

Custom Compact Closed Categories via Generalized Relations

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Compact Closed Categories and Diagrammatic Reasoning

The multi-disciplinary world of string diagrams and compact closed categories:

- ▶ Pure state quantum mechanics **FdHilb**, mixed state quantum mechanics **CPM(FdHilb)**
- ▶ Distributional semantics of language **FdVect** and semantics of linguistic ambiguity **CPM(FdVect)**
- ▶ Non-deterministic computation - **Rel**, also a popular toy and counter-model
- ▶ Decorated Cospans and Corelations - Networks and beyond
- ▶ Spans - Distributed Systems

Where do we find good settings for new applications?

Motivating Example I

Convexity

Mathematical models of cognition (Gärdenfors) emphasize convexity, how can we address this in a compact closed setting?

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- ▶ The finite distribution monad D

$$X \mapsto \{d : X \rightarrow [0, 1] \mid d \text{ has finite support and } \sum d(x) = 1\}$$

- ▶ Algebras in $\mathbf{EM}(D)$ are sets with a “convex mixing operation” $D(X) \rightarrow X$
- ▶ $\mathbf{EM}(D)$ is regular so we can form a compact closed category $\mathbf{Rel}(D)$ in which morphisms are binary relations $R : A \rightarrow B$ such that

$$R(a_1, b_1) \wedge \dots \wedge R(a_n, b_n) \Rightarrow R\left(\sum_i p_i a_i, \sum_i p_i b_i\right)$$

Motivating Example II

Metrics

Cognition also requires metrics, but categories of metric spaces are not regular, so we cannot use the previous trick. What to do?

Motivating Example II

Metrics

Cognition also requires metrics, but categories of metric spaces are not regular, so we cannot use the previous trick. What to do? A binary relation on sets can be identified with a map

$$A \times B \rightarrow 2$$

Using this formulation, we generalize the truth values to an arbitrary quantale Q

$$A \times B \rightarrow Q$$

$$1_A(a, a') = \bigvee \{k \mid a = a'\} \quad (S \circ R)(a, c) = \bigvee_b R(a, b) \otimes S(b, c)$$

These **Q -relations** form a category **$\mathbf{Rel}(Q)$** , and if Q is a commutative quantale then **$\mathbf{Rel}(Q)$** is a compact closed category.

Motivating Example II

Metrics

Order Structure

We can order Q -relations pointwise using the quantale ordering

$$R \subseteq R' \quad \text{iff} \quad \forall a, b. R(a, b) \leq R'(a, b)$$

This ordering makes $\mathbf{Rel}(Q)$ a **Poset**-enriched category. We can therefore talk about $\mathbf{Rel}(Q)$ -internal monads. These are endomorphisms satisfying

$$1 \subseteq R \quad \text{and} \quad R \circ R \subseteq R$$

This parallels the usual notion of a monad on a category being an endofunctor $T : \mathcal{C} \rightarrow \mathcal{C}$ with

$$\eta : 1 \Rightarrow T \quad \text{and} \quad T \circ T \Rightarrow T$$

Motivating Example II

Metrics

Example (Internal monads in $\mathbf{Rel}(Q)$)

- ▶ For quantale $B = \{0, 1\}$

$$R(a, a) \quad \text{and} \quad R(a, b) \wedge R(b, c) \Rightarrow R(a, c)$$

- ▶ For quantale $I = [0, 1]$

$$R(a, a) = 1 \quad \text{and} \quad R(a, b) \wedge R(b, c) \leq R(a, c)$$

- ▶ For quantale $C = ([0, \infty], \bigvee = \inf, k = 0, \otimes = +)$

$$R(a, a) = 0 \quad \text{and} \quad R(a, b) + R(b, c) \geq R(a, c)$$

- ▶ For quantale $F = ([0, \infty], \bigvee = \inf, k = 0, \otimes = \max)$

$$R(a, a) = 0 \quad \text{and} \quad \max(R(a, b), R(b, c)) \geq R(a, c)$$

Generalizing

From the ad-hoc to theory

- ▶ We used a couple of ad-hoc tricks
 - ▶ Relations in regular categories, particularly from algebraic structure
 - ▶ Relations with truth values in a commutative quantale
 - ▶ We observe that both approaches actually produce ordered hypergraph categories
- ▶ Questions
 - ▶ Can we relate / combine these two schemes?
 - ▶ Can relations be varied in other ways to generate yet more examples?
 - ▶ How can we relate constructions with different parameters?

Generalized Relations

Algebraic Structure and Generalized Truth

Starting with Q -relations, we aim to incorporate algebraic structure.

Algebraic Structure for Q -relations

To incorporate algebraic structure, we fix a signature of operation symbols Σ , and a set of equations over that signature. We need to generalize the condition

$$R(a_1, b_1) \wedge \dots \wedge R(a_n, b_n) \Rightarrow R(\sigma(a_1, \dots, a_n), \sigma(b_1, \dots, b_n))$$

We exploit the operations of our quantale, leading to the following condition for each $\sigma \in \Sigma$

$$R(a_1, b_1) \otimes \dots \otimes R(a_n, b_n) \leq R(\sigma(a_1, \dots, a_n), \sigma(b_1, \dots, b_n))$$

We refer to such a Q -relation as **algebraic**.

Generalized Relations

The General Construction

We note that we can interpret our definitions in an arbitrary topos.

Theorem

If \mathcal{E} is a topos, Q an internal commutative quantale, and (Σ, E) an algebraic variety in \mathcal{E} then

- ▶ *There is a category $\mathbf{Rel}_{(\Sigma, E)}(Q)$ with objects (Σ, E) -algebras, and morphisms algebraic Q -relations.*
- ▶ *$\mathbf{Rel}_{(\Sigma, E)}(Q)$ has a symmetric monoidal structure given by products in \mathcal{E} .*
- ▶ *$\mathbf{Rel}_{(\Sigma, E)}(Q)$ is poset enriched with respect to the ordering*

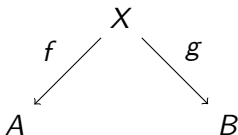
$$R \subseteq R' \quad \text{iff} \quad \forall a, b. R(a, b) \leq R'(a, b)$$

- ▶ *$\mathbf{Rel}_{(\Sigma, E)}(Q)$ is a hypergraph category*

Spans

Spans are Constructive Relations I

A span of sets consists of the data



Interpretation - Constructive Relations

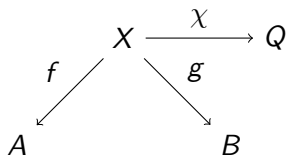
Elements of the apex X are proof witnesses for relatedness, we write

$$\begin{array}{c} x \\ \swarrow \quad \searrow \\ a \qquad \qquad b \end{array} \quad \text{if} \quad f(x) = a \quad \text{and} \quad g(x) = b$$

Spans

Incorporating Truth Values

We introduce a monoid $Q = (Q, \otimes, k)$ of truth values, and a characteristic morphism



We call such a span a Q -span, and write

$$a \begin{array}{c} \swarrow x^q \\ \searrow \end{array} b \quad \text{if} \quad f(x) = a \quad \text{and} \quad g(x) = b \quad \text{and} \quad \chi(x) = q$$

Spans

Algebraic Structure

We fix variety (Σ, E) . An **algebraic** Q -span is a Q -span with domain and codomain (Σ, E) -algebras, satisfying the condition that if for every $\sigma \in \Sigma$

$$a_1 \overset{x_1^{q_1}}{\swarrow \searrow} b_1 \quad \wedge \dots \wedge \quad a_n \overset{x_n^{q_n}}{\swarrow \searrow} b_n$$

Then there exists x such that

$$\begin{array}{ccc} & x^q & \\ & \swarrow \searrow & \\ \sigma(a_1, \dots, a_n) & & \sigma(b_1, \dots, b_n) \end{array} \quad \text{and } q_1 \otimes \dots \otimes q_n \leq q$$

Note the need for order structure on the truth values.

Spans

Algebraic Q -spans

Theorem

If \mathcal{E} is a topos, (Σ, E) a variety in \mathcal{E} , and Q an internal partially ordered commutative monoid then

- ▶ Algebraic Q -spans form a category $\mathbf{Span}_{(\Sigma, E)}(Q)$
- ▶ The category $\mathbf{Span}_{(\Sigma, E)}(Q)$ is a hypergraph category
- ▶ $\mathbf{Span}_{(\Sigma, E)}(Q)$ is a **Preord**-enriched category with

$$(X_1, f_1, g_1, \chi_1) \subseteq (X_2, f_2, g_2, \chi_2)$$

if there is a \mathcal{E} -monomorphism $m : X_1 \rightarrow X_2$ such that

$$f_1 = f_2 \circ m \text{ and } g_1 = g_2 \circ m \text{ and } \forall x. \chi_1(x) \leq \chi_2(m(x))$$

Relations and Spans

Killing Witnesses

If Q is a commutative quantale, we can turn an algebraic Q -span into an algebraic Q -relation via

$$V(X, f, g, \chi)(a, b) = \bigvee \{ \chi(x) \mid f(x) = a \wedge g(x) = b \}$$

Relations and Spans

Killing Witnesses

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$$V(X, f, g, \chi)(a, b) = \bigvee \{ \chi(x) \mid f(x) = a \wedge g(x) = b \}$$

Theorem

Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} and Q an internal commutative quantale. There is a strict monoidal, identity and surjective on objects, preorder-functor

$$V : \mathbf{Span}_{(\Sigma, E)}(Q) \rightarrow \mathbf{Rel}_{(\Sigma, E)}(Q)$$

Parameterized Constructions

Summary

We have shown a conceptually motivated procedure for constructing preordered hypergraph categories. These categories can be customized along 4 axes of variation

1. The ambient mathematical universe
2. The truth values
3. The algebraic structure
4. Proof relevance versus provability

Generalized Relations

Examples

The following examples can be constructed using this procedure.

- ▶ **Rel**
- ▶ **Rel**(C) - internal monads the generalized metric spaces
- ▶ **Rel**(F) - internal monads the generalized ultrametric spaces
- ▶ The category **Rel**(**EM**(D)) of convex relations arises for a suitable choice of (Σ, E) and Q
- ▶ The category of linear relations used in models of linear dynamical systems

We get new examples worthy of further investigation

- ▶ Blending both convexity and metrics
- ▶ Models varying with context using presheaf toposes
- ▶ Models with witnesses for relatedness using spans

Conclusion

Further Results

- ▶ Our constructions are functorial in the choice of truth values
- ▶ They are also functorial in the algebraic structure - linearity
- ▶ Symmetric Monoidal graph functors

Looking Further

- ▶ Structure - zero objects, biproducts etc.
- ▶ Functoriality in the choice of topos
- ▶ Our category of spans should really be a symmetric monoidal bicategory
- ▶ We can take truth values in monoidal categories - unify the span and relation constructions