoids of $\mathbf{C}$

Example

The PROP of cocommutative comonoids $\mathbb{M}^{\mathrm{op}}$





The $\mathbb{M}^{\mathrm{op}}$-algebras in $\mathbf{C}$ are the comonoids of $\mathbf{C}$

COMPOSING PROPS

Dedicated to Aurelio Carboni on the occasion of his sixtieth birthday

STEPHEN LACK

ABSTRACT. A PROP is a way of encoding structure borne by an object of a symmetric monoidal category. We describe a notion of *distributive law* for PROPs, based on Beck's distributive laws for monads. A distributive law between PROPs allows them to be composed, and an algebra for the composite PROP consists of a single object with an algebra structure for each of the original PROPs, subject to compatibility conditions encoded by the distributive law. An example is the PROP for bialgebras, which is a composite of the PROP for coalgebras and that for algebras.

Lack, "Composing PROPs", Theory and Applications of Categories 13(9), 2004.

Composing PROPs

PROPs are monads in a certain (complicated) category. Distributive laws of monads produce composite monads — can do this for PROPs!

$$\lambda : \mathbb{T} ; \mathbb{S} \Rightarrow \mathbb{S} ; \mathbb{T}$$

This boils down to an equation

$$n \xrightarrow{t} k \xrightarrow{s} m$$

$$\Downarrow$$

$$n \xrightarrow[s']{} k' \xrightarrow[t']{} m$$

for every composable pair.

Lack, "Composing PROPs", Theory and Applications of Categories 13(9), 2004.

Composing PROPs

Proposition: Given a distributive law

$$\lambda : \mathbb{T} ; \mathbb{S} \Rightarrow \mathbb{S} ; \mathbb{T}$$

Then

$$f : n \to m \;=\; n \xrightarrow{s} k \xrightarrow{t} m$$

Proposition: if $\mathbb{S} = (\Sigma_{\mathbb{S}}, E_{\mathbb{S}})$ $\mathbb{T} = (\Sigma_{\mathbb{T}}, E_{\mathbb{T}})$

then $\mathbb{S} ; \mathbb{T} = (\Sigma_{\mathbb{S}} + \Sigma_{\mathbb{T}}, E_{\mathbb{S}} + E_{\mathbb{T}} + E_{\lambda})$

Lack, "Composing PROPs", Theory and Applications of Categories 13(9), 2004.

Composing PROPs

Proposition: Given a distributive law

$$\lambda : \mathbb{T};\mathbb{S} \Rightarrow \mathbb{S};\mathbb{T}$$

Then

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Proposition: if $\mathbb{S} = (\Sigma_{\mathbb{S}}, E_{\mathbb{S}})$ $\mathbb{T} = (\Sigma_{\mathbb{T}}, E_{\mathbb{T}})$

then $\mathbb{S};\mathbb{T} = (\mathbb{S} + \mathbb{T})/E_{\lambda}$

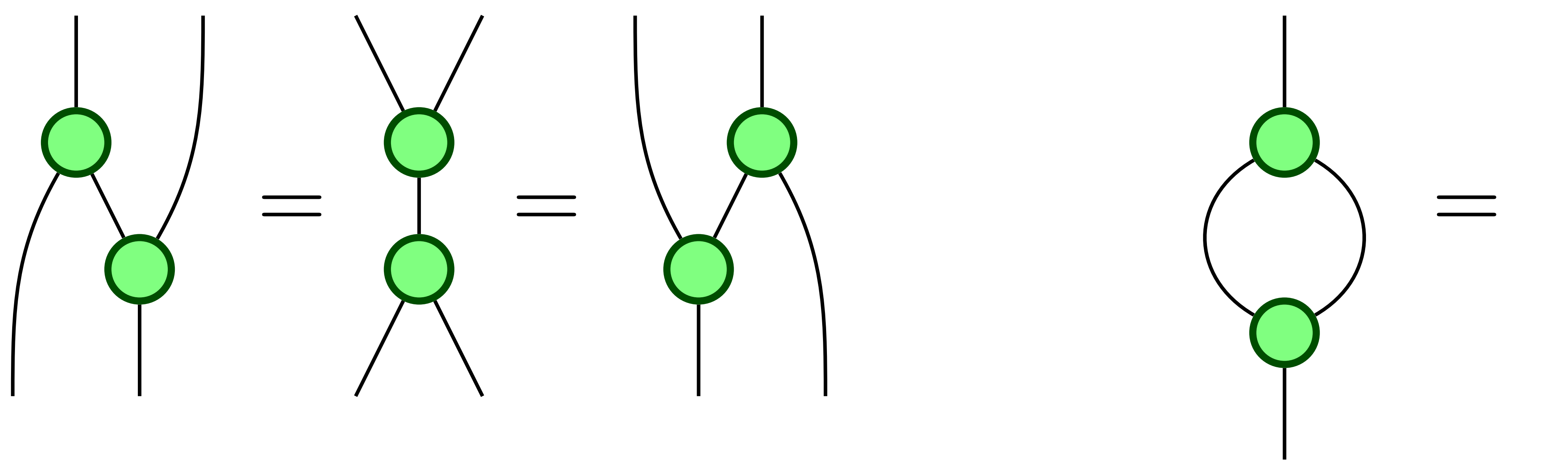
Lack, "Composing PROPs", Theory and Applications of Categories 13(9), 2004.

Frobenius Algebras

The PROP $\mathbb{F}$ of **special commutative Frobenius algebras** arises by a distributive law

$$\lambda_F : \mathbb{M}^{\mathrm{op}};\mathbb{M} \to \mathbb{M};\mathbb{M}^{\mathrm{op}}$$

generated by the equations

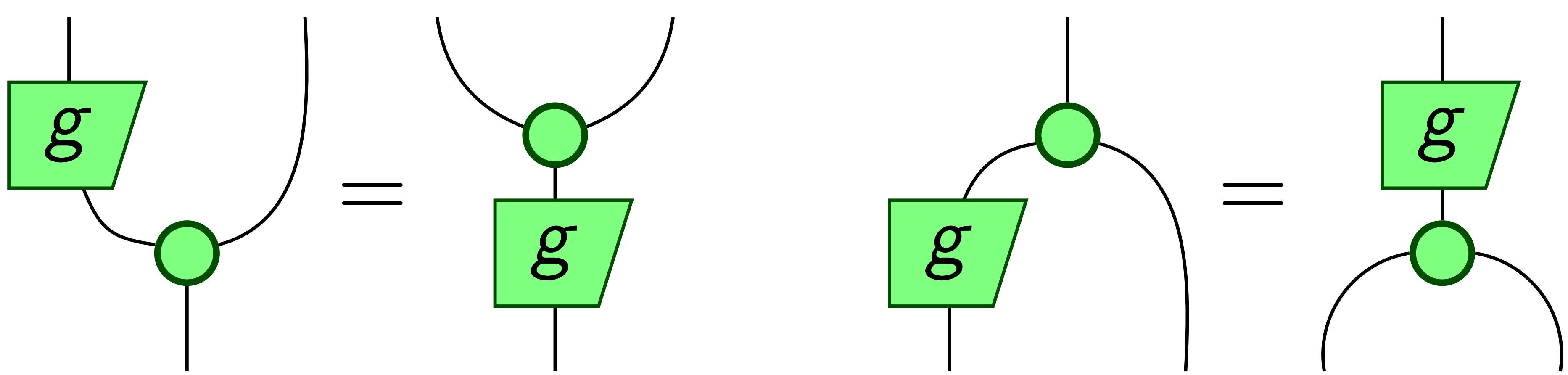


Phases

Let G be an abelian group; define the PROP $G^\times$ by

$$\Sigma = \{g : 1 \to 1 \mid g \in G\} \qquad E = \{g \circ h = gh\}$$

Quotient $\mathbb{F} + G^\times$ by the equations



(P)

Frob. algebras with phases

Recall $\mathbb{F}$ is itself a composite $\mathbb{M};\mathbb{M}^{\mathrm{op}}$ so we can view $\mathbb{F}G$ as an *iterated* distributive law for $\mathbb{M};G^{\times};\mathbb{M}^{\mathrm{op}}$.

This yields a factorisation:

$$f = n \xrightarrow[\mathbb{M}]{\nabla} m \xrightarrow[G^{\times}]{g} m \xrightarrow[\mathbb{M}^{\mathrm{op}}]{\Delta} n'$$

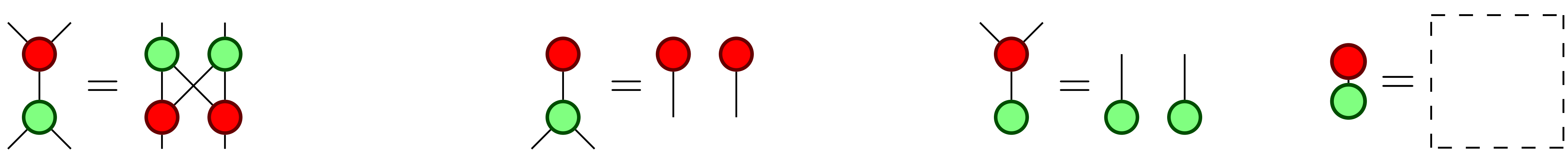
So $\mathbb{F}G$ is the PROP of Frob.algs. with *phases.*

Bialgebras

The PROP $\mathbb{B}$ of **bialgebras** arises by a distributive law

$$\lambda_B : \mathbb{M};\mathbb{M}^{\mathrm{op}} \to \mathbb{M}^{\mathrm{op}};\mathbb{M}$$

generated by the equations

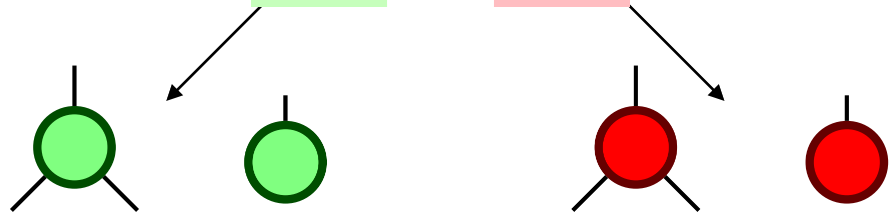


Can do the same for Hopf algebras.

Two Frobenius Algebras?

We can form the coproduct i.e. *non*-interacting Frobenius algebras with phases.

$$\mathbb{F}G + \mathbb{F}H$$



Factorisation:

$$f = n \xrightarrow{g_1} d_1 \xrightarrow{h_1} d_2 \xrightarrow{g_2} d_3 \xrightarrow{h_2} \cdots \xrightarrow{g_k} m$$

Sad Face :(

Theorem: $\mathbb{IF}$ does not arise as a distributive law

$$\lambda : \mathbb{F}G ; \mathbb{F}H \Rightarrow \mathbb{F}H ; \mathbb{F}G$$

Proof: Recall we need:

$$\begin{array}{c} n \xrightarrow{\;t\;} k \xrightarrow{\;s\;} m \\ \Downarrow \\ n \xrightarrow[\;s'\;]{} k' \xrightarrow[\;t'\;]{} m \end{array}$$

for every composable pair — including the phase groups

RD + Kevin Dunne, "Interacting Frobenius Algebras are Hopf", LiCS 2016.

But the news is still pretty good

- No distributive law for ZX-calculus
  — no nice normal forms for the full language
  — this would have been **very** surprising!

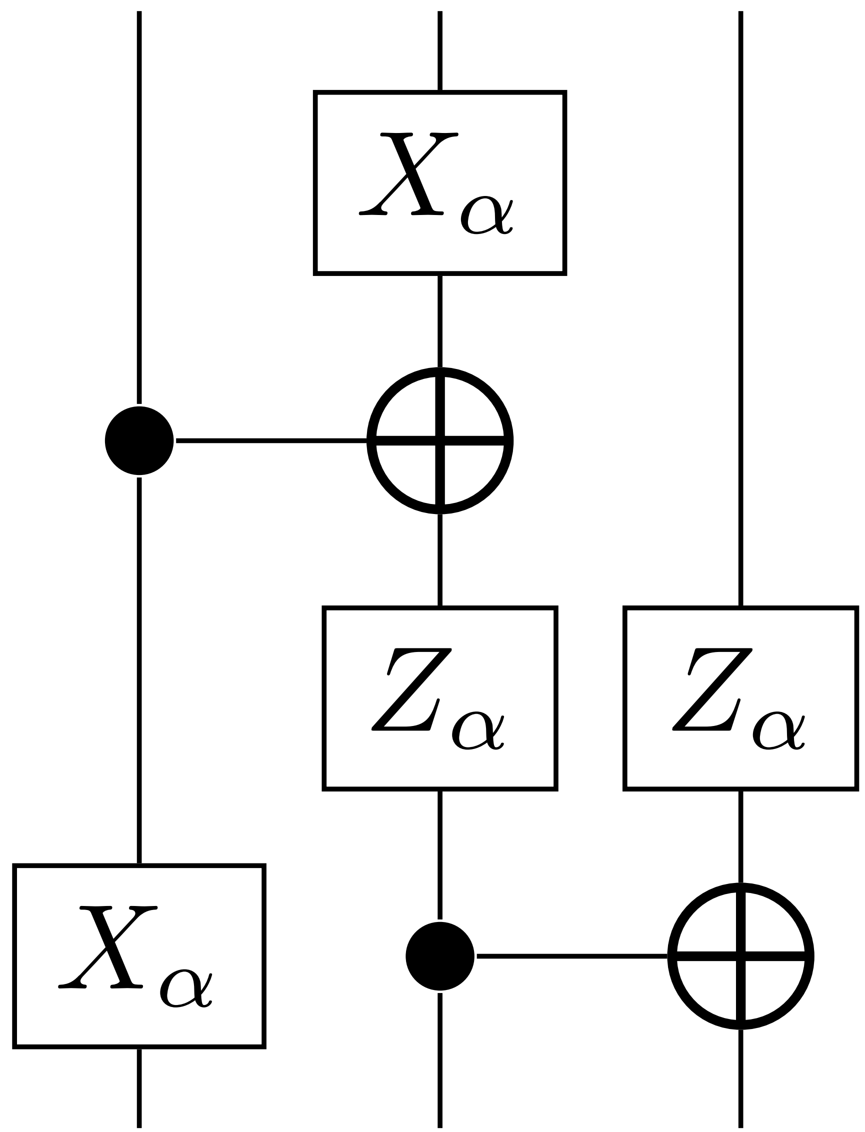
- **But** nice normal forms for every subtheory.
  — the monochrome theory = spiders
  — the phase-free theory = $\mathbb{Z}_2$-matrices
  — the Clifford fragment = *????*

- This will be enough for some interesting applications!

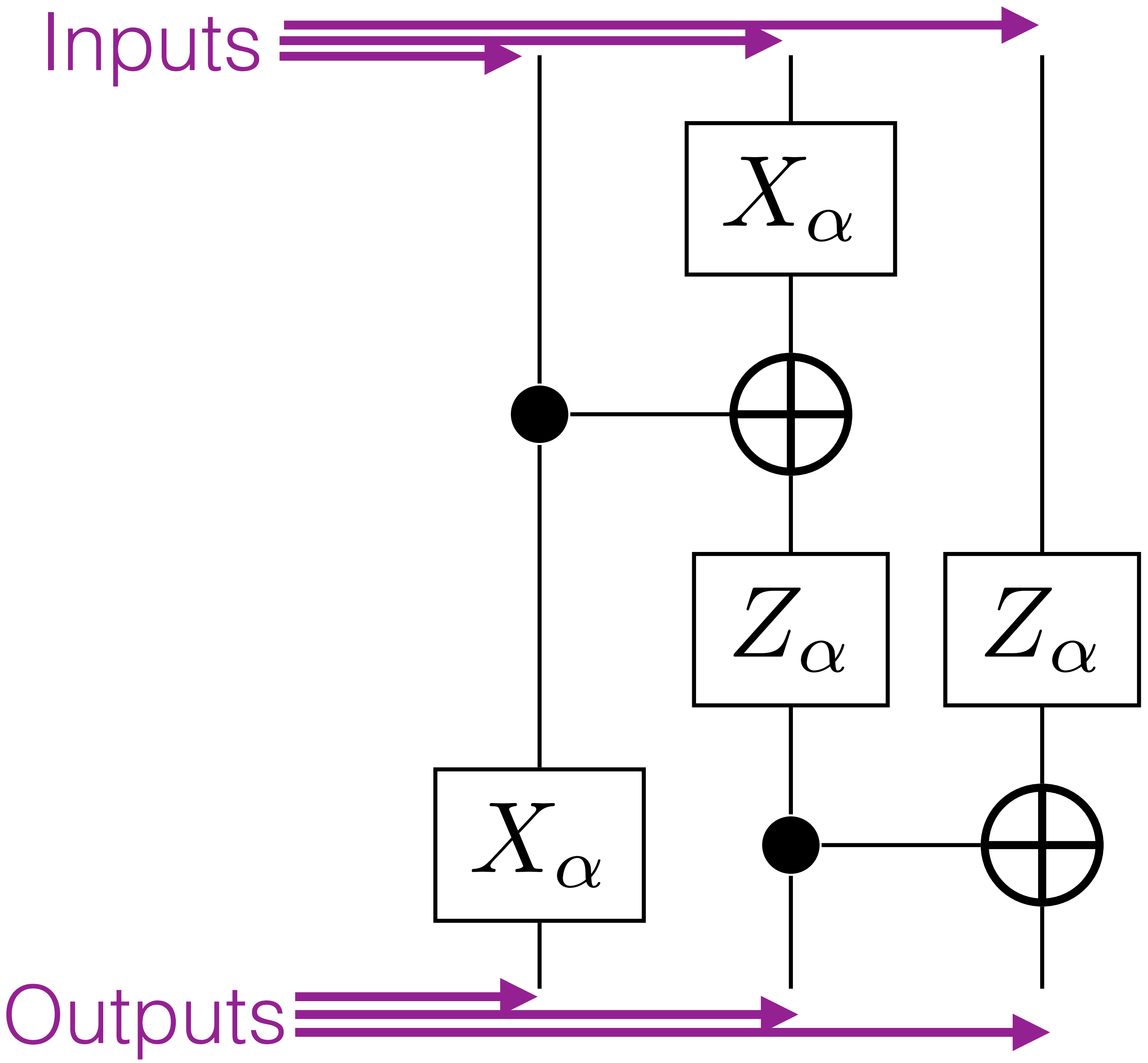
4. Compiling

Oh look, category theory can do something useful!

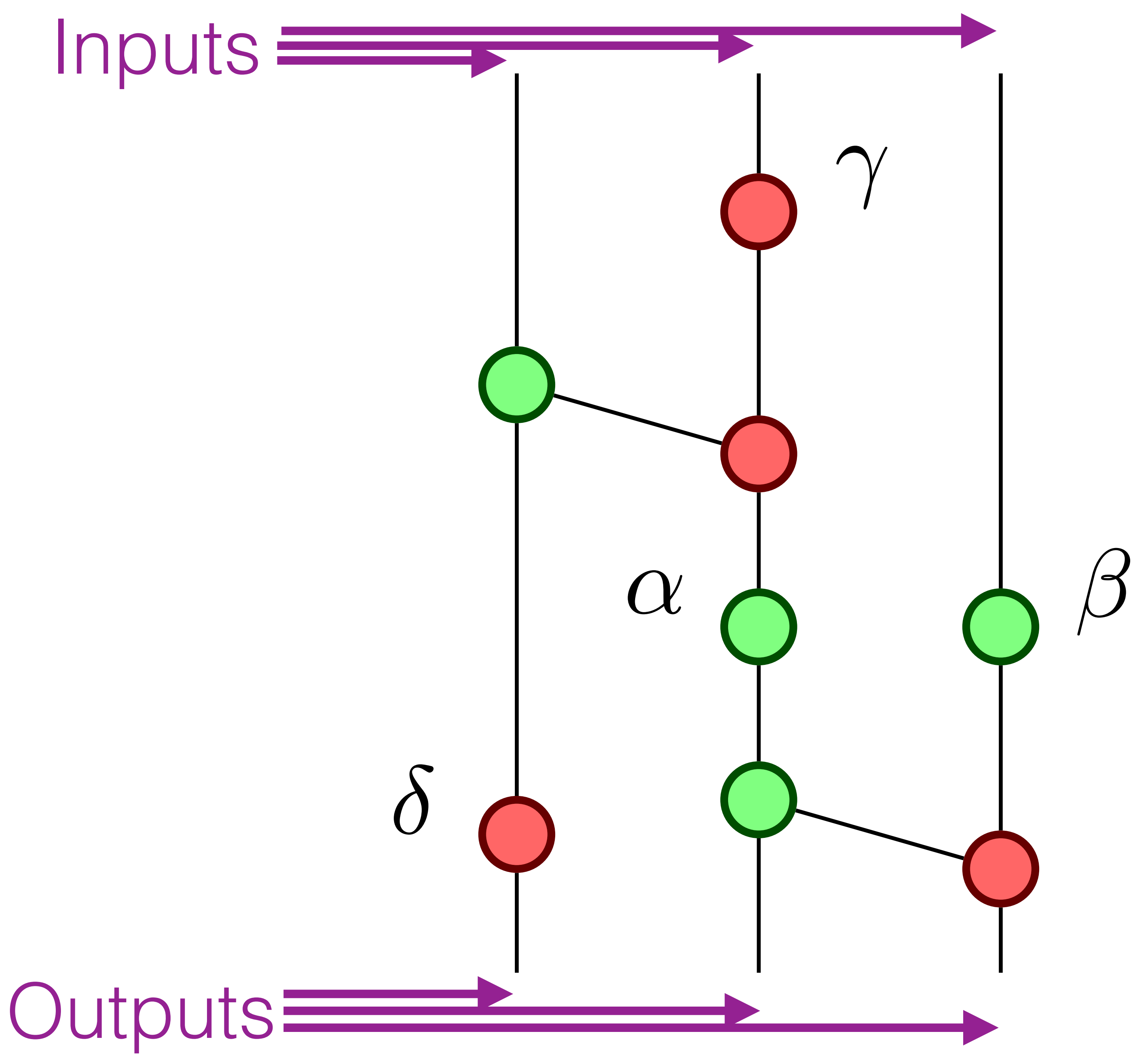
Circuit Perspective



Circuit Perspective



Circuit Perspective



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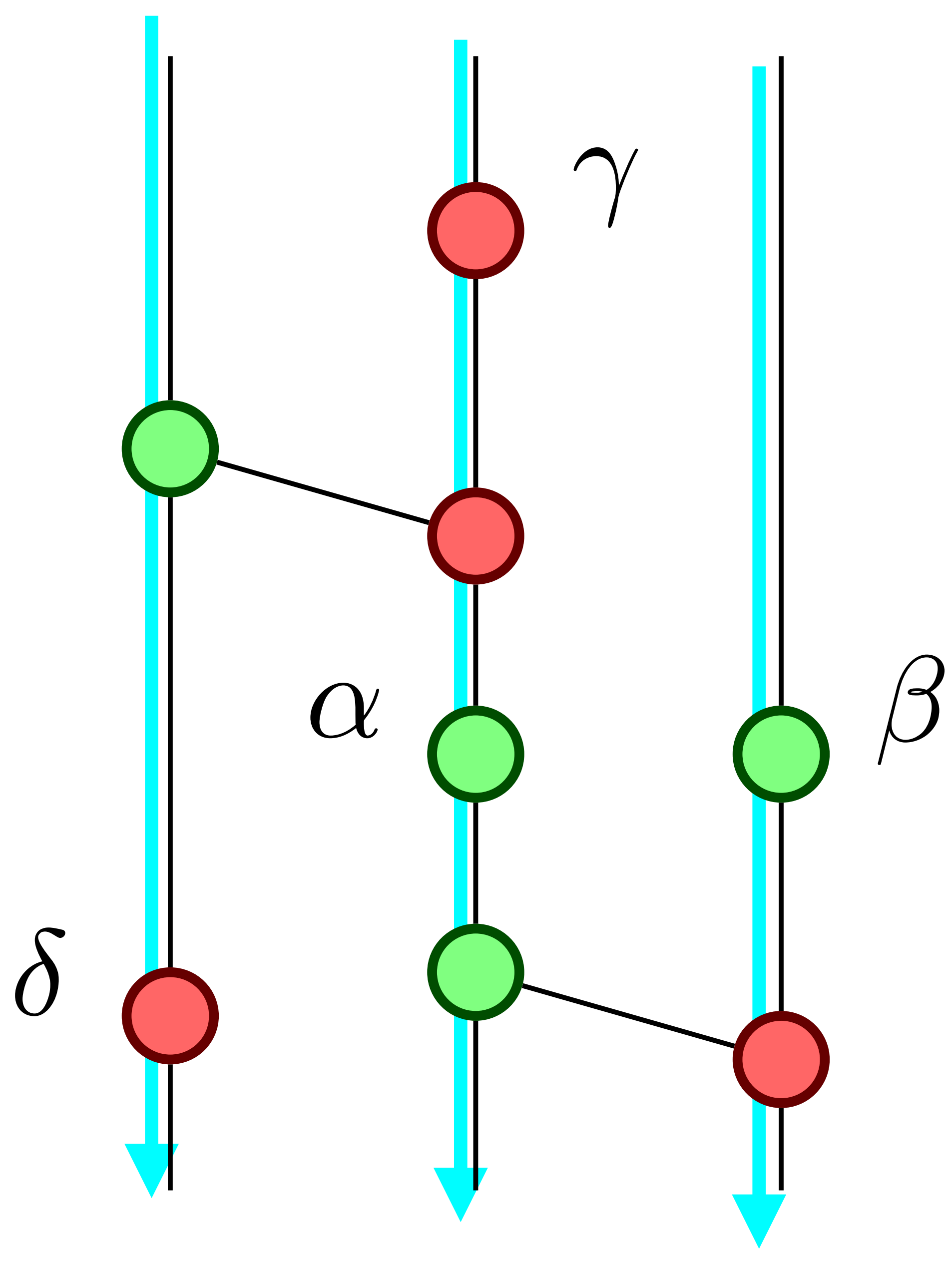

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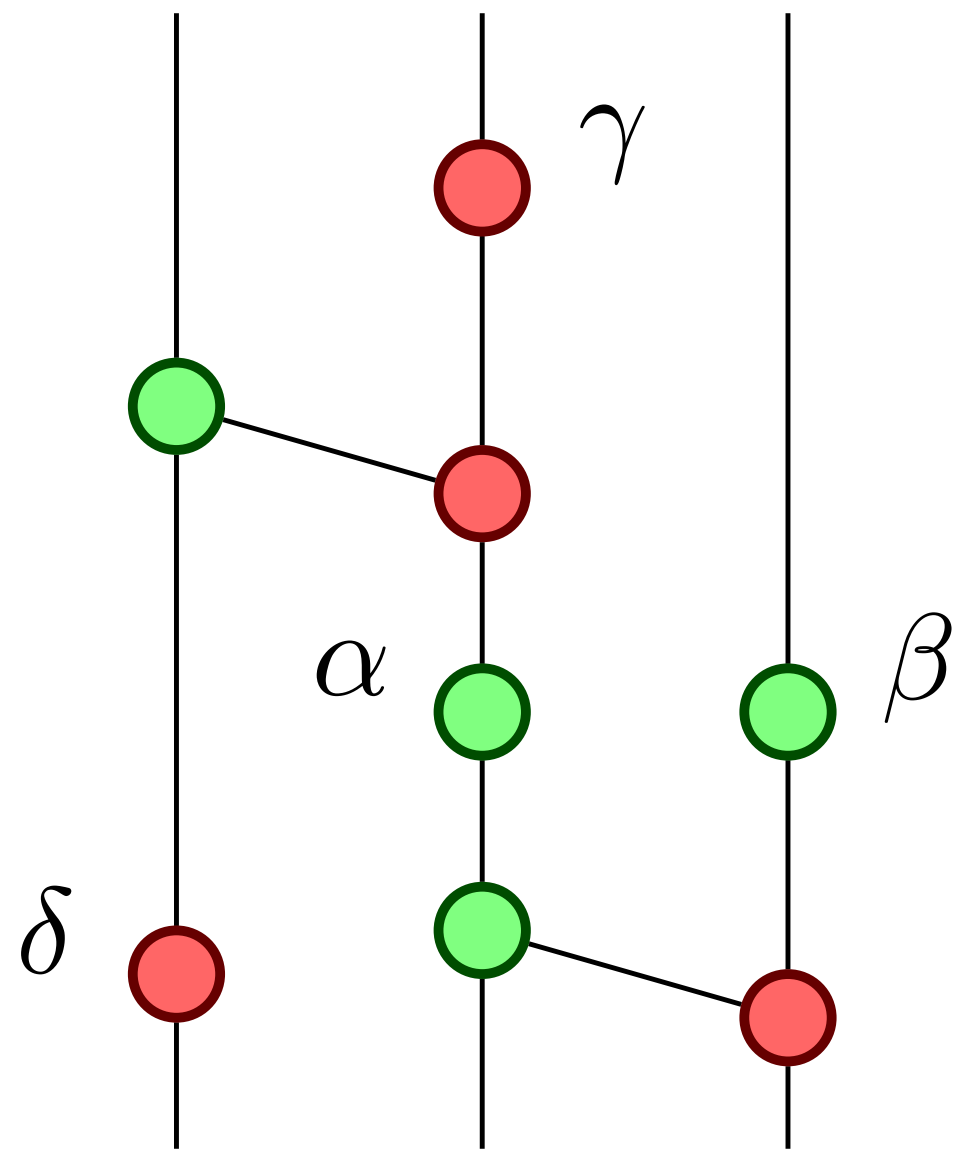
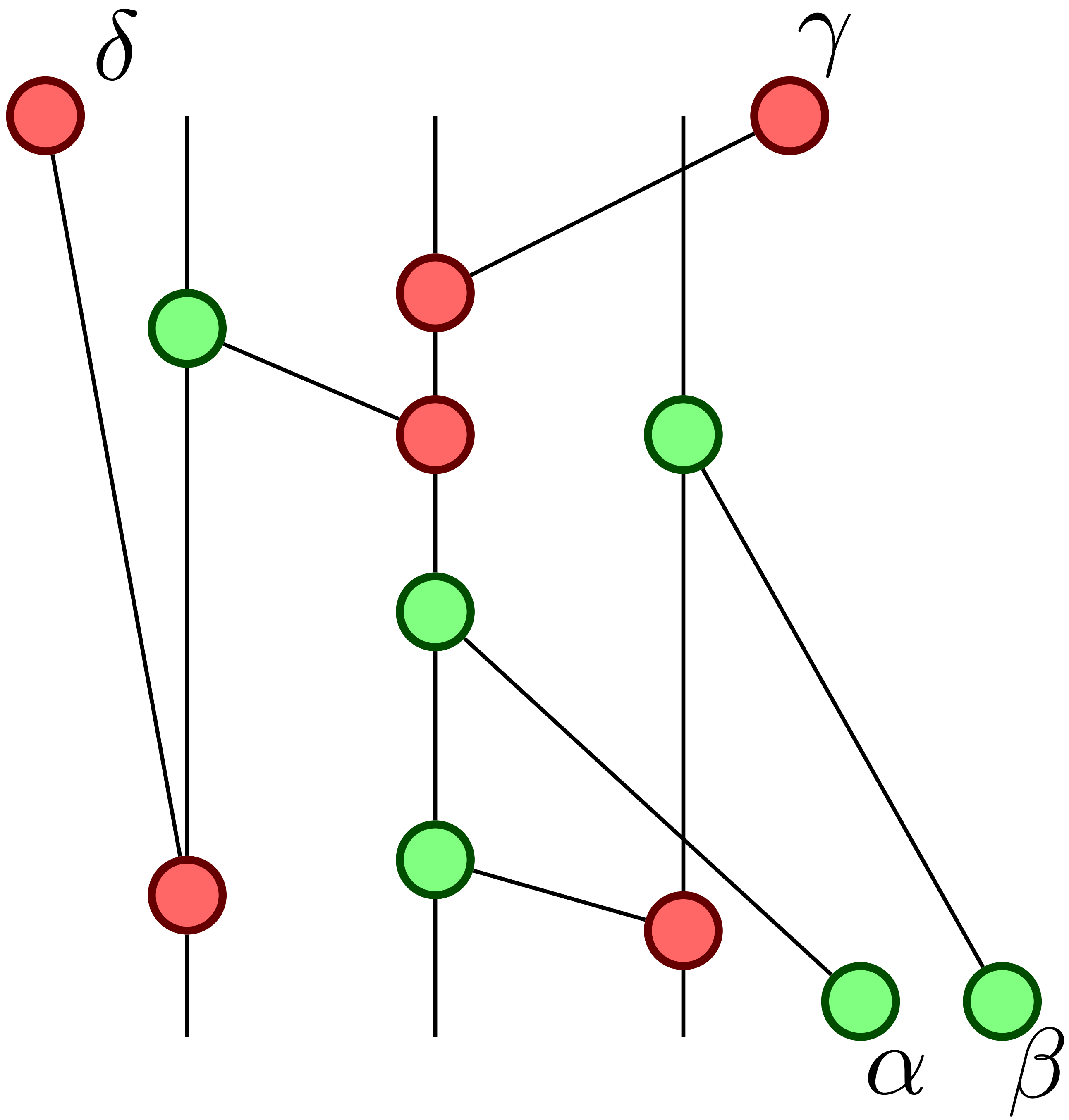
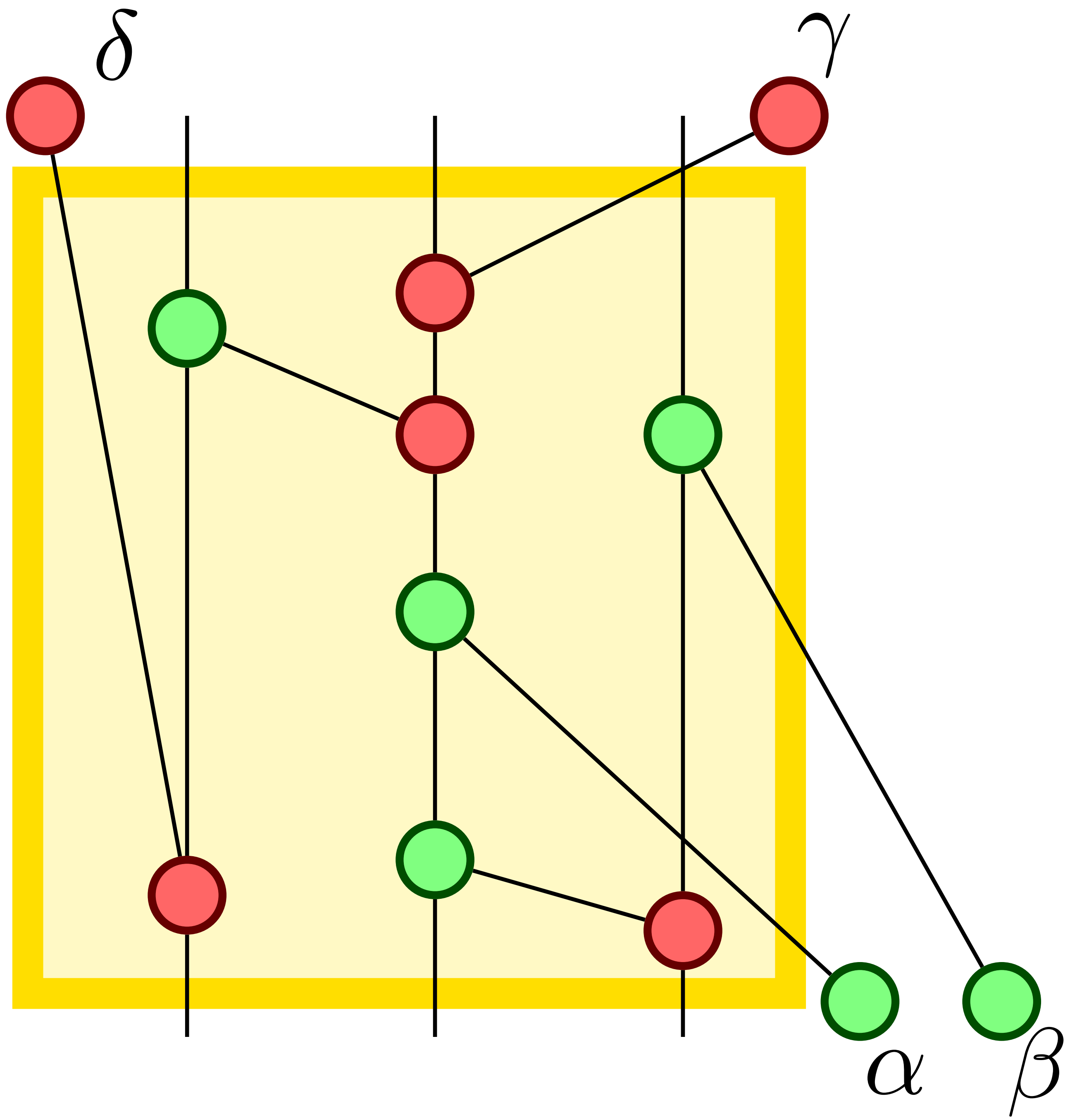


??? Perspective

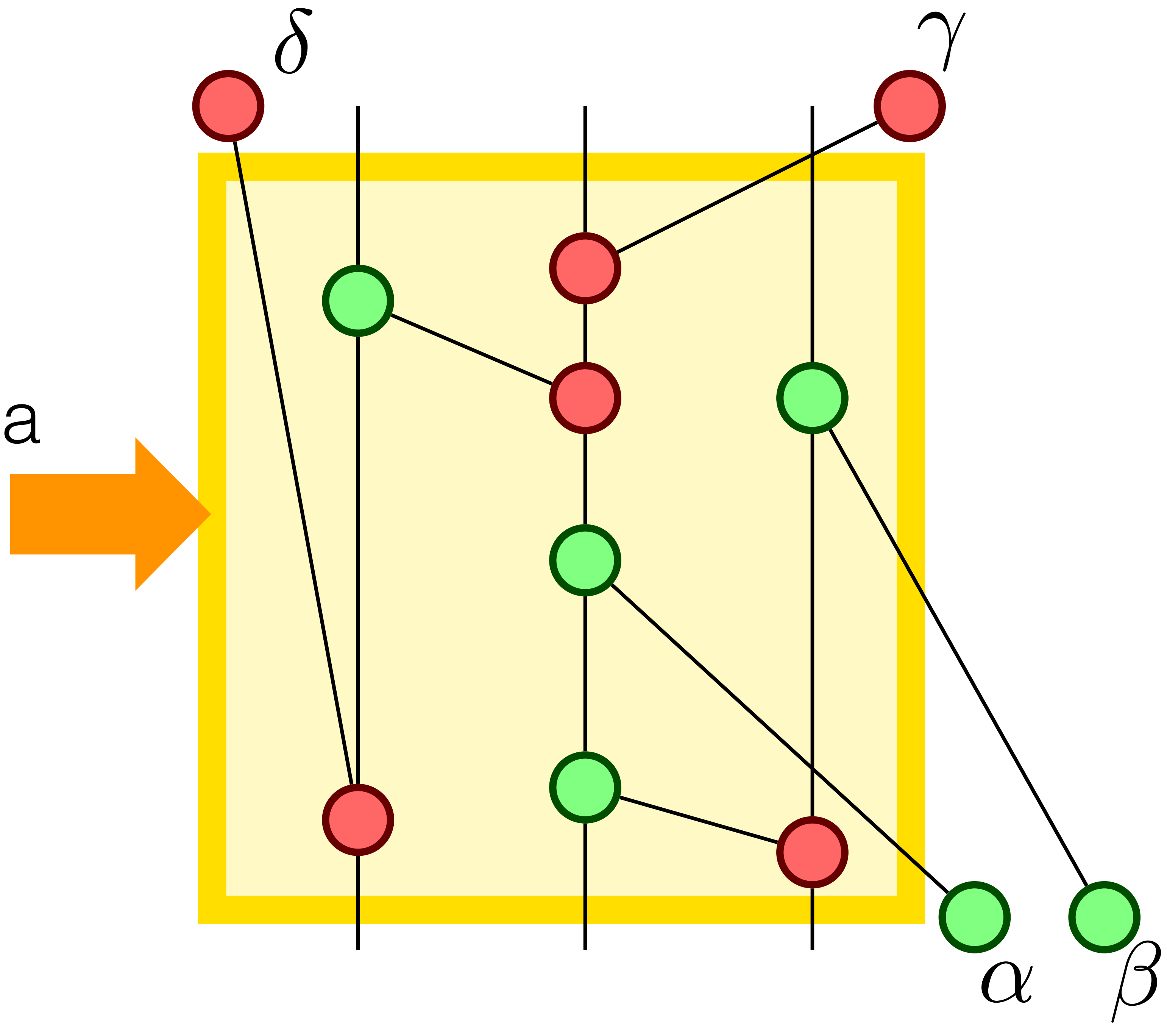


??? Perspective



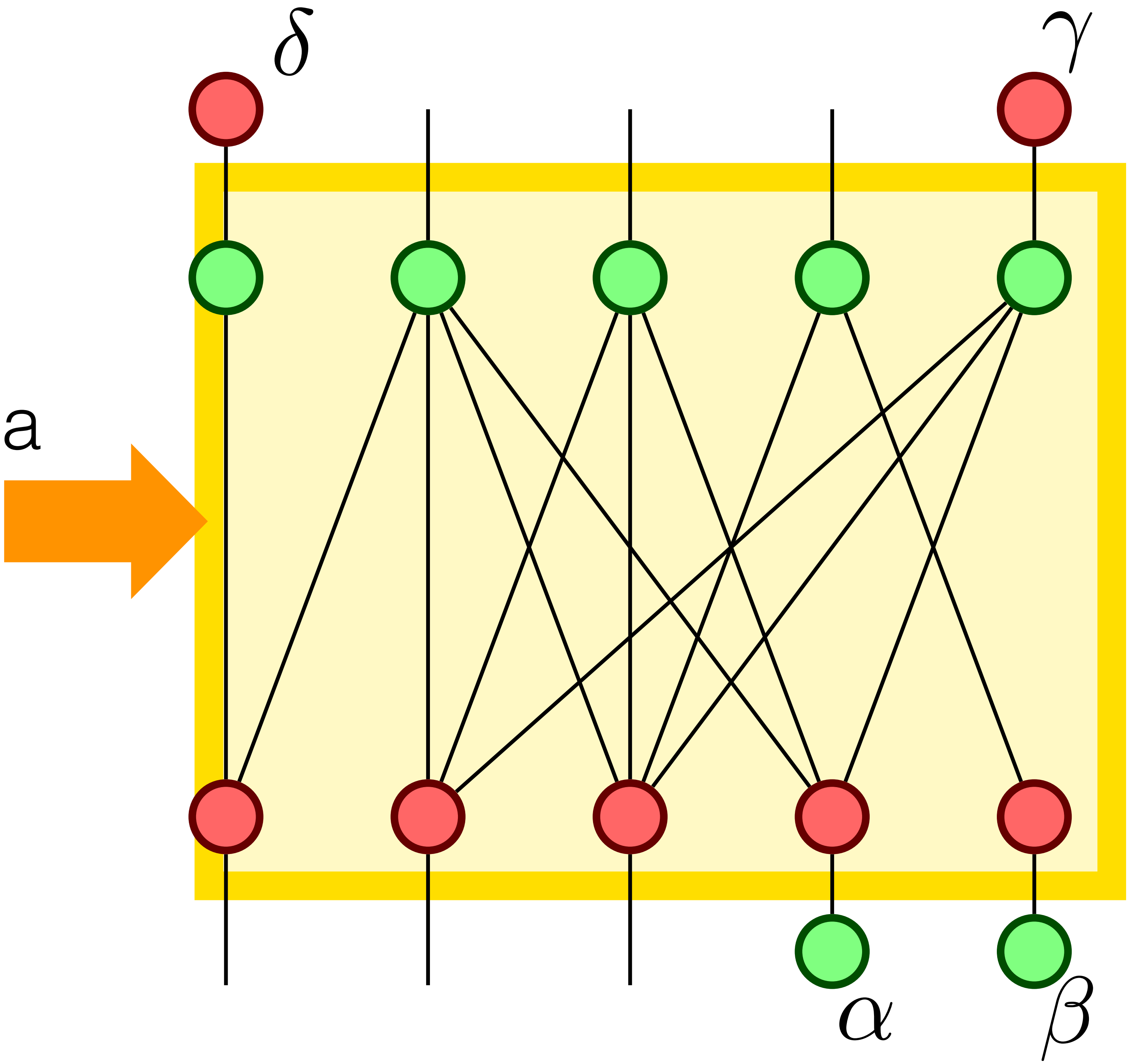
??? Perspective

Hopf algebra expression

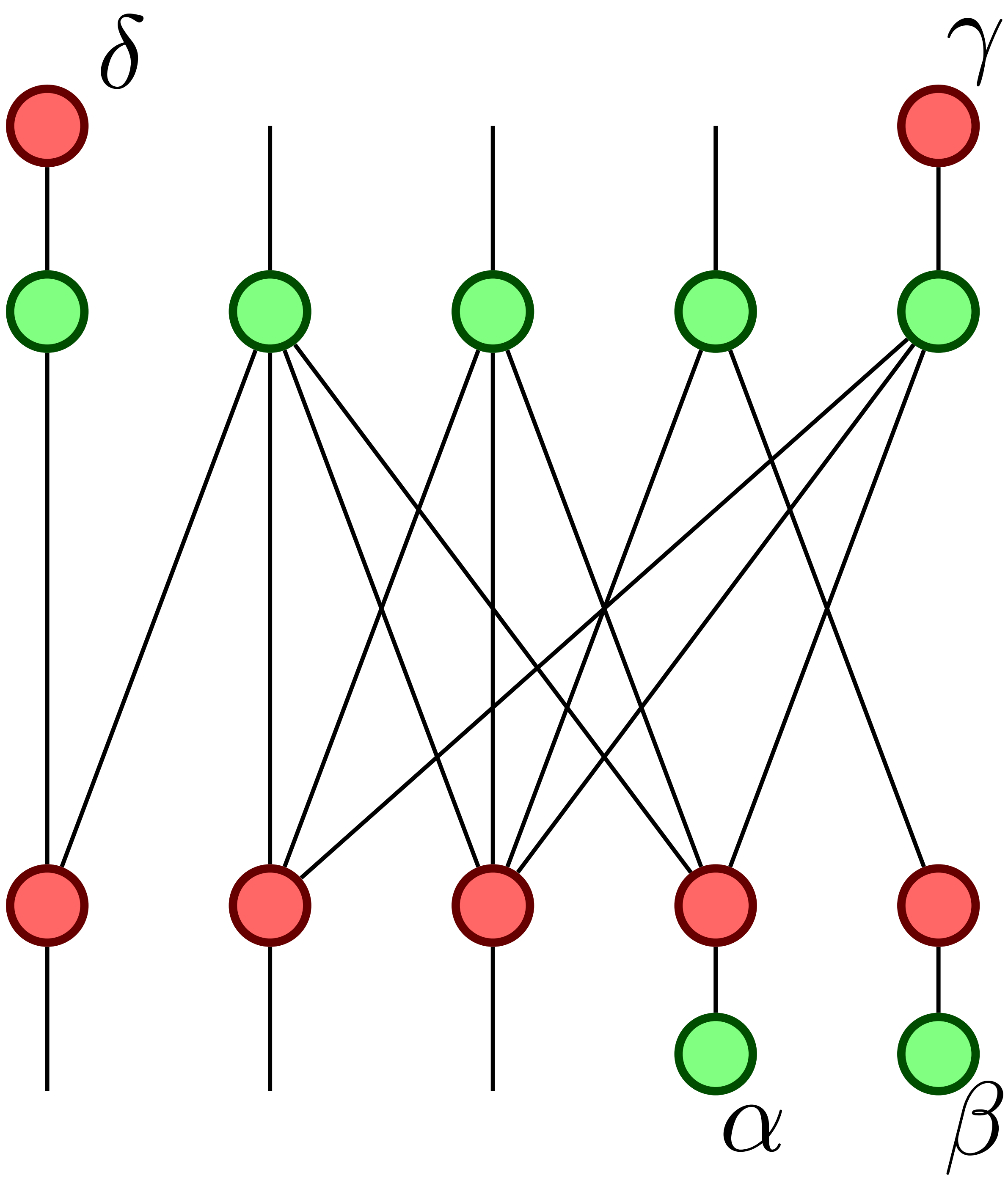


??? Perspective

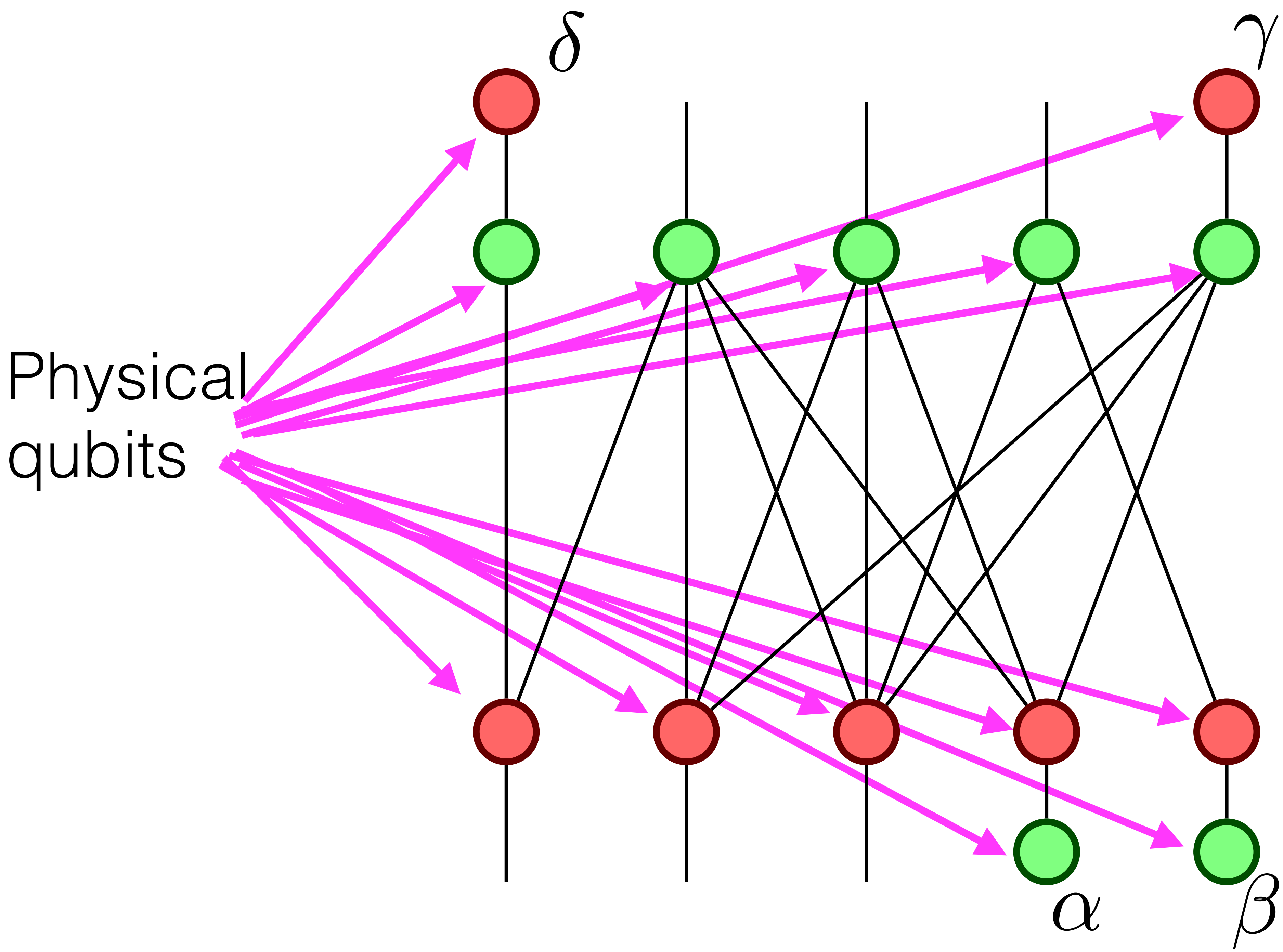
Hopf algebra normal form



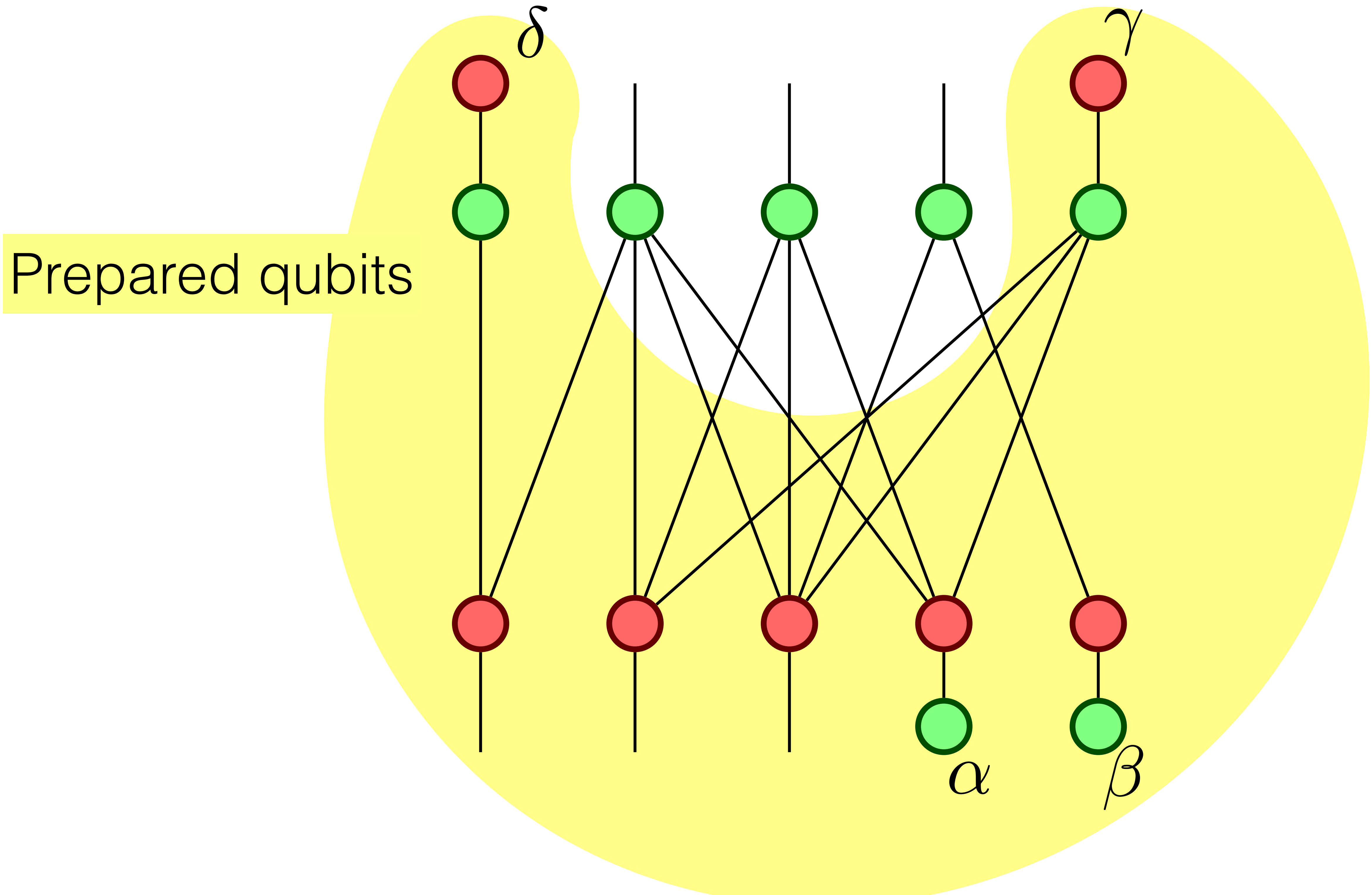
MBQC Perspective



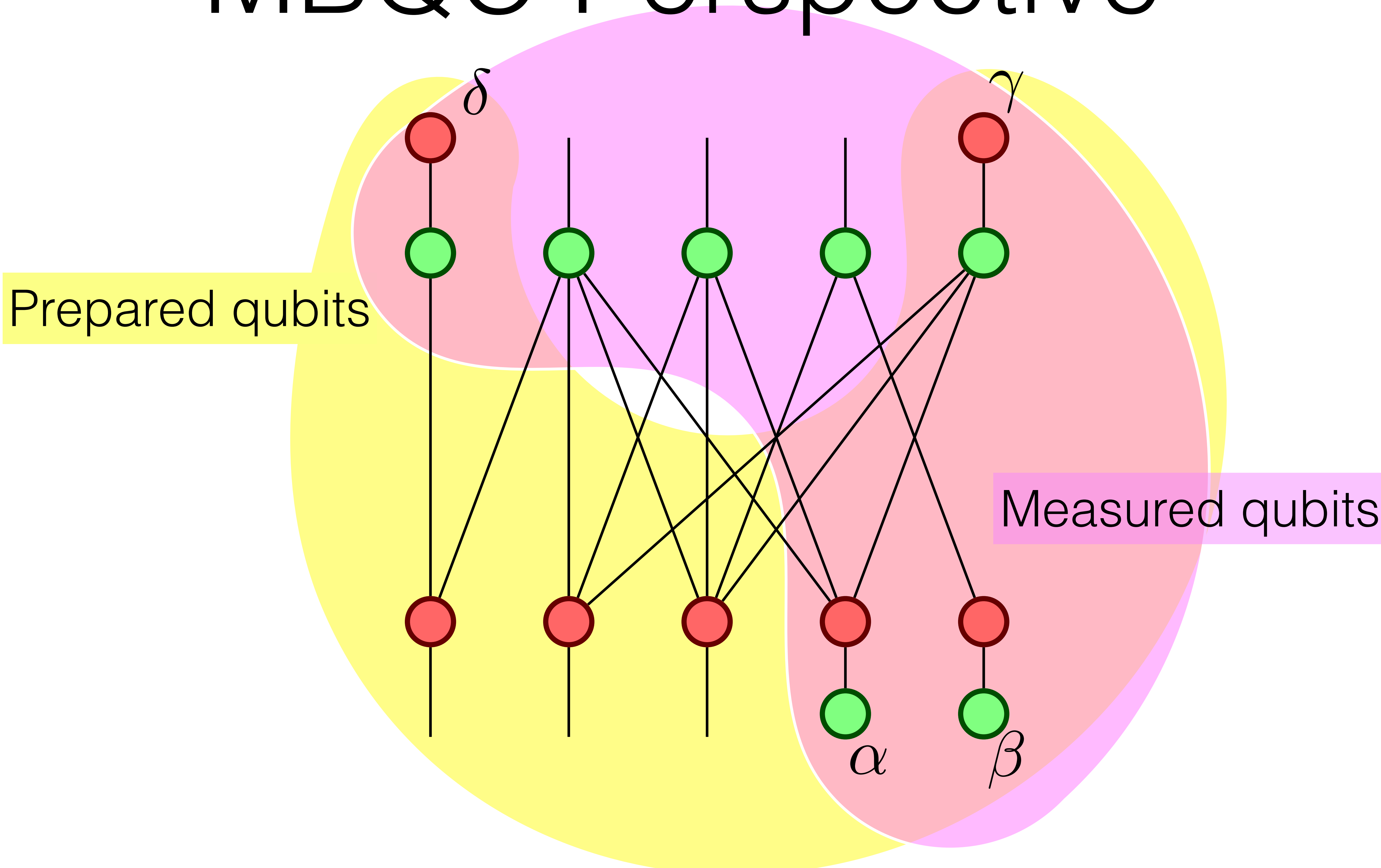
MBQC Perspective



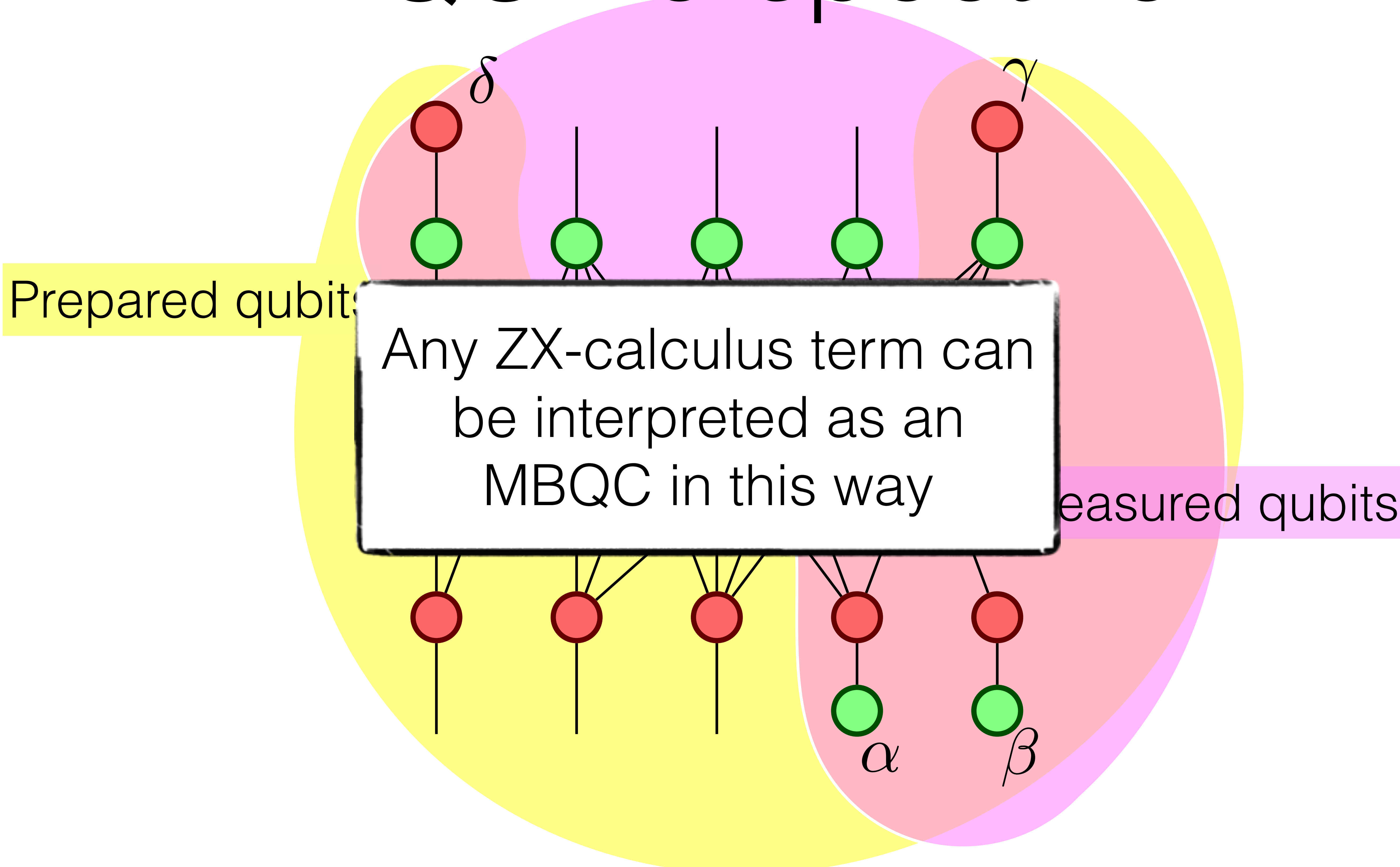
MBQC Perspective



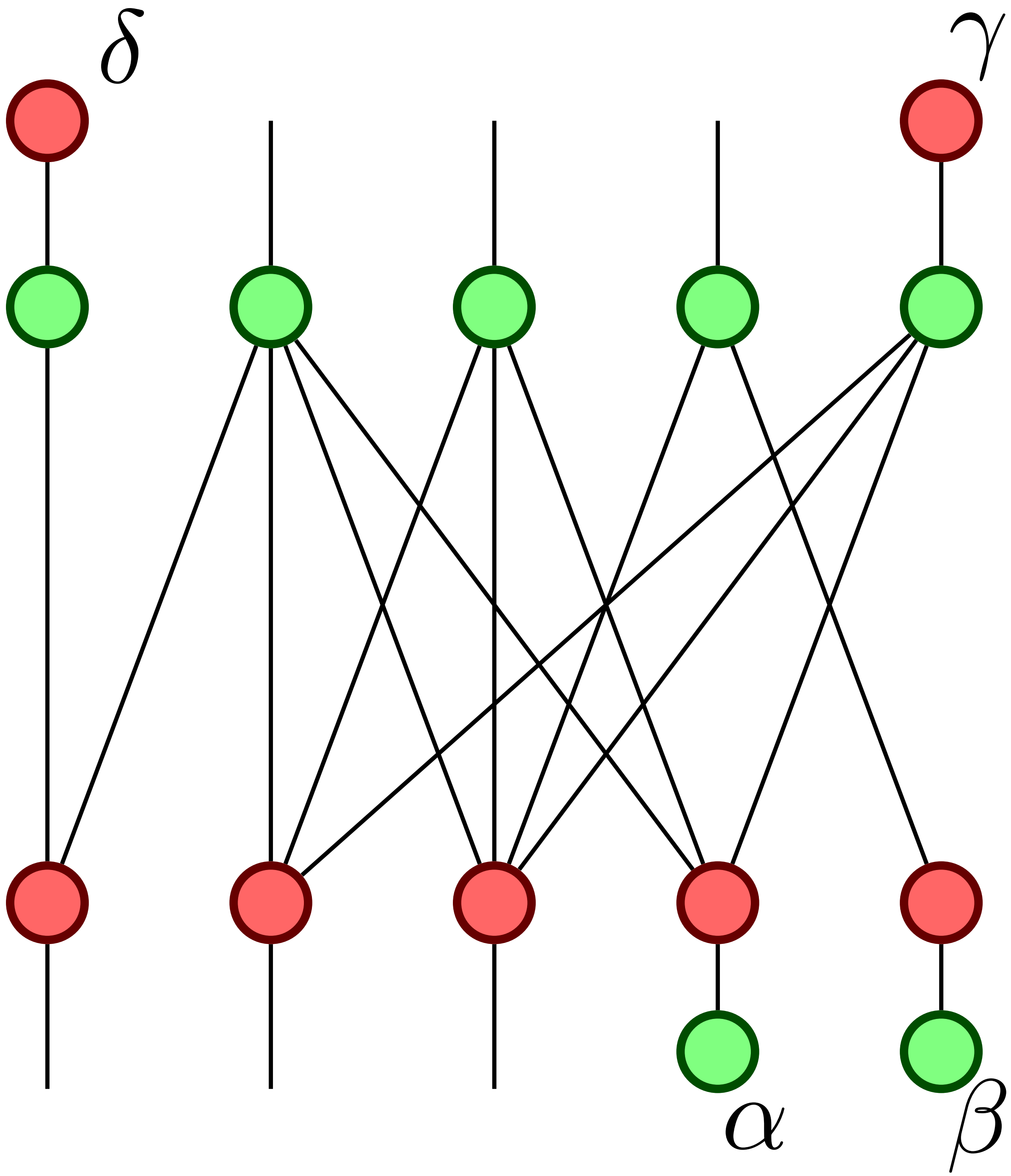
MBQC Perspective



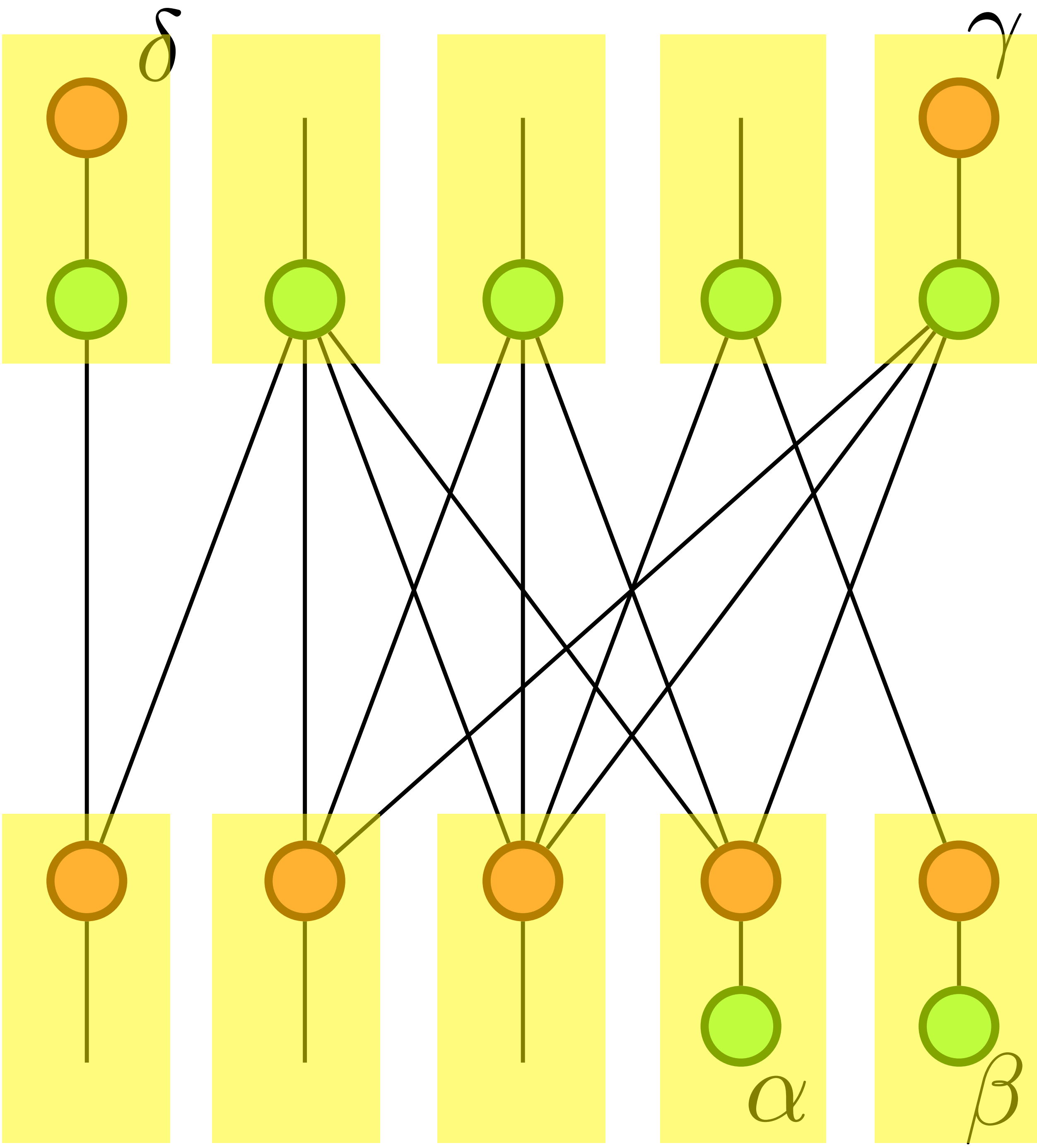
MBQC Perspective



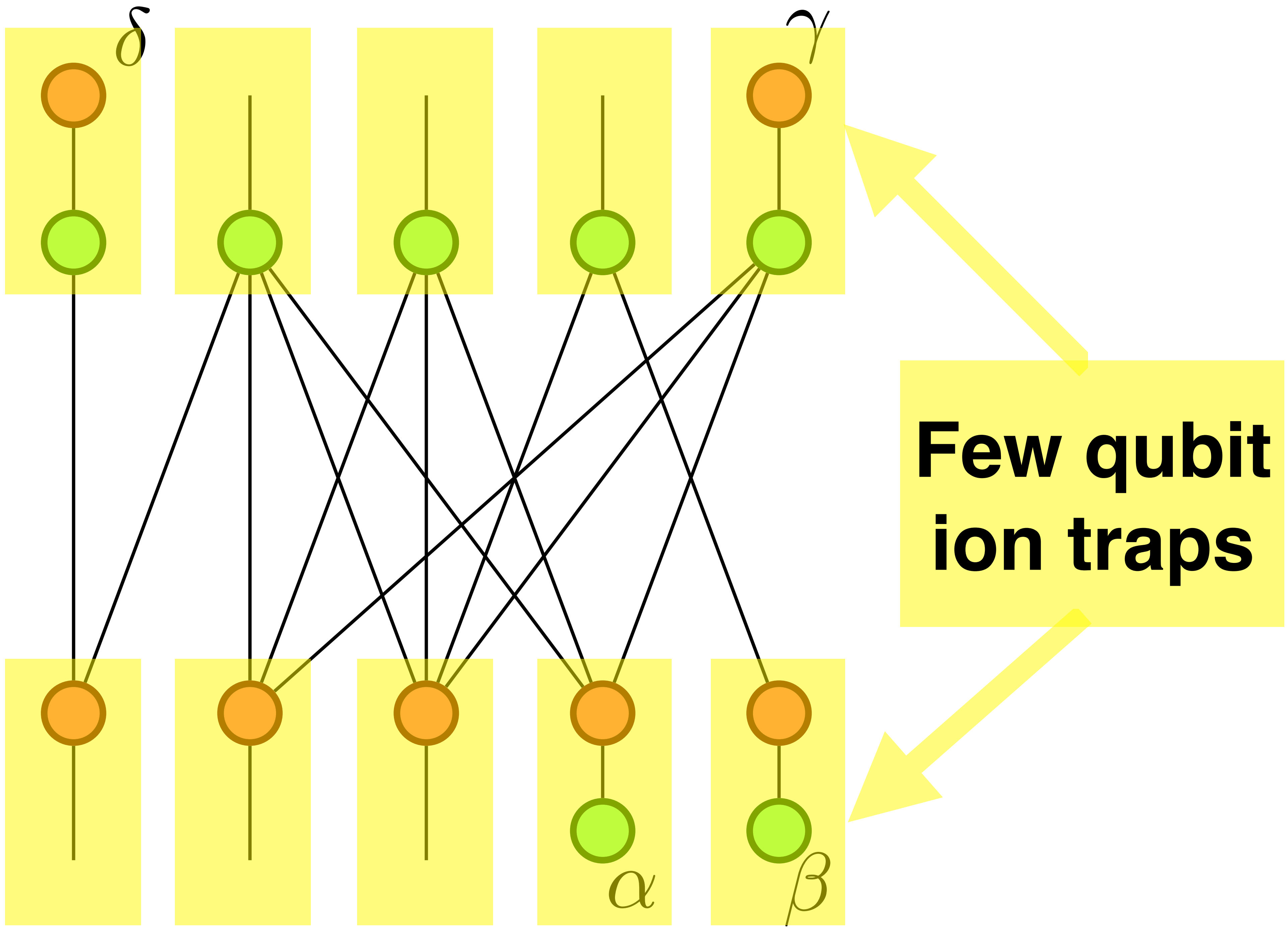
NQIT Perspective



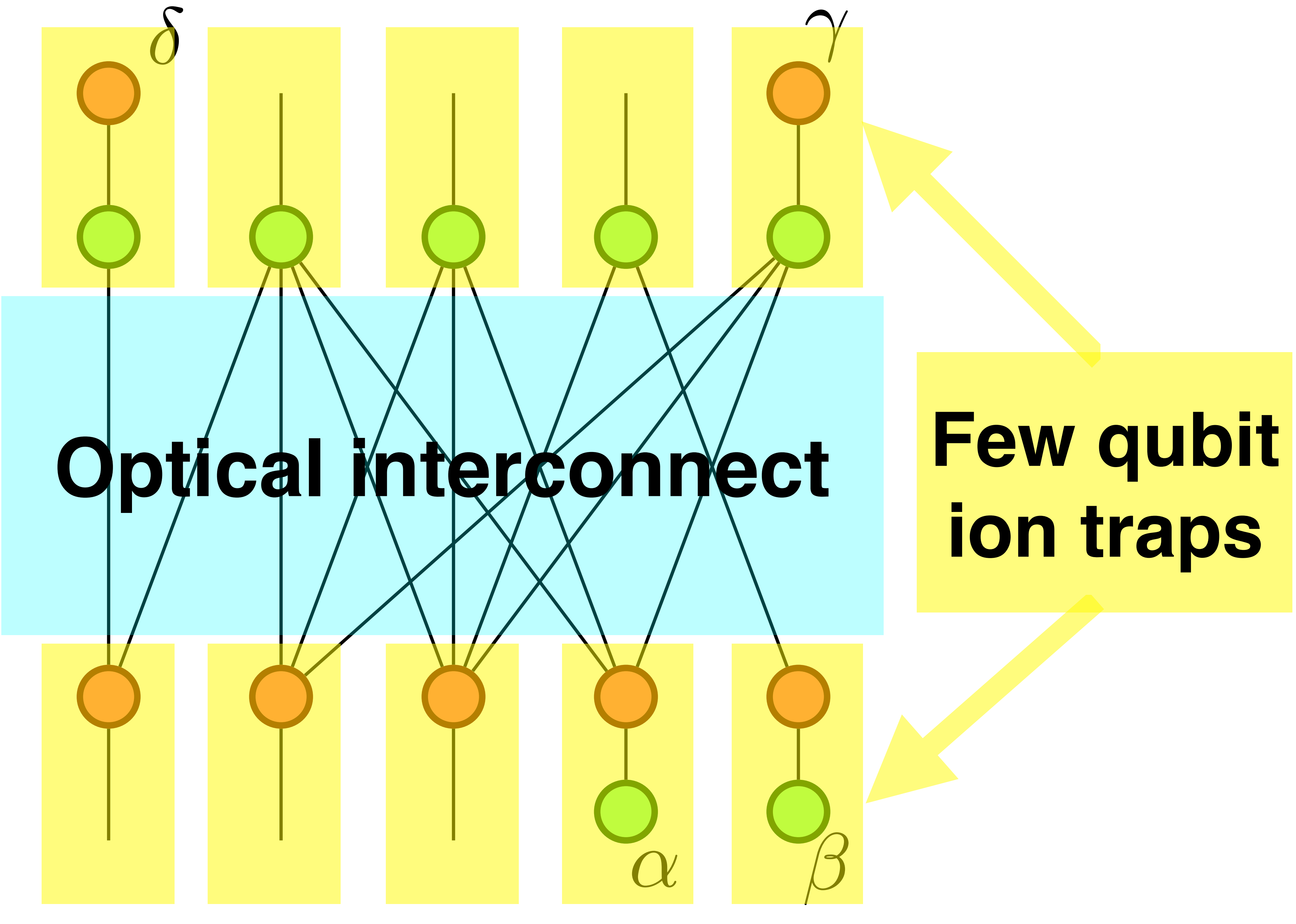
NQIT Perspective



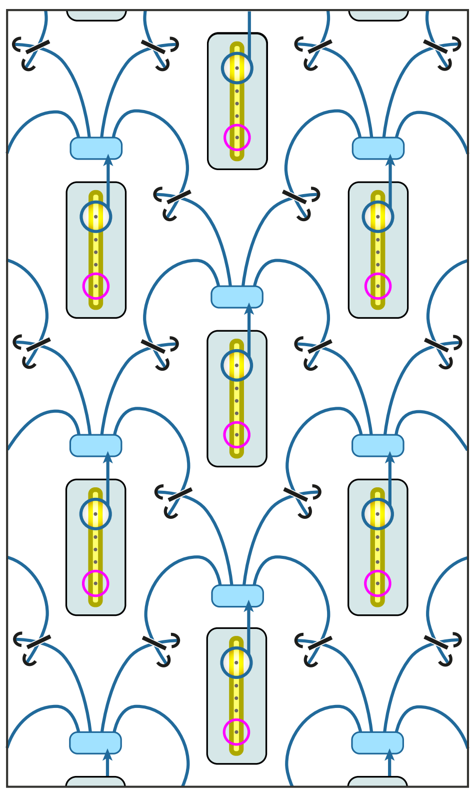
NQIT Perspective

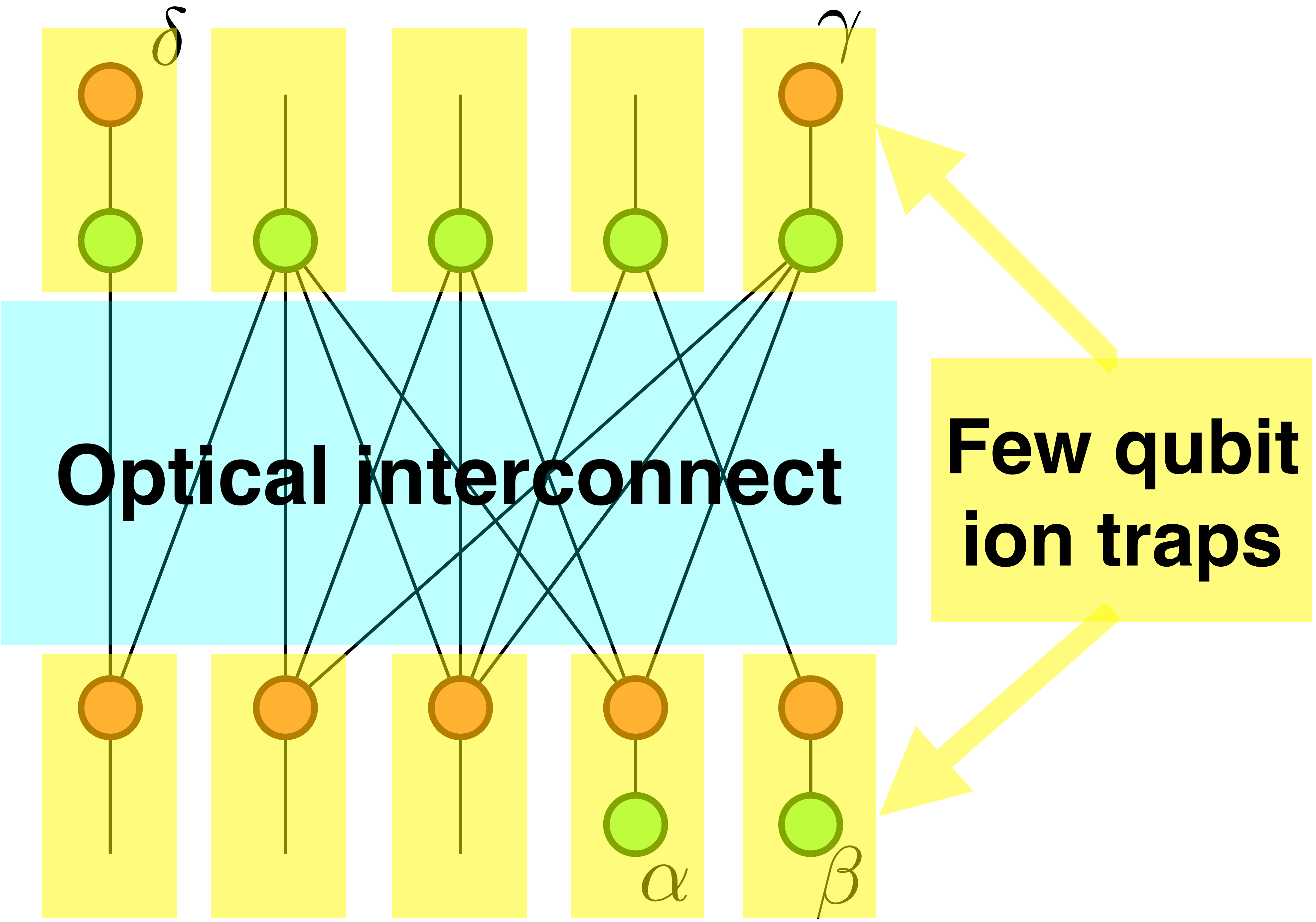


NQIT Perspective



NQIT Perspective





NQIT perspective(?)

- What about determinism?
  — unknown in general
  — use standard techniques for specific examples

- What are the trade-offs?
  — non-Clifford gates vs physical qubits
  — circuit depth vs complexity of entanglement