Composition and quantum theory: a conjecture, and how it could fail

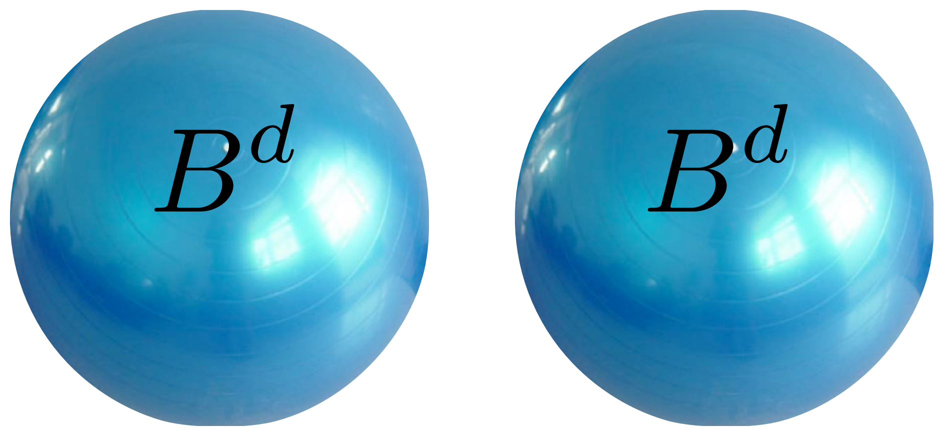
Markus P. Müller* and Marius Krumm
Departments of Applied Mathematics and Philosophy, UWO
Perimeter Institute for Theoretical Physics, Waterloo

Western
PERIMETER INSTITUTE FOR THEORETICAL PHYSICS
Canada Research Chairs
NSERC CRSNG

Outline

1. The conjecture tomographic locality + reversibility $\Rightarrow$ QT

2. Evidence



$\Rightarrow$ *d=3* and QT

3. Multipartite interaction beyond QT?

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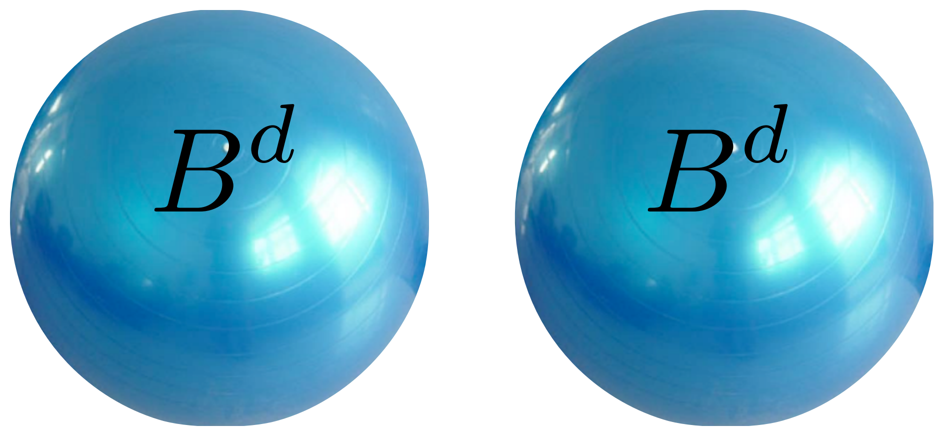

⇒ *d*=*3* and QT

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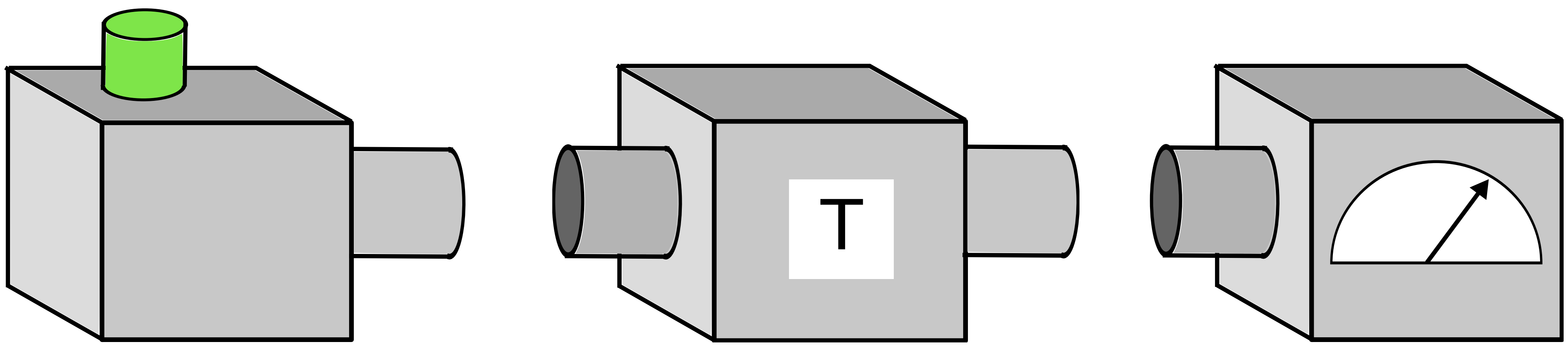
Starting point: convex-operational theory



Preparation,
transformation,
measurement.

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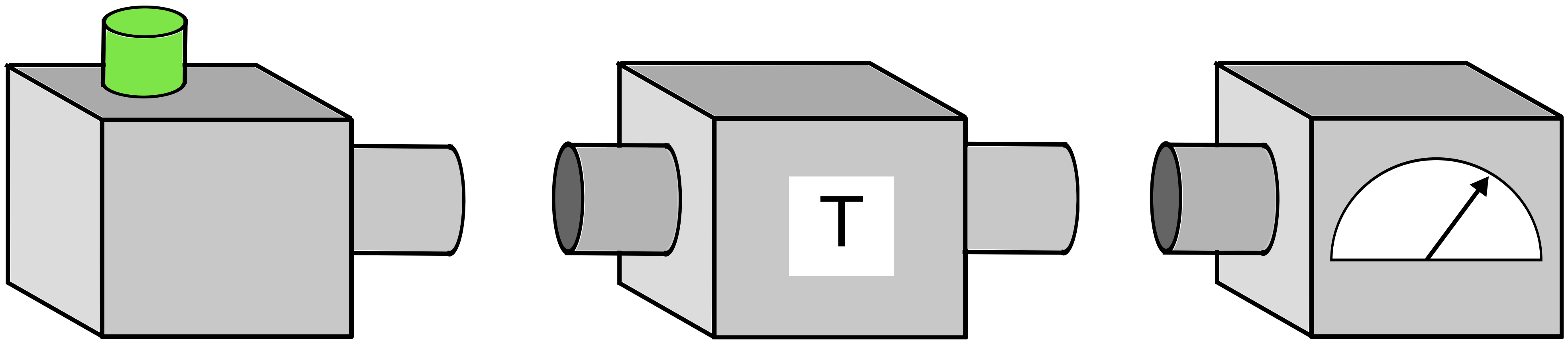

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State ω = equivalence class of **preparation procedures**

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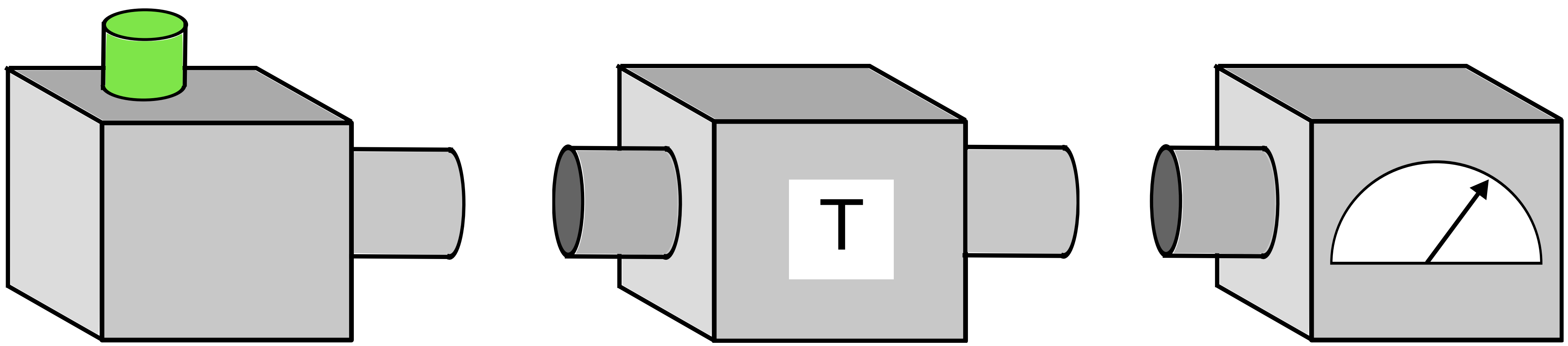

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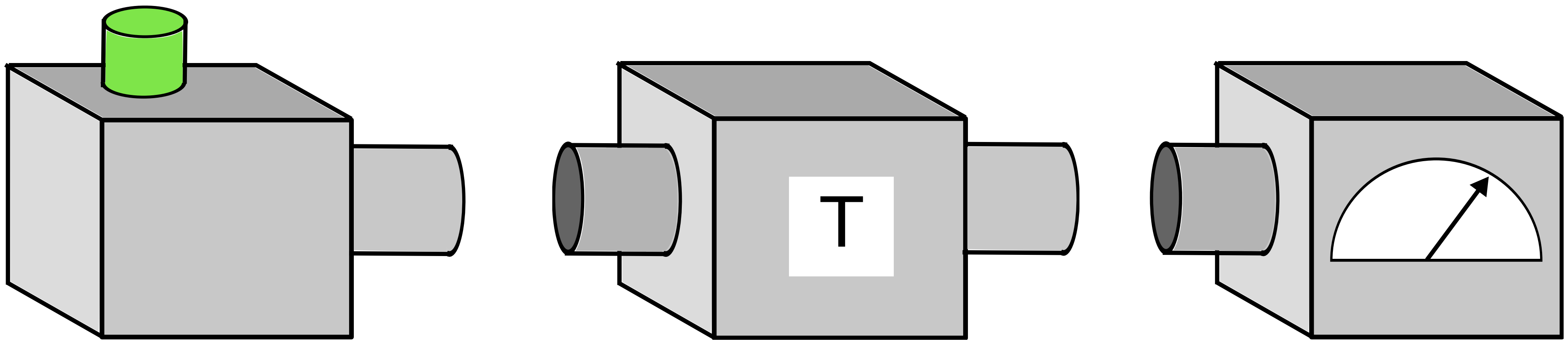

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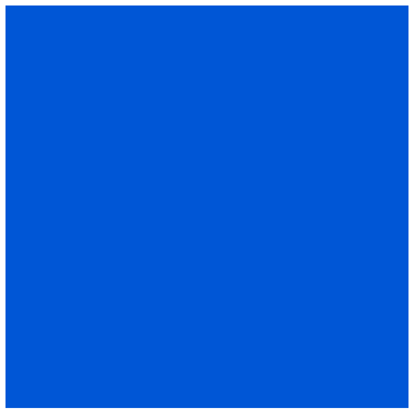
QT: Ω_N = set of N×N density matrices
CPT: Ω_N = set of prob. distributions $(p_1, \ldots, p_N)$.

1. The conjecture

Any compact convex set (in a real space) is a possible state space:

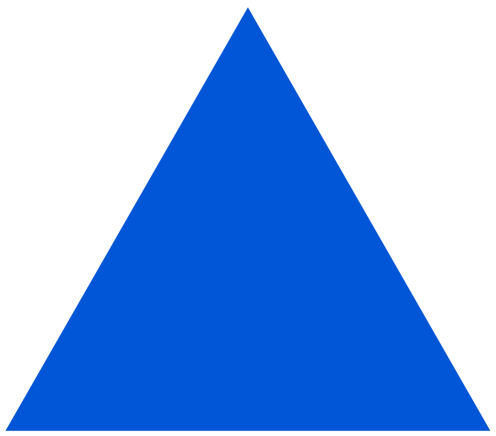
classical
bit

quantum
bit



"gbit"

Arbitrary convex
state space



Classical "trit"
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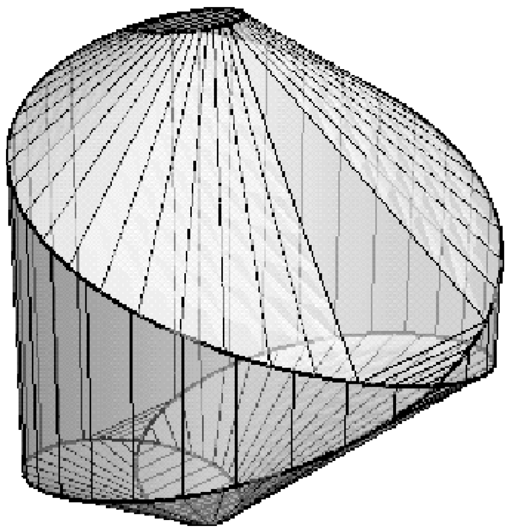


Quantum "trit".
Complicated, 8D!

1. The conjecture

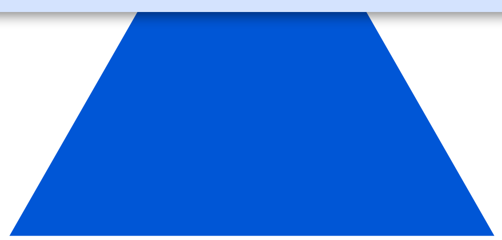
Any compact convex set (in a real space) is a possible state space:

clas
b

More **formal definition:** Set of unnormalized states is a closed, convex, pointed, generating cone in a (finite-dim. here) real vector space.
Transformations, measurements: analogous def's.

All here is math. rigoros, but I keep this talk simple.

Arbitrary convex state space



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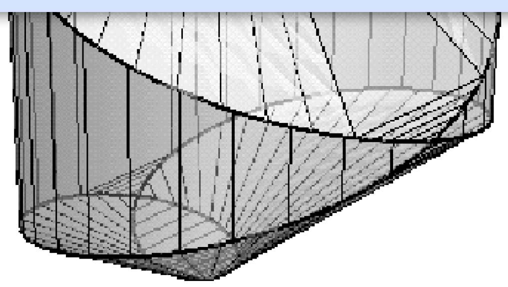

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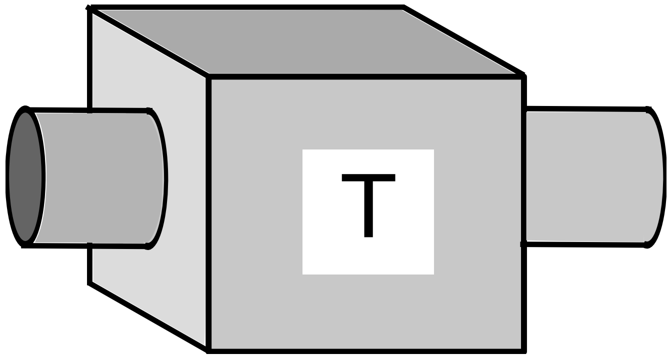
Reversibility and tomographic locality



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A transformation T is **reversible** if T^{-1} exists and is a transformation, too.

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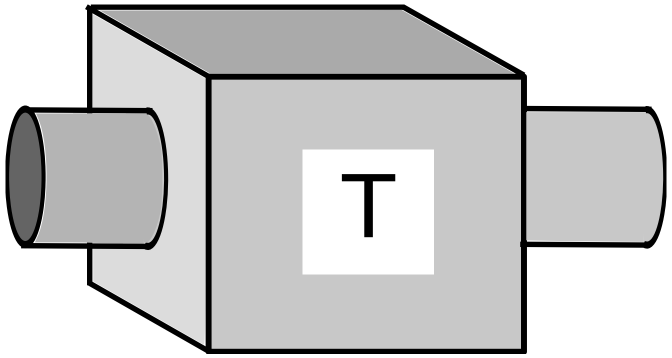

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QM: reversible transformations = unitaries, $\rho \mapsto U\rho U^\dagger$.

CPT: permutations, $(p_1, \ldots, p_n) \mapsto (p_{\pi(1)}, \ldots, p_{\pi(n)})$

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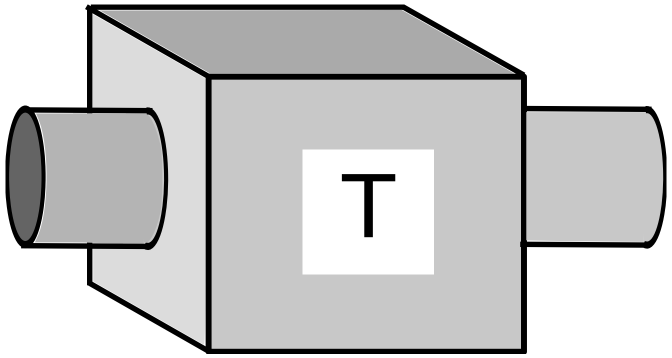

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Reversible transformations are **linear symmetries** of the state space.

They **map pure states to pure states** (pure state = extremal point of convex set).

Reversibility and tomographic locality

Reversibility postulate:

For every pair of pure states ω, φ,
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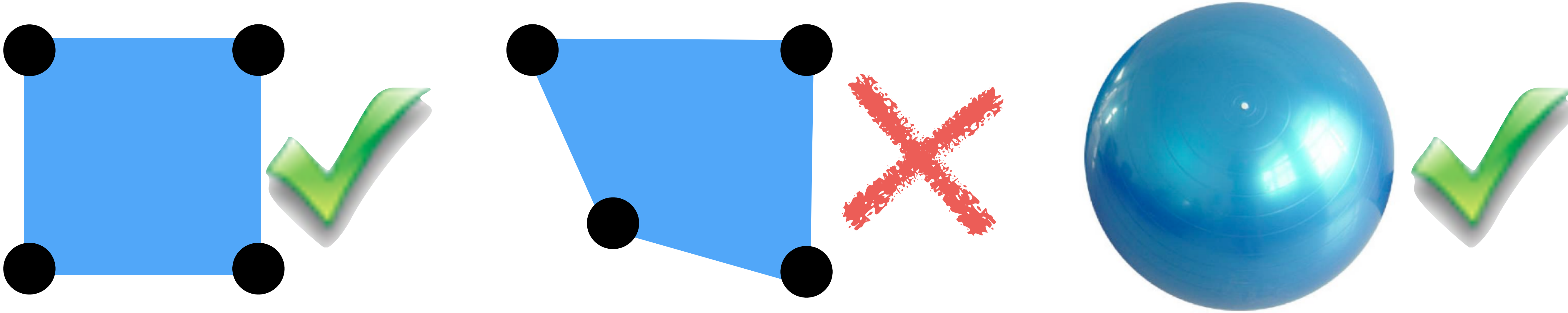
- Enforces some **symmetry** in the state space:



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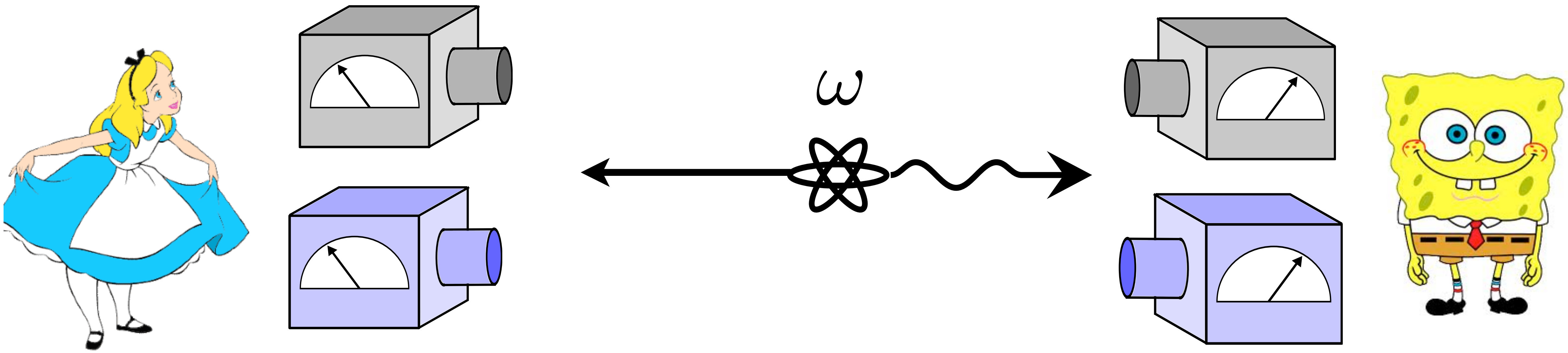

Alice+Bob are given (many copies) of a state.

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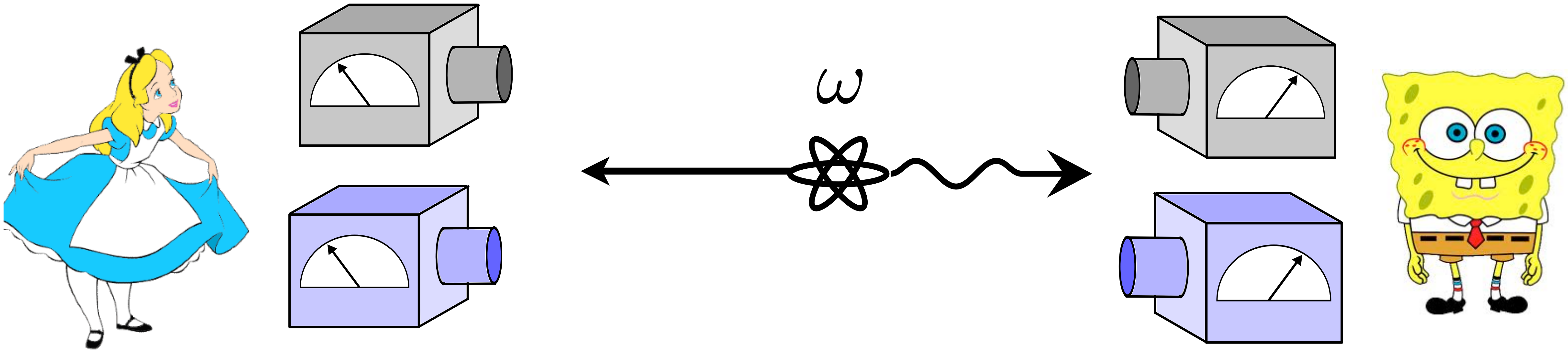
Reversibility and **tomographic locality**



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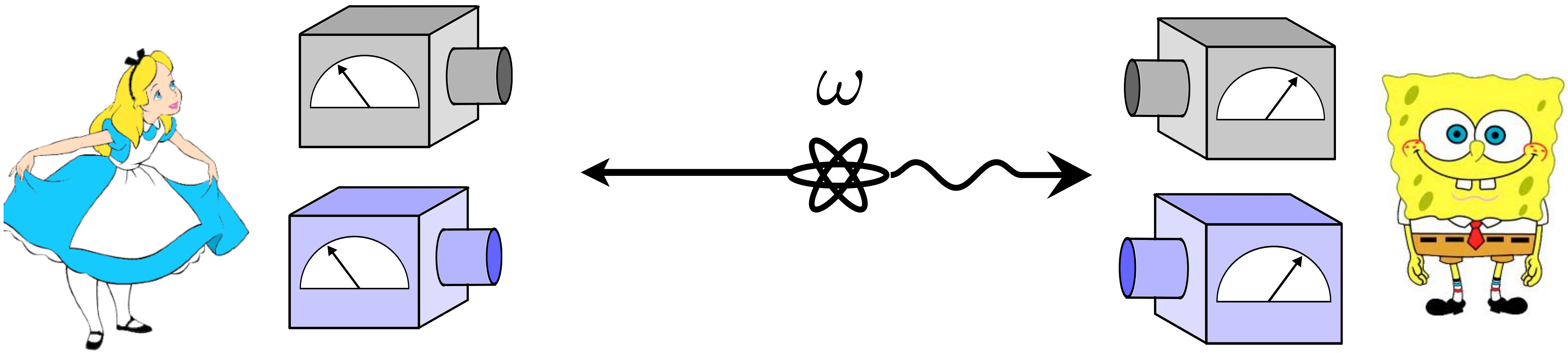

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Every state of a composite system is completely characterized by the correlations of measurements on the individual components.

Reversibility and **tomographic locality**

More mathematical perspective:

- Given two state spaces Ω_A and Ω_B there are always infinitely many possible composites Ω_{AB}.
- Only constraints: there are notions of "product states" and "product measurements".
- **Tomographic locality equivalent** to the following property of state-space-carrying **vector spaces:**

$$V_{AB} = V_A \otimes V_B .$$

Tomographic locality:

Every state of a composite system is completely characterized by the correlations of measurements on the individual components.

The conjecture

Conjecture:

If some Ω_{AB} is a **locally tomographic** composite of some Ω_A and Ω_B, and all three state spaces satisfy **reversibility**, and there is at least one reversible transformation $T_{AB} \neq T_A \otimes T_B$, then Ω_{AB} is a (subspace of a) **quantum** state space.

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- **If true:** Gives very clear idea of "*why the quantum?*".
- **If wrong** (which I actually hope):
  Physically interesting: counterexamples describe possible alternative/new physics.
  Mathematically interesting: interplay convex geometry/ group theory/ multilinear algebra.
  Computersciency interesting: contrast that new theory to quantum computation!

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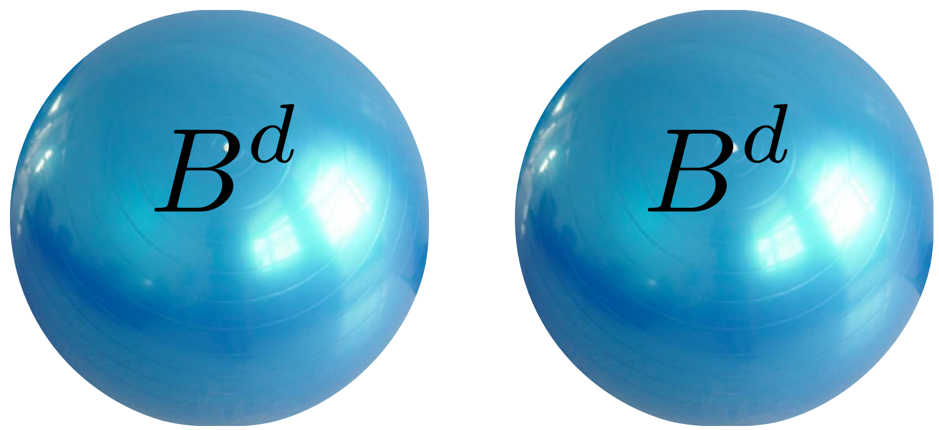

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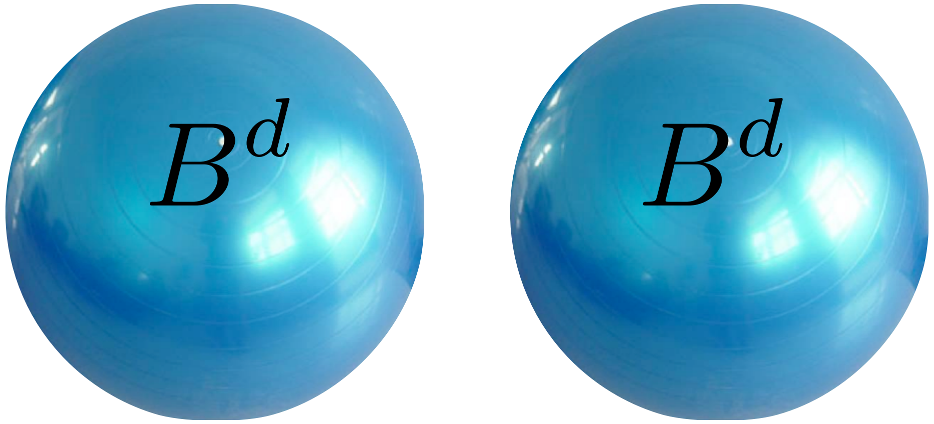

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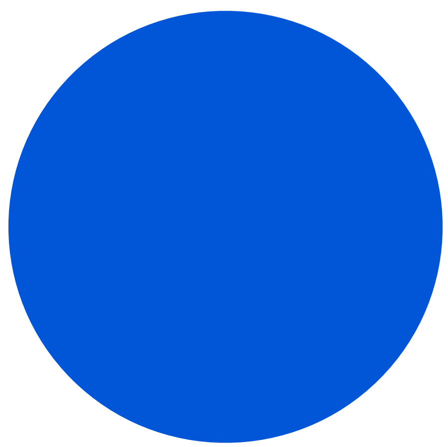
$d = 1$

classical bit



$d = 2$

$d = 3$

quantum bit

...

$d = 2, 5, 9$ are bits in quantum theory over $\mathbb{R}$, $\mathbb{H}$, $\mathbb{O}$.

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Ll. Masanes, MM, D. Pérez-García, R. Augusiak, *Entanglement and the three-dimensionality of the Bloch ball*, J. Math. Phys. **55**, 122203 (2014).

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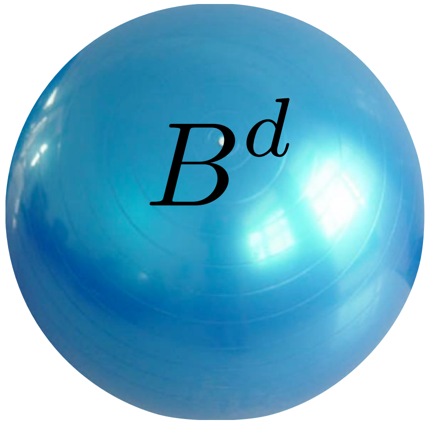
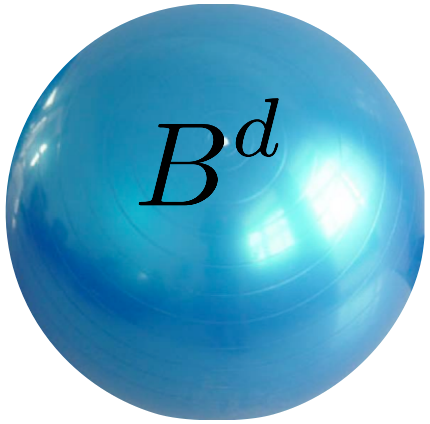


A



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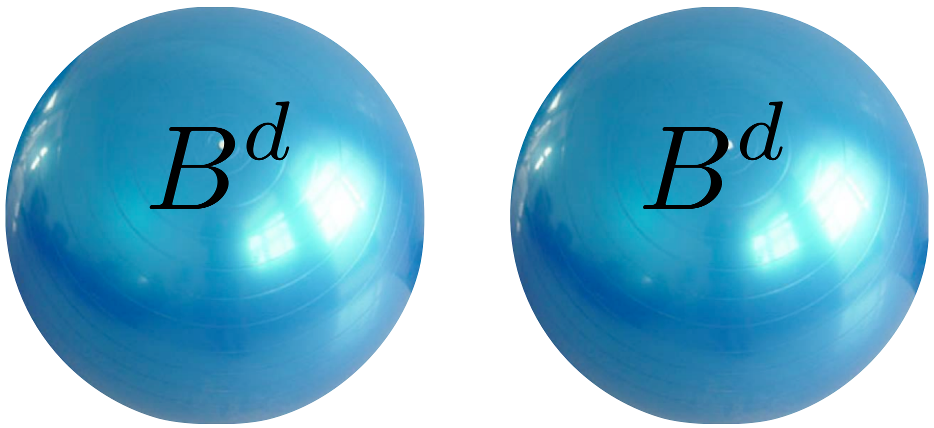

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Assume tomographic locality, and reversibility for A, B, AB.

$\Rightarrow$ group of reversible transformations $\mathcal{G}_A = \mathcal{G}_B$ must be transitive on $\partial B^d = S^{d-1}$.

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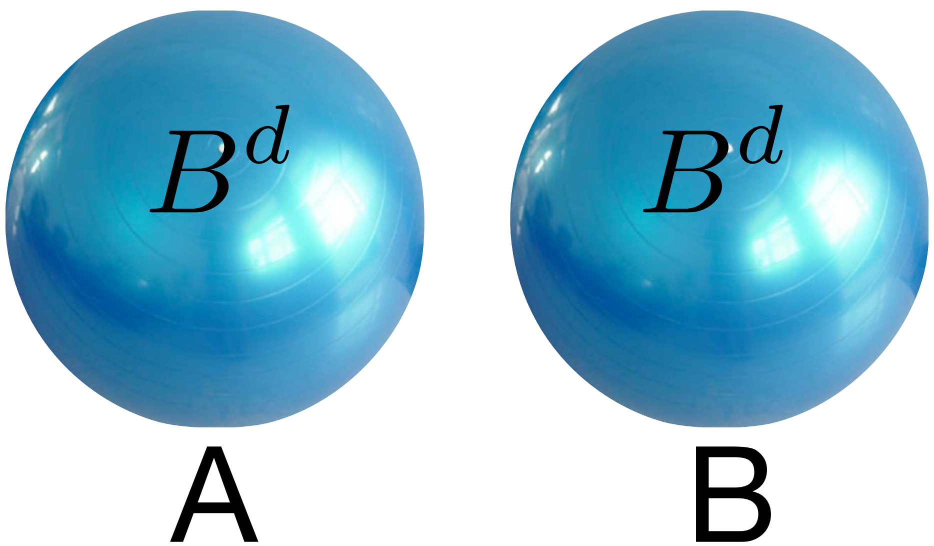

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abstract groups	d
$\mathrm{SO}(d)$	$3, 4, 5 \ldots$
$\mathrm{SU}(d/2)$	$4, 6, 8 \ldots$
$\mathrm{U}(d/2)$	$2, 4, 6, 8 \ldots$
$\mathrm{Sp}(d/4)$	$8, 12, 16 \ldots$
$\mathrm{Sp}(d/4) \times \mathrm{U}(1)$	$8, 12, 16 \ldots$
$\mathrm{Sp}(d/4) \times \mathrm{SU}(2)$	$4, 8, 12 \ldots$
G_2	7
$\mathrm{Spin}(7)$	8
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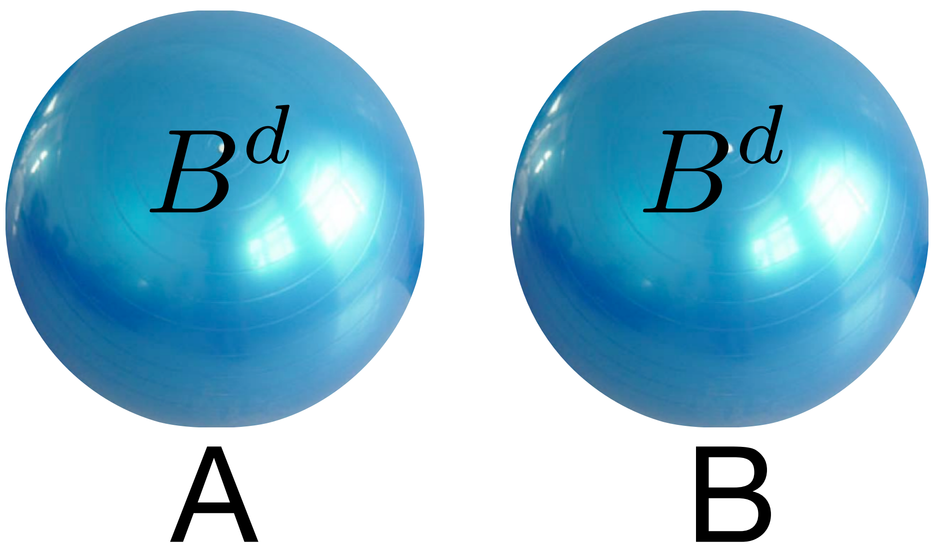

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Additional assumption:
$\mathcal{G}_{AB}$ is a **connected** group.

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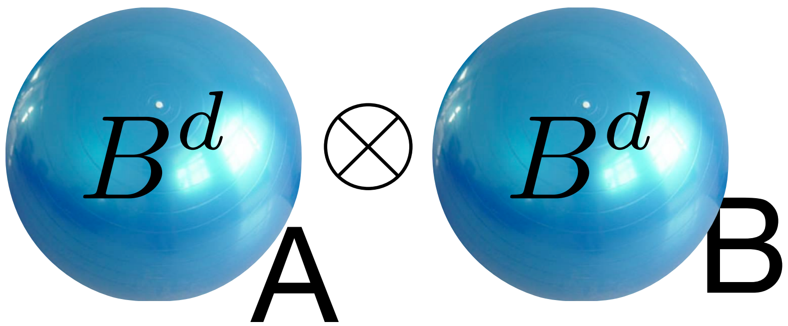

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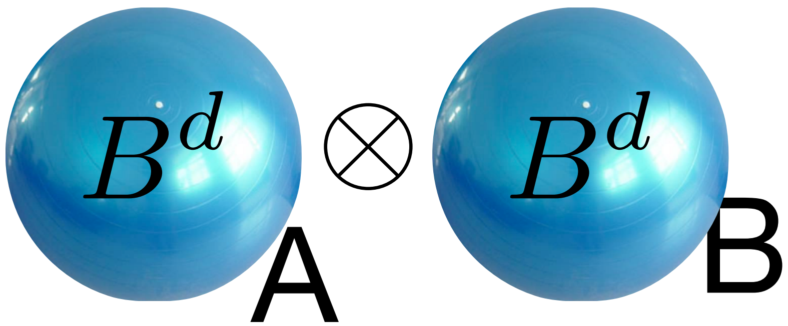

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Theorem. Among all dimensions d and all groups $\mathcal{G}_A$, there are only the following possibilities:

- The trivial solution: $\mathcal{G}_{AB} = \mathcal{G}_A \otimes \mathcal{G}_B$.
- $d = 3$, $\mathcal{G}_A = \mathrm{SO}(3)$ (i.e. the quantum bit), $\mathcal{G}_{AB} \simeq \mathrm{PU}(4)$, and Ω_{AB} is equivalent to the two-qubit quantum state space.

In particular, **continuous reversible interaction** is only possible for $d = 3$, in standard complex two-qubit quantum theory.

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Proof sketch:

- Use Lie algebra properties to get generators that look "simple":

$$X \in \mathfrak{g}_{AB} \;\Rightarrow\; X' := \int_{\mathcal{G}_A \otimes \mathcal{G}_B} (A \otimes B) X (A^{-1} \otimes B^{-1}) dA\, dB \in \mathfrak{g}_{AB}.$$

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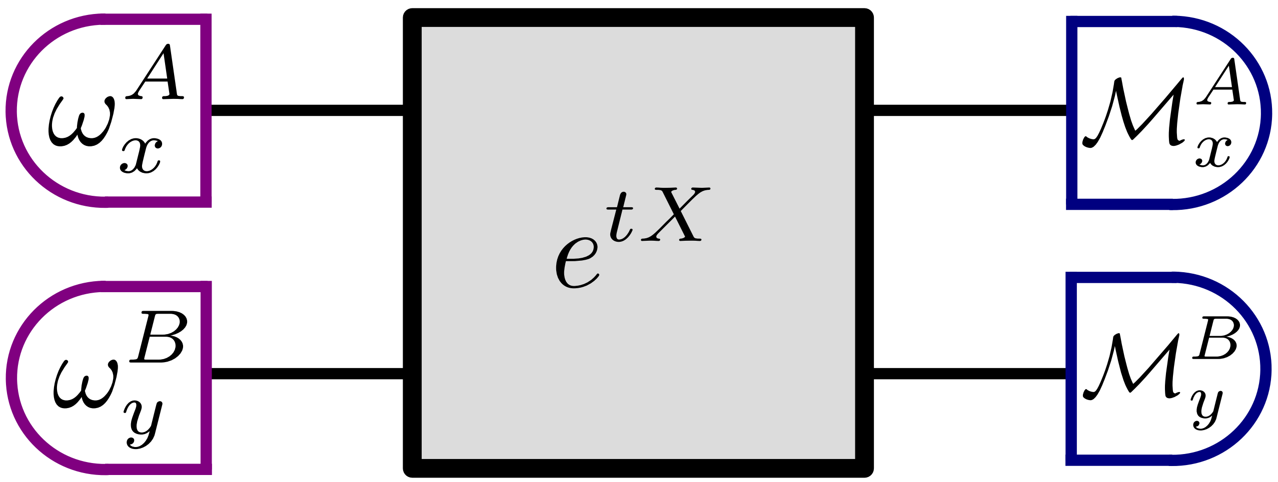

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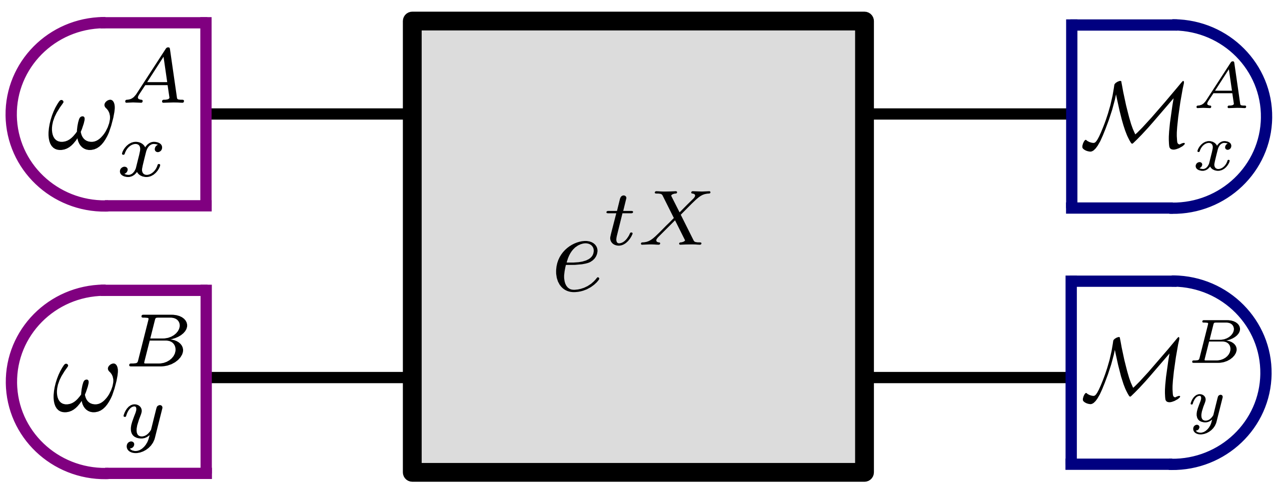

$$\mathcal{M}_x^A(\omega_x^A) = 0 \;\Rightarrow\; \left(\mathcal{M}_x^A \otimes \mathcal{M}_y^B\right) e^{tX} (\omega_x^A \otimes \omega_y^B) \restriction_{t=0} = 0$$

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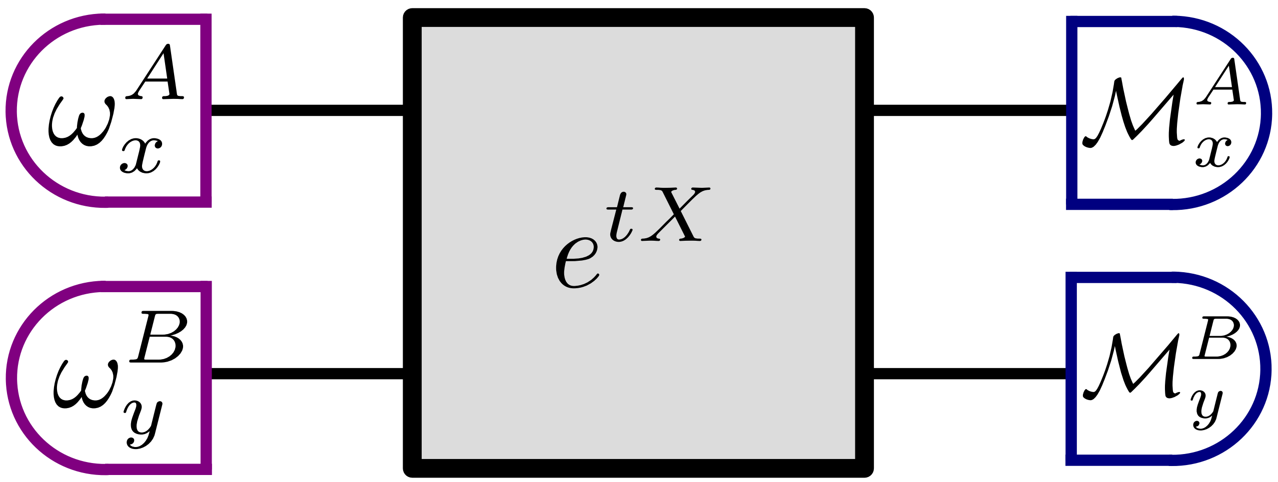

- If $d \neq 3$ then it follows $X = X_A \otimes \mathbf{1}_B + \mathbf{1}_A \otimes X_B$.

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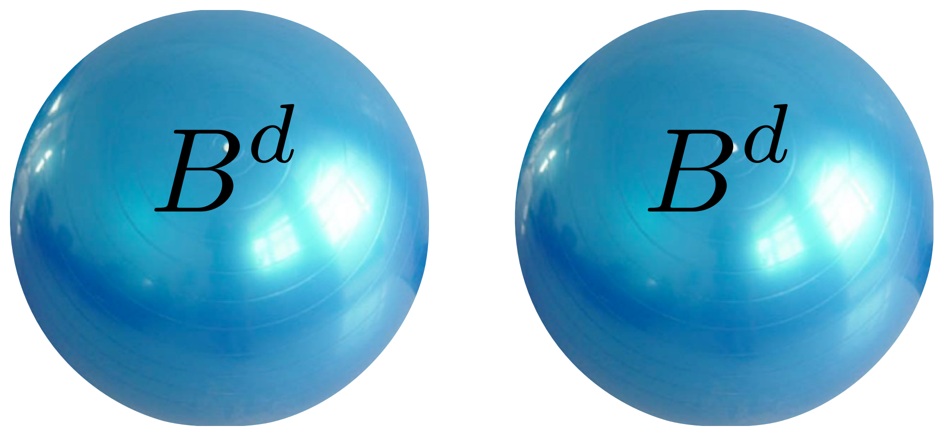

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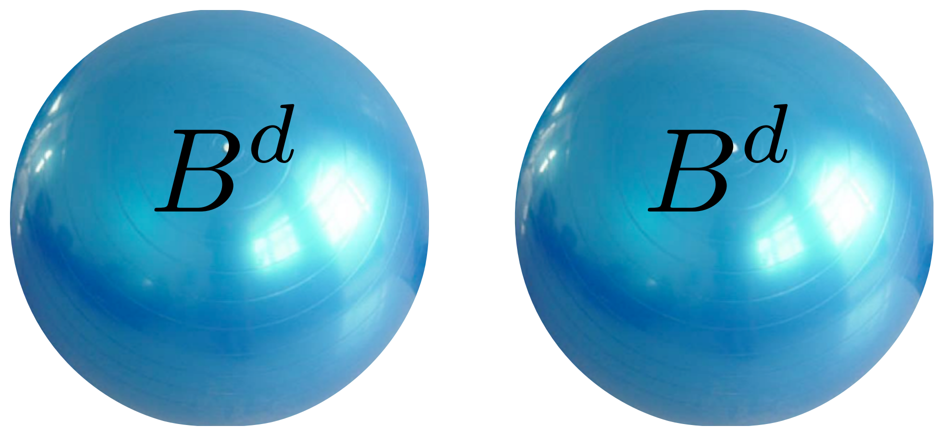

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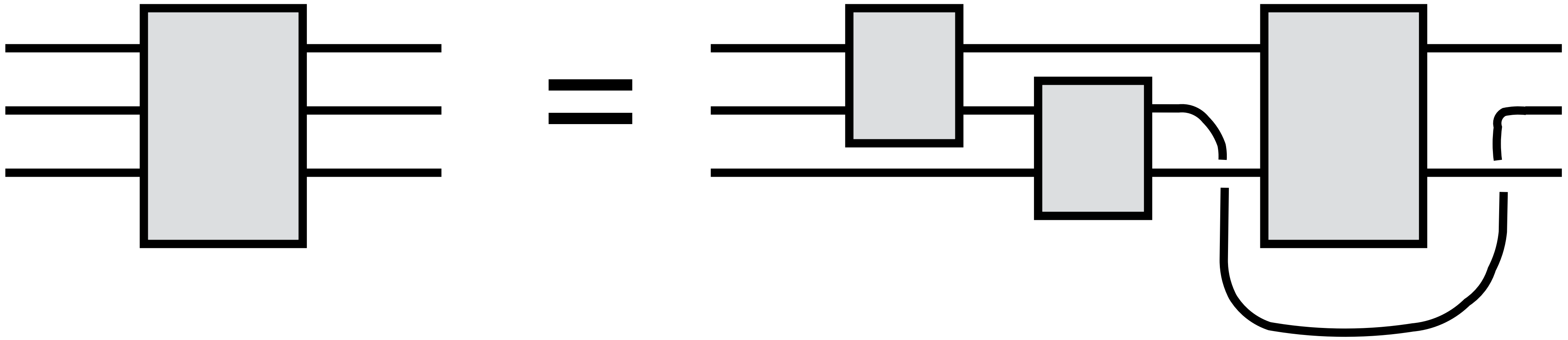

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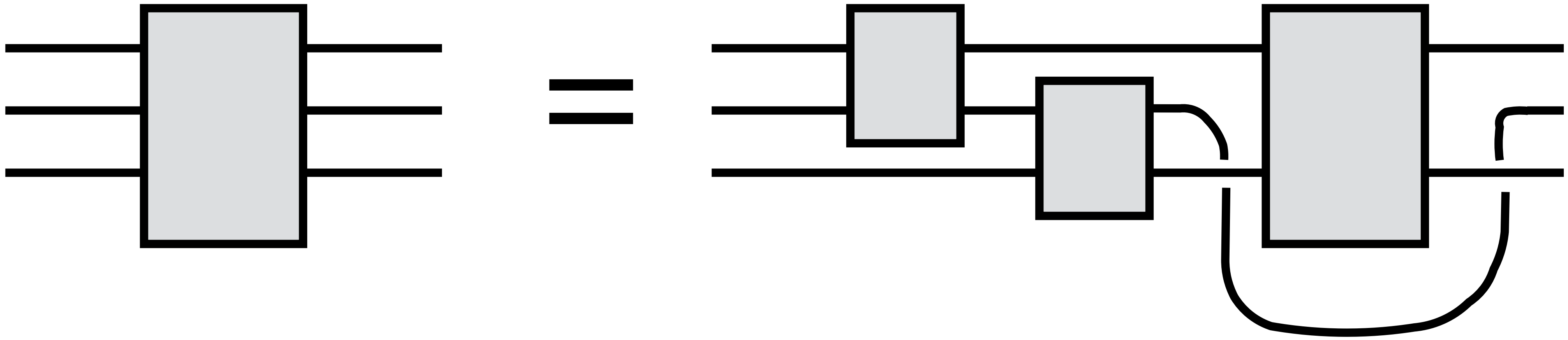

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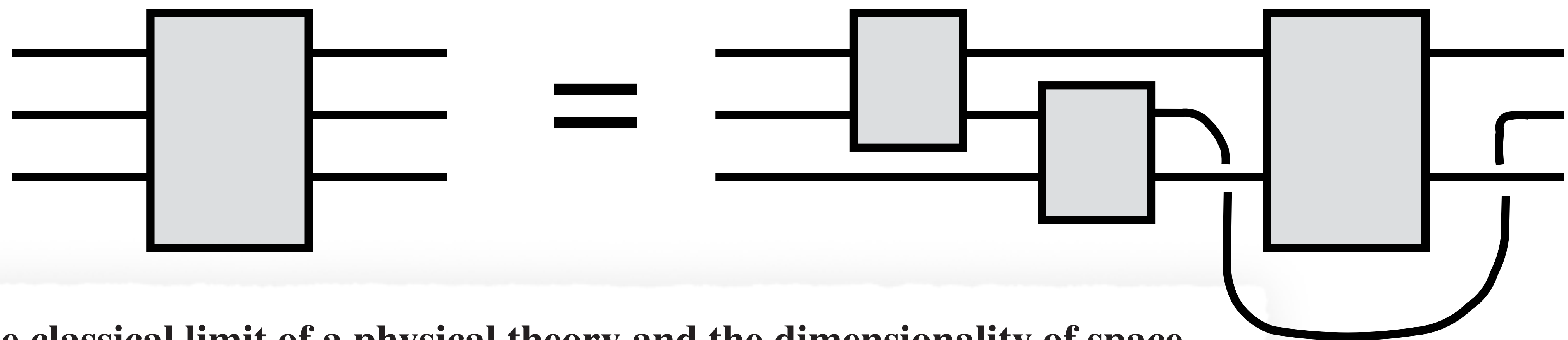

The classical limit of a physical theory and the dimensionality of space

Borivoje Dakić[1,2] and Časlav Brukner[1,3]

[1]*Vienna Center for Quantum Science and Technology (VCQ), Faculty of Physics, University of Vienna, Boltzmanngasse 5, A-1090 Vienna, Austria*
[2]*Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore 117543*
[3]*Institute of Quantum Optics and Quantum Information (IQOQI), Austrian Academy of Sciences, Boltzmanngasse 3, A-1090 Vienna, Austria*

In the operational approach to general probabilistic theories one distinguishes two spaces, the state space of

embedded in a higher-dimensional physical space? We show that as long as the interaction is pairwise, this is impossible, and quantum mechanics and the three-dimensional space remain the only solution. However, having multi-particle interactions and a generalized notion of "classical field" may open up such a possibility.

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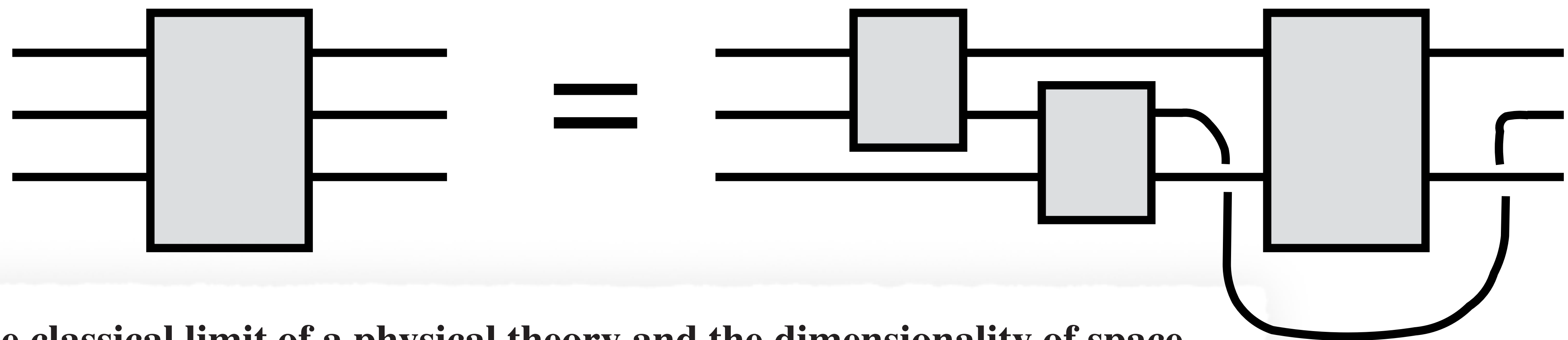
3. Multipartite interaction?

In QT, multipartite interactions can be decomposed into pairs:



The classical limit of a physical theory and the dimensionality of space

Borivoje Dakić[1,2] and Časlav Brukner[1,3]

[1] *Vienna Center for Quantum Science and Technology (VCQ), Faculty of Physics, University of Vienna, Boltzmanngasse 5, A-1090 Vienna, Austria*
[2] *Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore 117543*
[3] *Institute of Quantum Optics and Quantum Information (IQOQI), Austrian Academy of Sciences, Boltzmanngasse 3, A-1090 Vienna, Austria*

In the operational approach to general probabilistic theories one distinguishes two spaces, the state space of

embedded in a higher-dimensional physical space? We show that as long as the interaction is pairwise, this is impossible, and quantum mechanics and the three-dimensional space remain the only solution. However, having multi-particle interactions and a generalized notion of "classical field" may open up such a possibility.

But maybe this is **false** in many "nice" theories beyond QT?

Interaction among (d-1) many d-balls?

3. Multipartite interaction?

Joint work (in progress, unpublished)
with Marius Krumm

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Theorem: There is no tomographically local interaction among $n \geq 2$ many d-ball state spaces, if $d \in \{5, 7, 9, 11, 13, \ldots\}$ and if the group of local reversible transformations is $\mathrm{SO}(d)$.

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Theorem: There is no tomographically local interaction among $n \geq 2$ many d-ball state spaces, if $d \in \{5, 7, 9, 11, 13, \ldots\}$ and if the group of local reversible transformations is $\mathrm{SO}(d)$.

Disproves part of Brukner's and Dakic's conjecture.

Summary

Conjecture:

If some Ω_{AB} is a **locally tomographic** composite of some Ω_A and Ω_B, and all three state spaces satisfy **reversibility**, and there is at least one reversible transformation $T_{AB} \neq T_A \otimes T_B$, then Ω_{AB} is a (subspace of a) **quantum** state space.

- **Counterexamples** would be extremely **interesting** for physics, mathematics and computer science.
- **Evidence** for conjecture: only pairs of quantum Bloch balls can interact reversibly (singling out *d=3* and the quantum state space).

  Ll. Masanes, MM, D. Pérez-García, R. Augusiak, *Entanglement and the three-dimensionality of the Bloch ball*, J. Math. Phys. **55**, 122203 (2014).
- **Hope for "counterex".:** multipartite interaction. Being killed now. :(