Multipartite Composition of Contextuality Scenarios

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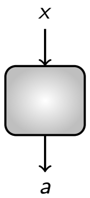
Perimeter Institute for Theoretical Physics

Contextuality scenarios

- Set of measurements
- Set of outcomes



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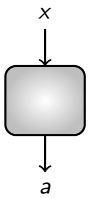
- Identify outcomes of different measurements: same probability



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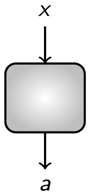

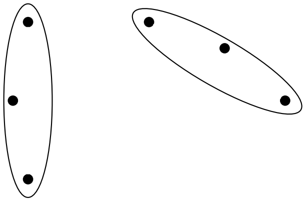

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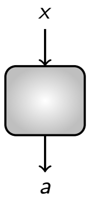

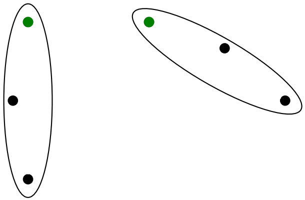

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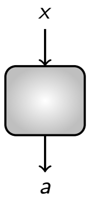

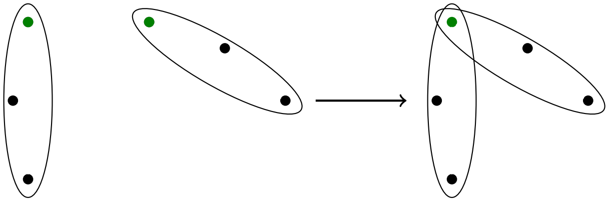

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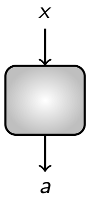
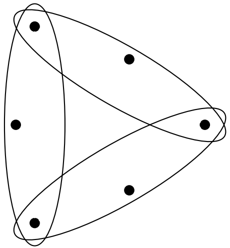


Hypergraph:

- Vertices → events – measurement outcome
- Hyperedges → complete measurements – set of outcomes



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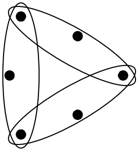

- **Probabilistic Model**: $\mathcal{G}(H)$
  $p : V \to [0,1], \quad$ such that $\quad \sum_{v \in e} p(v) = 1, \quad$ for each $e \in E$.

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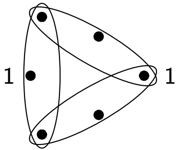

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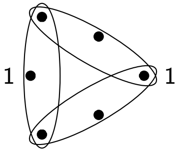

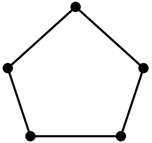

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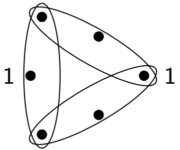

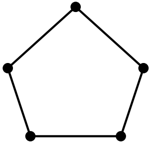

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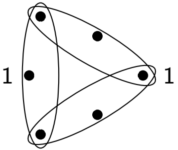

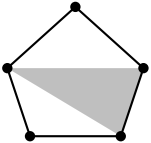

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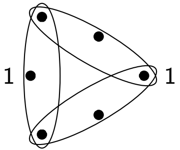

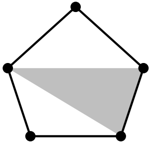

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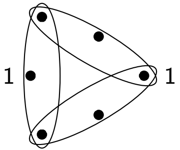

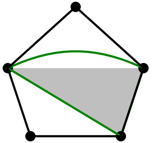

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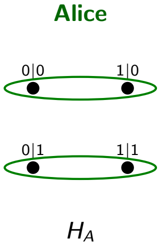
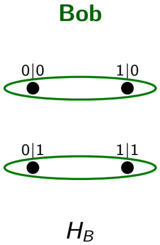
Composing contextuality scenarios

$\mathcal{G}(H_{AB})$ satisfies No Signalling





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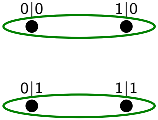
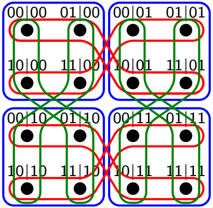
Alice



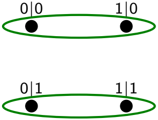
H_A



Foulis and Randall:

$$H_{AB} = H_A \otimes_{\mathrm{FR}} H_B$$

Bob



H_B

Multipartite systems

$$H_{ABC} \quad \rightarrow \quad (H_A \otimes_{\mathrm{FR}} H_B) \otimes_{\mathrm{FR}} H_C \neq H_A \otimes_{\mathrm{FR}} (H_B \otimes_{\mathrm{FR}} H_C)$$

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Minimal product	Common product	Maximal product
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$\bigcup (H_i \times H_j) \otimes_{\mathrm{FR}} H_k$	$\bigcup (H_i \otimes_{\mathrm{FR}} H_j) \otimes_{\mathrm{FR}} H_k$	Measurement protocols

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Classical, quantum and general probabilistic models:

Observationally equivalent

Almost quantum models

$\mathcal{Q}_1(H)$:

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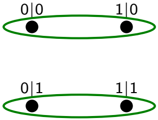
$$\mathcal{Q}_1(^{\text{min}} \otimes\ (H_A, H_B, H_C)) \supsetneq \mathcal{Q}_1(^{\textbf{com}} \otimes\ (H_A, H_B, H_C))$$

Example:

$H_A = H_B = H_C =$



Bell inequality: GYNI

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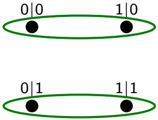

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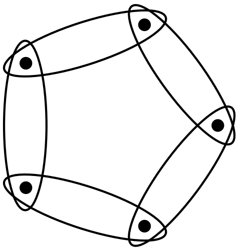
$$\mathcal{Q}_1(^{\text{com}} \otimes (H_A, H_B, H_C)) \to 1\,, \quad \mathcal{Q}_1(^{\text{min}} \otimes (H_A, H_B, H_C)) \to 1.156$$

$\mathcal{Q}_1(^{\text{min}}\otimes)$ **violates Local Orthogonality**

Completion scenario



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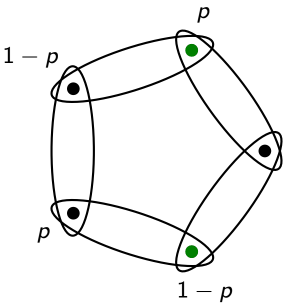

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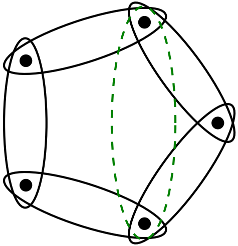

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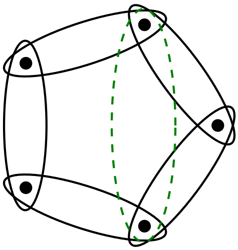

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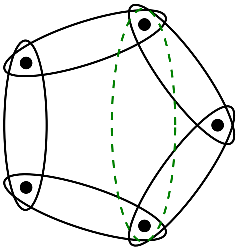

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$\mathcal{G}(H) = \mathcal{G}(\bar{H})$

$\mathcal{Q}(H) = \mathcal{Q}(\bar{H})$, $\mathcal{C}(H) = \mathcal{C}(\bar{H})$

$$H_{1\ldots N} := \overline{H_1 \otimes \ldots \otimes H_N} \neq {}^{\max} \otimes_j \bar{H}_j$$

Summary and open questions

- No unique way to compose many systems
- Classical, Quantum and General models are observationally invariant
- Almost quantum models display a gap between ${}^{\text{min}}\otimes$ and ${}^{\text{com}}\otimes$
- LO removes the gaps found

- Are there gaps that LO cannot remove?
- Are there gaps among ${}^{\text{com}}\otimes$, ${}^{\text{max}}\otimes$ and ‘completion’ ?
- physical intuition to settle for one product

Thanks !!!

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