Modelling interconnected systems with decorated corelations

Brendan Fong
University of Pennsylvania

Workshop on Compositionality, 5–9 December 2016
Simons Institute for the Theory of Computing, Berkeley

All **hypergraph categories** are **decorated corelation categories**.

Context

David (yesterday): Introduced hypergraph categories.

John (this morning): Introduced decorated cospans.

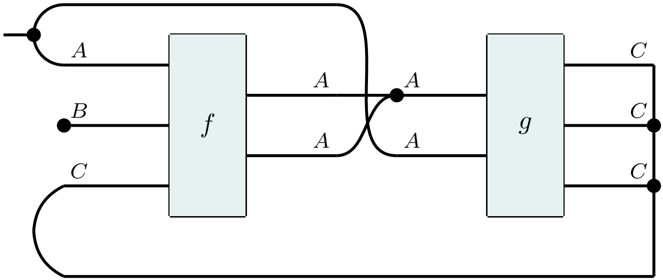
Me (now): All hypergraph categories are decorated corelation categories.

Dan (next): Hypergraph categories via relations.

Ross (tomorrow): Hypergraph categories in categorical quantum mechanics.

Hypergraph categories model network compositionality

Hypergraph categories model network compositionality



Hypergraph categories model network compositionality

A **hypergraph category** is a symmetric monoidal category in which each object is equipped with a special commutative Frobenius monoid in a way coherent with the monoidal product.

Hypergraph categories model network compositionality

A

A **hypergraph category** is a symmetric monoidal category in which each object is equipped with a special commutative Frobenius monoid in a way coherent with the monoidal product.

Hypergraph categories model network compositionality

A

B

A **hypergraph category** is a symmetric monoidal category in which each object is equipped with a special commutative Frobenius monoid in a way coherent with the monoidal product.

Hypergraph categories model network compositionality

A

B

C

A **hypergraph category** is a symmetric monoidal category in which each object is equipped with a special commutative Frobenius monoid in a way coherent with the monoidal product.

Hypergraph categories model network compositionality

A

A

B

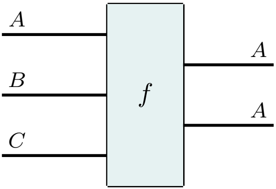
C

A **hypergraph category** is a symmetric monoidal category in which each object is equipped with a special commutative Frobenius monoid in a way coherent with the monoidal product.

Hypergraph categories model network compositionality

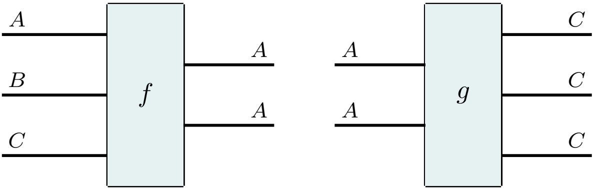
A **hypergraph category** is a symmetric monoidal category in which each object is equipped with a special commutative Frobenius monoid in a way coherent with the monoidal product.

Hypergraph categories model network compositionality



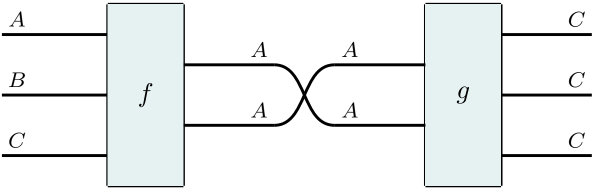
A **hypergraph category** is a symmetric monoidal category in which each object is equipped with a special commutative Frobenius monoid in a way coherent with the monoidal product.

Hypergraph categories model network compositionality



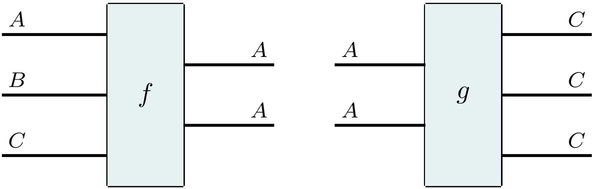
A **hypergraph category** is a symmetric monoidal category in which each object is equipped with a special commutative Frobenius monoid in a way coherent with the monoidal product.

Hypergraph categories model network compositionality



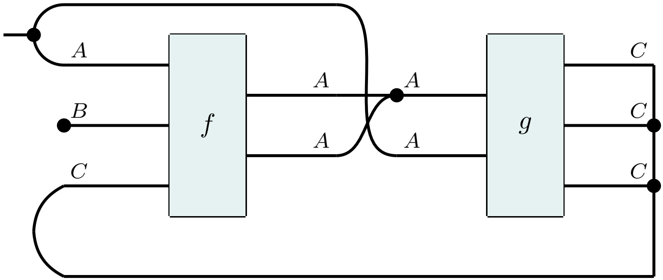
A **hypergraph category** is a symmetric monoidal category in which each object is equipped with a special commutative Frobenius monoid in a way coherent with the monoidal product.

Hypergraph categories model network compositionality

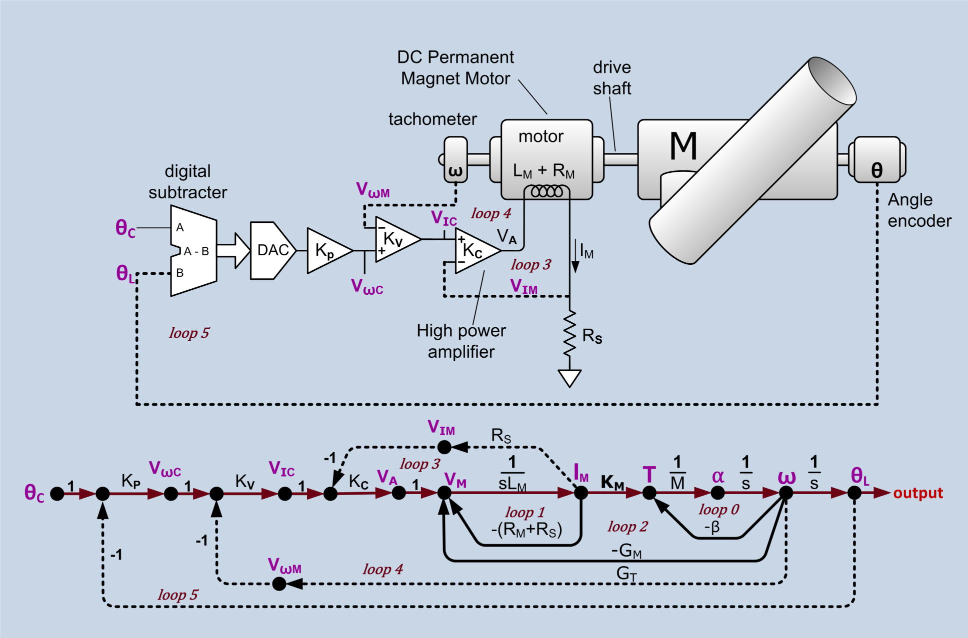


A **hypergraph category** is a symmetric monoidal category in which each object is equipped with a special commutative Frobenius monoid in a way coherent with the monoidal product.

Hypergraph categories model network compositionality



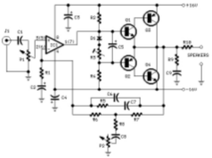
A **hypergraph category** is a symmetric monoidal category in which each object is equipped with a special commutative Frobenius monoid in a way coherent with the monoidal product.

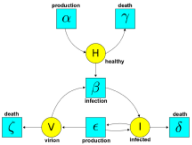


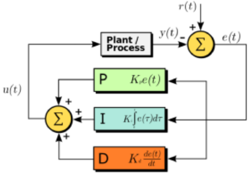
DC Permanent Magnet Motor
drive shaft
tachometer
motor
LM + RM
M
ω
θ
Angle encoder
digital subtracter
θC
θL
A
A - B
B
DAC
Kp
VωM
KV
VIC
KC
VA
loop 4
loop 3
IM
VωC
VIM
High power amplifier
RS
loop 5
VIM
RS
-1
loop 3
1
sLM
IM
KM
T
1
M
α
1
s
ω
1
s
θL
output
θC
1
KP
VωC
1
KV
VIC
1
KC
VA
1
VM
loop 1
-(RM+RS)
loop 2
loop 0
-β
-GM
GT
-1
-1
VωM
loop 4
loop 5

Recall from John Baez's talk. . .

In many areas of science and engineering, people use *networks*, drawn as boxes connected by wires:







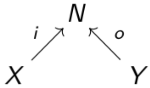
We need a good general theory of these!

Decorated cospans build hypergraph categories

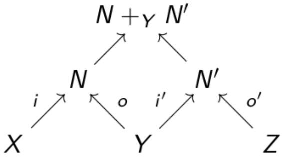
Decorated cospans build hypergraph categories

Also recall. . .

Say we start with a category **C** with finite colimits: in our example, **C** = **FinSet**. We can build a bicategory where morphisms are cospans in **C**:



and composition is done by pushout:



Decorated cospans build hypergraph categories

Let $\mathcal{C}$ have finite colimits. Then $\mathrm{Cospan}(\mathcal{C})$ is a hypergraph category.

Decorated cospans build hypergraph categories

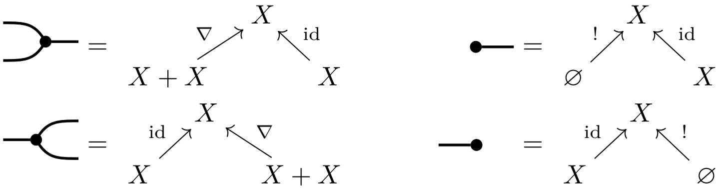
Let $\mathcal{C}$ have finite colimits. Then $\mathrm{Cospan}(\mathcal{C})$ is a hypergraph category. The monoidal product is the coproduct $+$ in $\mathcal{C}$.

Decorated cospans build hypergraph categories

Let $\mathcal{C}$ have finite colimits. Then $\mathrm{Cospan}(\mathcal{C})$ is a hypergraph category.

The monoidal product is the coproduct $+$ in $\mathcal{C}$.

The Frobenius maps are given by the codiagonal map $\nabla\colon X+X\to X$ and the initial map $!\colon \varnothing \to X$.

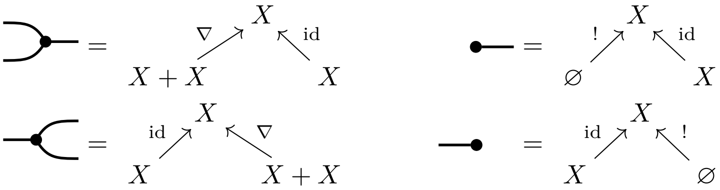


Decorated cospans build hypergraph categories

Let $\mathcal{C}$ have finite colimits. Then $\mathrm{Cospan}(\mathcal{C})$ is a hypergraph category.

The monoidal product is the coproduct $+$ in $\mathcal{C}$.

The Frobenius maps are given by the codiagonal map $\nabla\colon X+X \to X$ and the initial map $!\colon \varnothing \to X$.

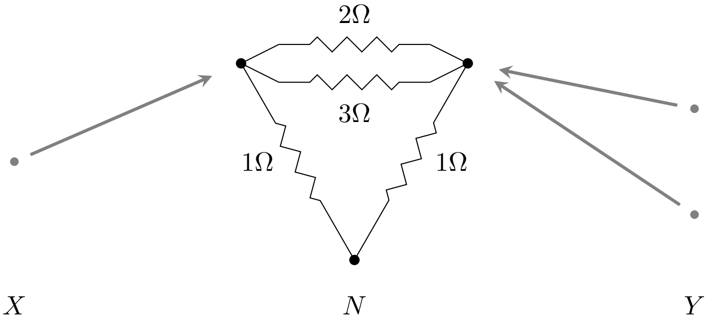


Decorated cospan categories inherit this hypergraph structure via the embedding $\mathrm{Cospan}(\mathcal{C}) \to F\mathrm{Cospan}$.

Decorated cospan categories are good for syntax

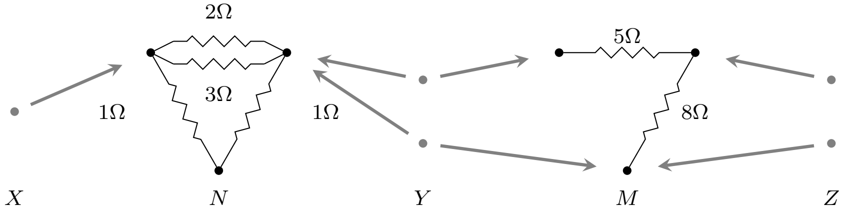
But when composing decorated cospans, the morphism grows

Decorated cospan categories are good for syntax



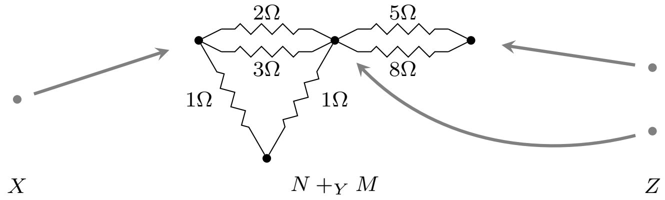
But when composing decorated cospans, the morphism grows

Decorated cospan categories are good for syntax



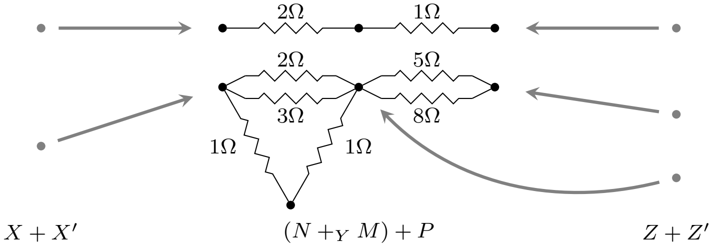
But when composing decorated cospans, the morphism grows

Decorated cospan categories are good for syntax



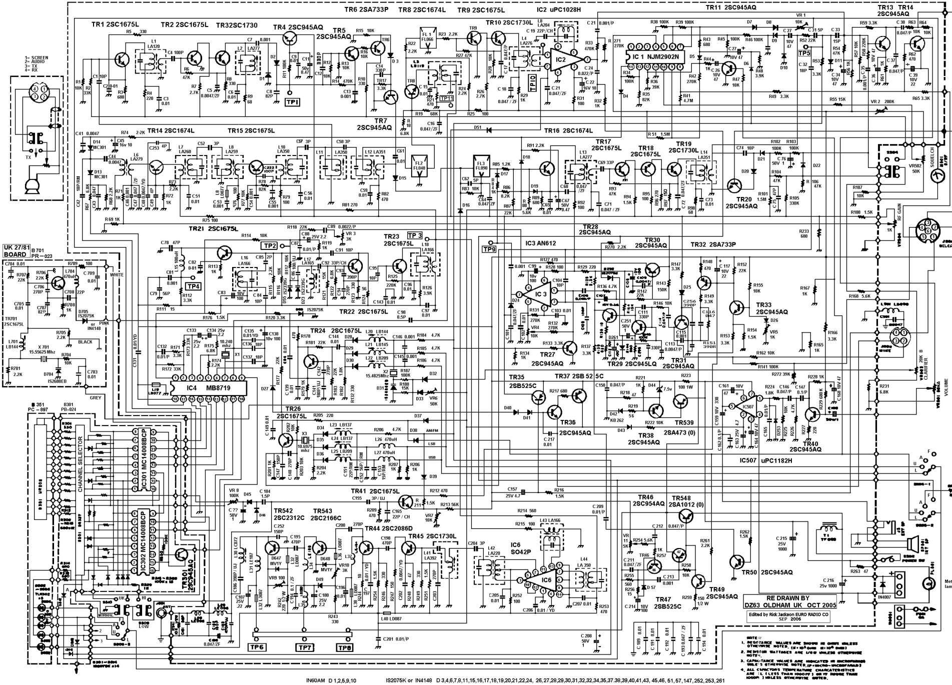
But when composing decorated cospans, the morphism grows, and grows

Decorated cospan categories are good for syntax



But when composing decorated cospans, the morphism grows, and grows, and grows. . .

PC 893 STALKER 9F DX

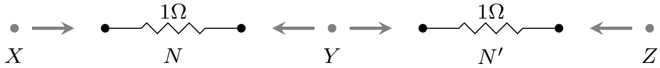


IN60AM D 1,2,5,9,10 IS2075K or IN4148 D 3,4,6,7,8,11,15,16,17,18,19,20,21,22,24, 26,27,28,29,30,31,32,32,34,35,37.38,39,40,41,43, 45,46, 51,57,147,252,253,261

What about hypergraph categories for semantics?

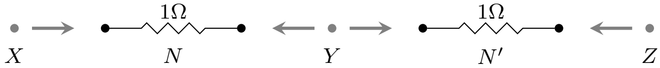
Decorated corelations are better for semantics

Consider the pair of decorated cospans

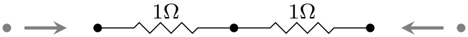


Decorated corelations are better for semantics

Consider the pair of decorated cospans

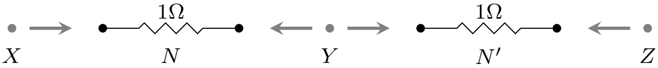


Their composite is

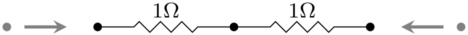


Decorated corelations are better for semantics

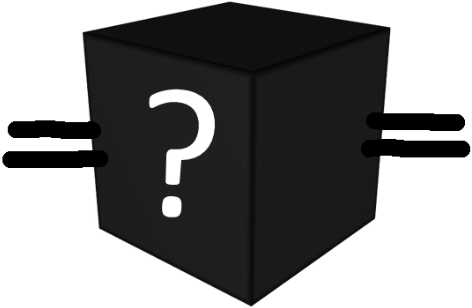
Consider the pair of decorated cospans



Their composite is

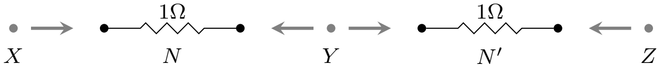


But this is, in an extensional sense, the same as

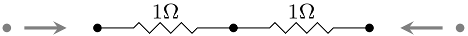


Decorated corelations are better for semantics

Consider the pair of decorated cospans



Their composite is

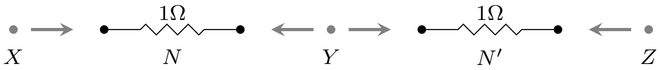


But this is, in an extensional sense, the same as

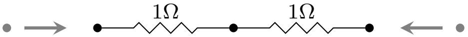
To construct a category which does not see the difference between these two circuits, we use decorated corelations.

Decorated corelations are better for semantics

Consider the pair of decorated cospans



Their composite is

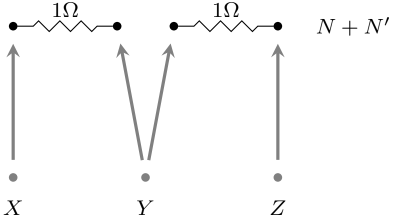


But this is, in an extensional sense, the same as

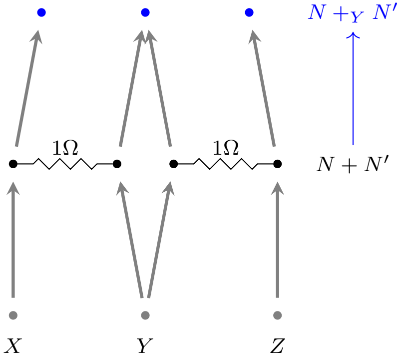
To construct a category which does not see the difference between these two circuits, we use decorated corelations.

The key idea is that we only want the part of a decoration that lives on the boundary.

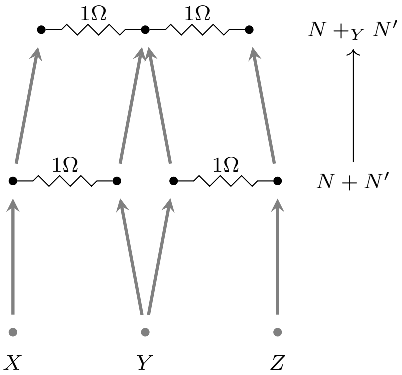
Decorated corelations are better for semantics



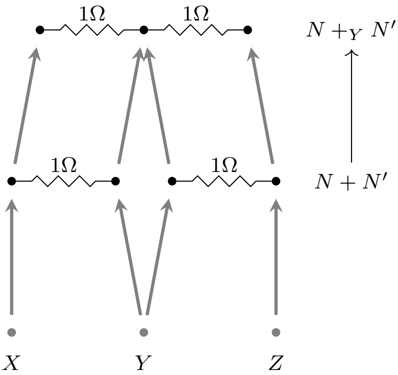
Decorated corelations are better for semantics



Decorated corelations are better for semantics

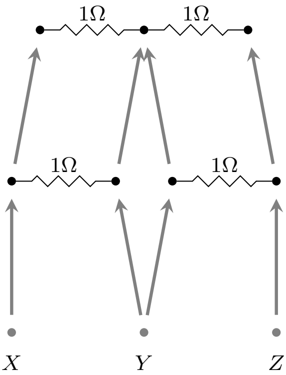


Decorated corelations are better for semantics



$$\Big\{\big(x, y, x-y, y-x\big)\Big\} \subseteq \mathbb{R}^N \oplus \mathbb{R}^N \qquad \Big\{\big(y', z, y'-z, z-y'\big)\Big\} \subseteq \mathbb{R}^{N'} \oplus \mathbb{R}^{N'}$$

Decorated corelations are better for semantics

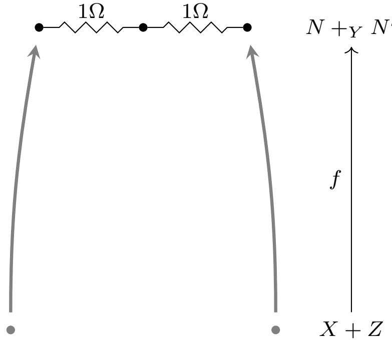


$$N +_Y N' \qquad \left\{ \Big(x, y, z, x-y, 2y-(x+z), z-y \Big) \right\} \subseteq \mathbb{R}^{N+_Y N'} \oplus \mathbb{R}^{N+_Y N'}$$

$$\uparrow$$

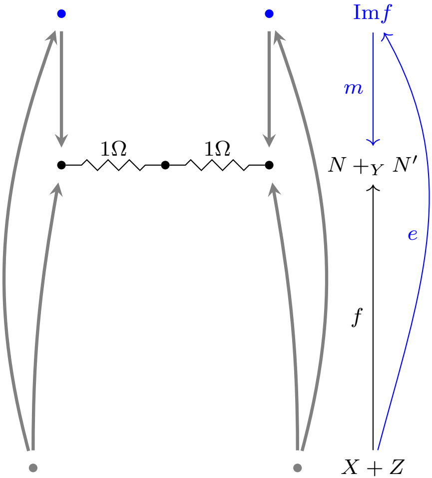
$$N + N' \qquad \left\{ \Big(x, y, x-y, y-x \Big) \right\} \subseteq \mathbb{R}^{N} \oplus \mathbb{R}^{N} \qquad \left\{ \Big(y', z, y'-z, z-y' \Big) \right\} \subseteq \mathbb{R}^{N'} \oplus \mathbb{R}^{N'}$$

Decorated corelations are better for semantics



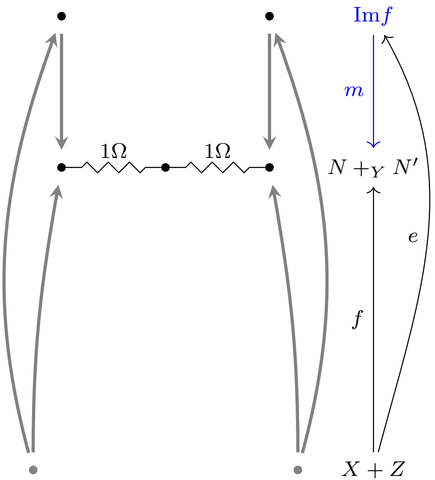
$$\Big\{ \Big(x, y, z, x-y, 2y-(x+z), z-y\Big) \Big\} \subseteq \mathbb{R}^{N+_Y N'} \oplus \mathbb{R}^{N+_Y N'}$$

Decorated corelations are better for semantics



$$\Big\{\Big(x, y, z, x-y, 2y-(x+z), z-y\Big)\Big\} \subseteq \mathbb{R}^{N+_Y N'} \oplus \mathbb{R}^{N+_Y N'}$$

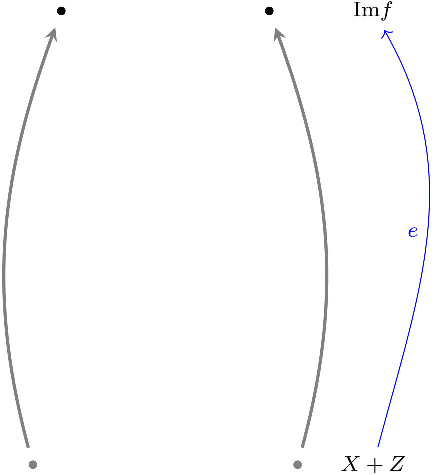
Decorated corelations are better for semantics



$$\left\{\left(x, z, \tfrac{1}{2}(x-z), \tfrac{1}{2}(z-x)\right)\right\} \subseteq \mathbb{R}^{\mathrm{Im} f} \oplus \mathbb{R}^{\mathrm{Im} f}$$

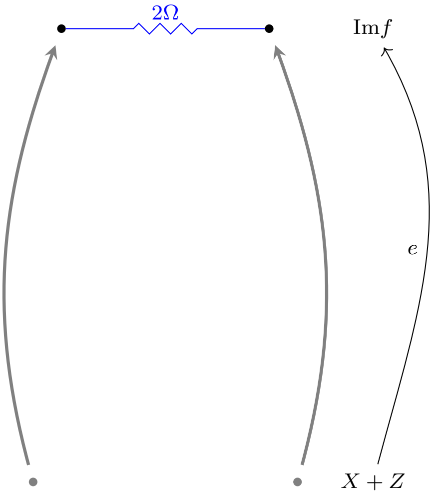
$$\left\{\left(x, y, z, x-y, 2y-(x+z), z-y\right)\right\} \subseteq \mathbb{R}^{N+_Y N'} \oplus \mathbb{R}^{N+_Y N'}$$

Decorated corelations are better for semantics



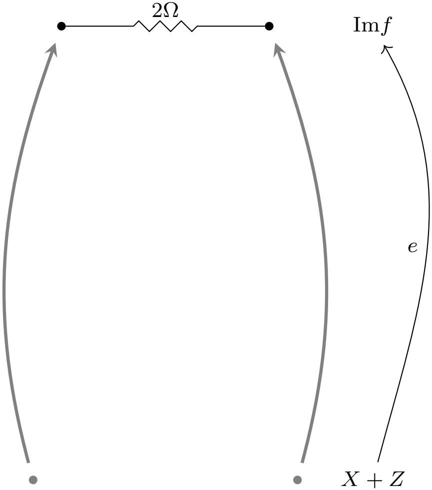
$$\left\{ \left(x, z, \tfrac{1}{2}(x-z), \tfrac{1}{2}(z-x)\right) \right\} \subseteq \mathbb{R}^{\mathrm{Im} f} \oplus \mathbb{R}^{\mathrm{Im} f}$$

Decorated corelations are better for semantics



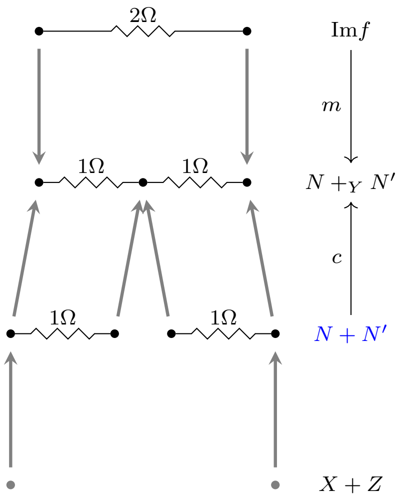
$$\left\{\left(x, z, \tfrac{1}{2}(x-z), \tfrac{1}{2}(z-x)\right)\right\} \subseteq \mathbb{R}^{\mathrm{Im}f} \oplus \mathbb{R}^{\mathrm{Im}f}$$

Decorated corelations are better for semantics



$$\left\{ \left(x, z, \tfrac{1}{2}(x-z), \tfrac{1}{2}(z-x) \right) \right\} \subseteq \mathbb{R}^{\mathrm{Im} f} \oplus \mathbb{R}^{\mathrm{Im} f}$$

Decorated corelations are better for semantics

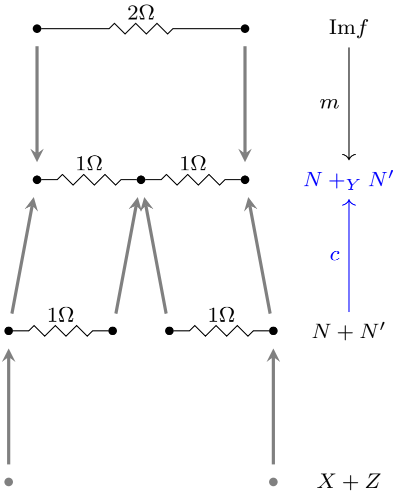


$$\left\{\left(x, z, \tfrac{1}{2}(x-z), \tfrac{1}{2}(z-x)\right)\right\} \subseteq \mathbb{R}^{\mathrm{Im} f} \oplus \mathbb{R}^{\mathrm{Im} f}$$

$$\left\{\left(x, y, z, x-y, 2y-(x+z), z-y\right)\right\} \subseteq \mathbb{R}^{N+_Y N'} \oplus \mathbb{R}^{N+_Y N'}$$

$$\left\{\left(x, y, x-y, y-x\right)\right\} \subseteq \mathbb{R}^{N} \oplus \mathbb{R}^{N} \quad \left\{\left(y', z, y'-z, z-y'\right)\right\} \subseteq \mathbb{R}^{N'} \oplus \mathbb{R}^{N'}$$

Decorated corelations are better for semantics

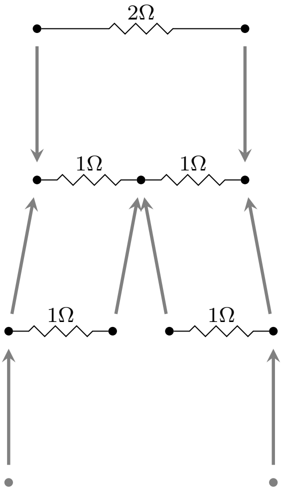


$$\left\{\left(x, z, \tfrac{1}{2}(x-z), \tfrac{1}{2}(z-x)\right)\right\} \subseteq \mathbb{R}^{\mathrm{Im}f} \oplus \mathbb{R}^{\mathrm{Im}f}$$

$$\left\{\left(x, y, z, x-y, 2y-(x+z), z-y\right)\right\} \subseteq \mathbb{R}^{N+_Y N'} \oplus \mathbb{R}^{N+_Y N'}$$

$$\left\{\left(x, y, x-y, y-x\right)\right\} \subseteq \mathbb{R}^{N} \oplus \mathbb{R}^{N} \quad \left\{\left(y', z, y'-z, z-y'\right)\right\} \subseteq \mathbb{R}^{N'} \oplus \mathbb{R}^{N'}$$

Decorated corelations are better for semantics



$\operatorname{Im} f$ $\quad \{(x, z, \frac{1}{2}(x-z), \frac{1}{2}(z-x))\} \subseteq \mathbb{R}^{\operatorname{Im} f} \oplus \mathbb{R}^{\operatorname{Im} f}$

$\downarrow m$

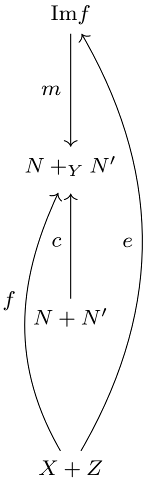
$N +_Y N'$ $\quad \{(x, y, z, x-y, 2y-(x+z), z-y)\} \subseteq \mathbb{R}^{N+_Y N'} \oplus \mathbb{R}^{N+_Y N'}$

$\uparrow c$

$N + N'$ $\quad \{(x, y, x-y, y-x)\} \subseteq \mathbb{R}^{N} \oplus \mathbb{R}^{N}$ $\quad \{(y', z, y'-z, z-y')\} \subseteq \mathbb{R}^{N'} \oplus \mathbb{R}^{N'}$

$X + Z$

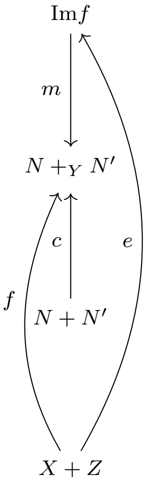
Decorated corelation categories



To recap:

- Write cospan $f\colon X+Z \longrightarrow N +_Y N'$.

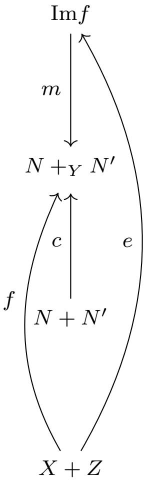
Decorated corelation categories



To recap:

- Write cospan $f\colon X+Z \longrightarrow N+_Y N'$.
- Factor $f = m \circ e$, where $m \in \mathrm{Inj}$ and $e \in \mathrm{Sur}$.

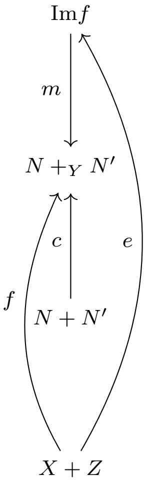
Decorated corelation categories



To recap:

- Write cospan $f\colon X + Z \longrightarrow N +_Y N'$.
- Factor $f = m \circ e$, where $m \in \mathrm{Inj}$ and $e \in \mathrm{Sur}$.
- Transfer decoration along $\xrightarrow{c}\xleftarrow{m}$.

Decorated corelation categories

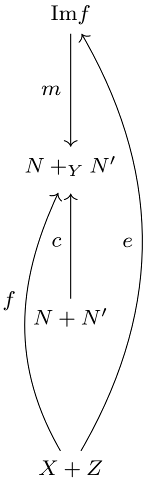


To recap:

- Write cospan $f\colon X+Z \longrightarrow N +_Y N'$.
- Factor $f = m \circ e$, where $m \in \mathrm{Inj}$ and $e \in \mathrm{Sur}$.
- Transfer decoration along $\xrightarrow{c}\xleftarrow{m}$.

More generally, we need a costable factorisation system $(\mathcal{E}, \mathcal{M})$ on $\mathcal{C}$.

Decorated corelation categories



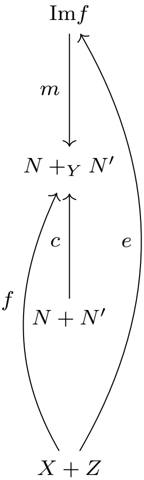
To recap:

- Write cospan $f\colon X + Z \longrightarrow N +_Y N'$.
- Factor $f = m \circ e$, where $m \in \mathrm{Inj}$ and $e \in \mathrm{Sur}$.
- Transfer decoration along $\xrightarrow{c}\xleftarrow{m}$.

More generally, we need a costable factorisation system $(\mathcal{E}, \mathcal{M})$ on $\mathcal{C}$.

Here $\mathcal{C} = \mathrm{FinSet}$, $(\mathcal{E}, \mathcal{M}) = (\mathrm{Sur}, \mathrm{Inj})$.

Decorated corelation categories



To recap:

- Write cospan $f\colon X+Z \longrightarrow N +_Y N'$.
- Factor $f = m \circ e$, where $m \in \mathrm{Inj}$ and $e \in \mathrm{Sur}$.
- Transfer decoration along $\xrightarrow{c}\xleftarrow{m}$.

More generally, we need a costable factorisation system $(\mathcal{E}, \mathcal{M})$ on $\mathcal{C}$.

Here $\mathcal{C} = \mathrm{FinSet}$, $(\mathcal{E}, \mathcal{M}) = (\mathrm{Sur}, \mathrm{Inj})$.

We write $\mathcal{C};\mathcal{M}^{\mathrm{op}}$ for the category with $\xrightarrow{c}\xleftarrow{m}$ as morphisms.

Decorated corelation categories

Theorem
Suppose that $\mathcal{C}$ has finite colimits and a costable factorisation system $(\mathcal{E}, \mathcal{M})$, and

$$F\colon (\mathcal{C}; \mathcal{M}^{\mathrm{op}}, +) \longrightarrow (\mathrm{Set}, \times)$$

is a lax symmetric monoidal functor.

Decorated corelation categories

Theorem
Suppose that $\mathcal{C}$ has finite colimits and a costable factorisation system $(\mathcal{E}, \mathcal{M})$, and

$$F\colon (\mathcal{C}; \mathcal{M}^{\mathrm{op}}, +) \longrightarrow (\mathrm{Set}, \times)$$

is a lax symmetric monoidal functor. Then there is a hypergraph category of F-decorated corelations, $F\mathrm{Corel}$ where

- *an object is an object of $\mathcal{C}$*
- *a morphism from X to Y is a cospan*

$$X \xrightarrow{f} N \xleftarrow{g} Y$$

such that $[f, g]\colon X + Y \to N$ lies in $\mathcal{E}$, together with a decoration $d \in F(N)$. (Actually, an isomorphism class of these!)

Decorated corelation functors

A hypergraph functor is a strong monoidal functor that preserves the Frobenius maps.

Decorated corelation functors

A hypergraph functor is a strong monoidal functor that preserves the Frobenius maps.

Theorem
Suppose that $\mathcal{C}$ has finite colimits and a costable factorisation system $(\mathcal{E}, \mathcal{M})$, and

$$F, G\colon (\mathcal{C}; \mathcal{M}^{\mathrm{op}}) \longrightarrow (\mathrm{Set}, \times)$$

are lax symmetric monoidal functors, and

$$\theta\colon F \Rightarrow G$$

is a monoidal natural transformation.

Decorated corelation functors

A hypergraph functor is a strong monoidal functor that preserves the Frobenius maps.

Theorem
Suppose that $\mathcal{C}$ has finite colimits and a costable factorisation system $(\mathcal{E}, \mathcal{M})$, and

$$F, G \colon (\mathcal{C}; \mathcal{M}^{\mathrm{op}}) \longrightarrow (\mathrm{Set}, \times)$$

are lax symmetric monoidal functors, and

$$\theta \colon F \Rightarrow G$$

is a monoidal natural transformation. Then we obtain a hypergraph functor

$$T_\theta \colon F\mathrm{Corel} \longrightarrow G\mathrm{Corel}.$$

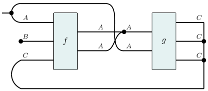
Theorem
Every hypergraph category is equivalent, as a hypergraph category, to a decorated corelation category.

Theorem
Every hypergraph category is equivalent, as a hypergraph category, to a decorated corelation category.

In fact, allowing changes of the base category $\mathcal{C}$ and factorisation system $(\mathcal{E}, \mathcal{M})$, we can define a category of decorated corelation categories. This category is equivalent to the category of hypergraph categories.

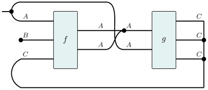
Summary

Hypergraph categories model network compositionality.

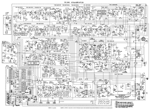


Summary

Hypergraph categories model network compositionality.

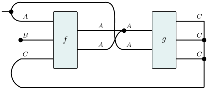


Decorated cospans give hypergraph categories, but ‘freely so’.

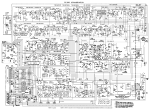


Summary

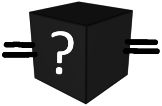
Hypergraph categories model network compositionality.



Decorated cospans give hypergraph categories, but ‘freely so’.

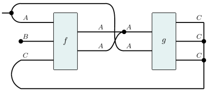


For coarser, ‘black box’ semantics, we can use decorated corelations.

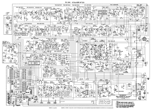


Summary

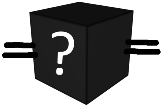
Hypergraph categories model network compositionality.



Decorated cospans give hypergraph categories, but ‘freely so’.



For coarser, ‘black box’ semantics, we can use decorated corelations.



This solution is general:

All **hypergraph categories** are **decorated corelation categories**.

Thanks for listening.

For more
Paper on circuits (with John Baez): arXiv:1504.05625
My thesis: arXiv:1609.05382
John Baez's network theory program: http://math.ucr.edu/baez/networks/
These slides are available at: http://www.brendanfong.com/fcorel.pdf/