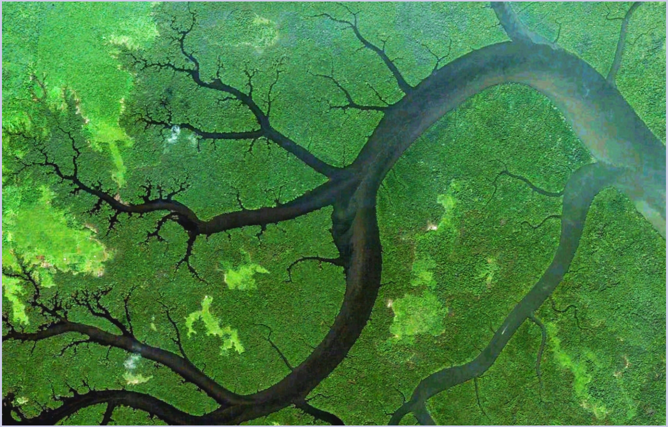
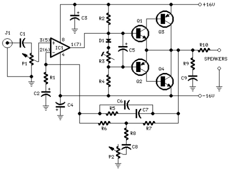
The Mathematics of Networks

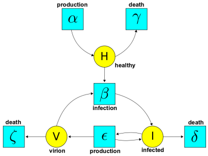


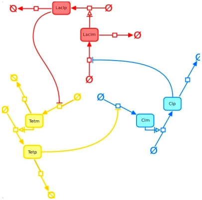
John Baez, U. C. Riverside

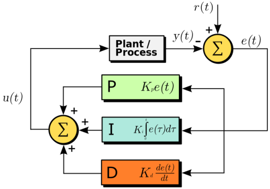
Simons Institute of Computing,
workshop on *Compositionality*, 6 December 2016

In many areas of science and engineering, people use *networks*, drawn as boxes connected by wires:



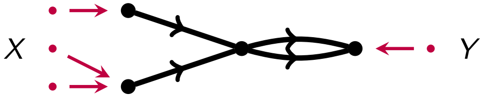






We need a good general theory of these!

Networks of some particular kind, with specified inputs and outputs, can be seen as morphisms in some symmetric monoidal category:



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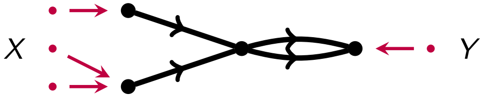

Such networks let us describe *open* systems, meaning systems where:

- stuff can flow in or out;
- we can combine systems to form larger systems by *composition* and *tensoring*.

To use networks as a 'syntax' for open systems, we follow the ideas of 'functorial semantics':

- Networks of some kind, with specified input and outputs, will be morphisms in some symmetric monoidal category $\mathbf{X}$.
- To 'interpret' these networks we use a symmetric monoidal functor $F\colon \mathbf{X} \to \mathbf{Y}$, where $\mathbf{Y}$ is a symmetric monoidal category good for semantics, e.g. **Set** or **Rel**.

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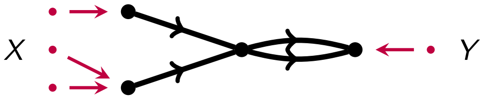
- a set of generating objects;
- a set of generating morphisms;
- a set of relations between morphisms.

We can then specify a (strict) symmetric monoidal functor by sending generators to generators in such a way that relations are preserved.

Another way, pioneered by Brendan Fong, is to use decorated cospans. For example, this:

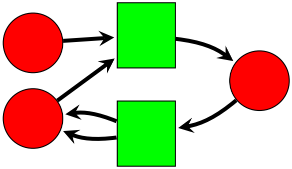


is really a cospan of finite sets:

$$X \xrightarrow{i} S \xleftarrow{o} Y$$

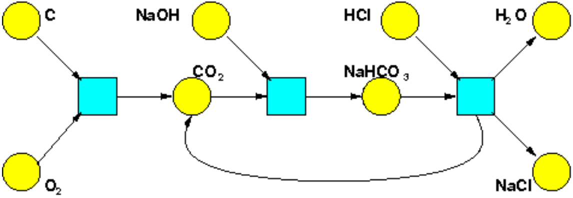
where S is 'decorated' with extra structure making it into the set of vertices of a graph: $E \underset{t}{\overset{s}{\rightrightarrows}} S$.

Let's look at a more interesting example: Petri nets.



A **Petri net** is a bipartite graph. The two kinds of vertices are called **places** and **transitions**.

In computer science, Petri nets became popular as models of concurrency starting in the 1970s. But they were invented for chemistry in 1939:



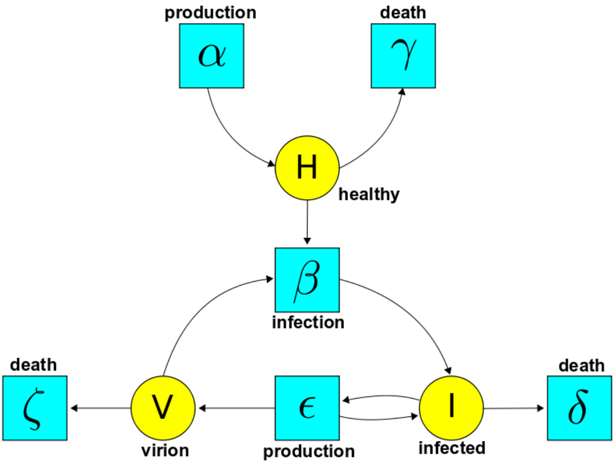
as an alternative to the more familiar reaction networks:

$$C + O_2 \to CO_2$$

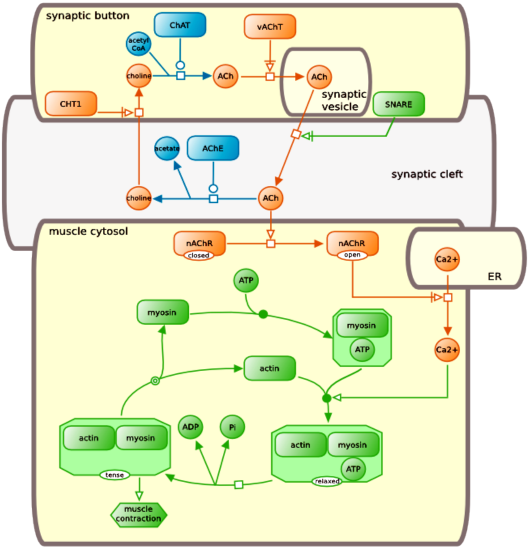
$$CO_2 + NaOH \to NaHCO_3$$

$$NaHCO_3 + HCl \to H_2O + NaCl + CO_2$$

Now they're used in epidemiology...

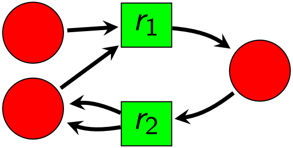


...systems biology...



... and many other fields.

In a Petri net **with rates**, each transition is assigned a **rate constant**: a positive real number. We can then write down a **rate equation** describing dynamics. For example, this Petri net with rates:



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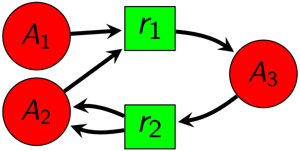

gives this rate equation:

$$\frac{dA_1}{dt} = -r_1\, A_1 A_2$$

$$\frac{dA_2}{dt} = -r_1\, A_1 A_2 + 2 r_2\, A_3$$

$$\frac{dA_3}{dt} = r_1\, A_1 A_2 - r_2\, A_3$$

So far these Petri nets describe *closed* systems.

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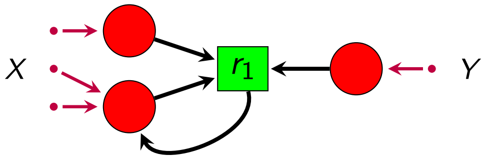
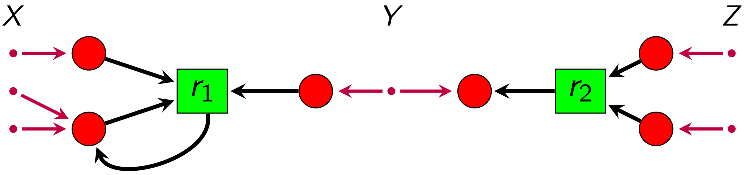
But there's a symmetric monoidal category of **open Petri nets with rates**, called **Petri**, where:

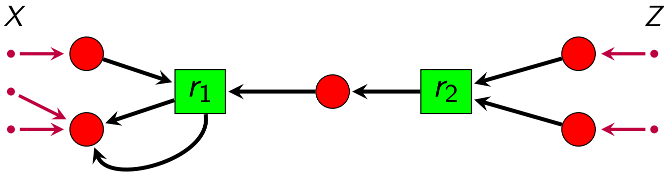
- an object is a finite set;
- a morphism $f: X \to Y$ is a Petri net with rates together with functions from X and Y to its set of places:



- To compose morphisms $f\colon X \to Y$ and $g\colon Y \to Z$:



we put them in series, identifying outputs of f with inputs of g:



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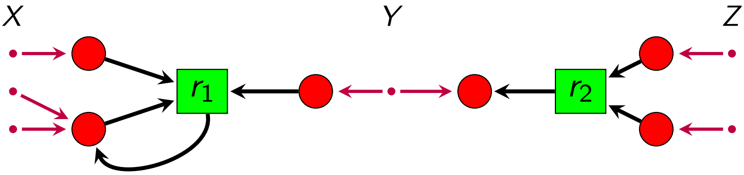

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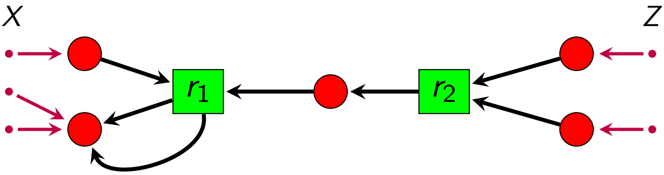
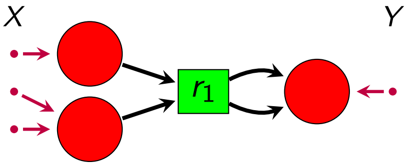


- To tensor morphisms, we put them in parallel.

An open Petri net with rates $f\colon X \to Y$ gives an **open rate equation** involving flows in and out, which can be arbitrary smooth functions of time. For example this:



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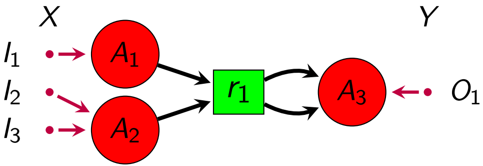

gives:

$$\frac{dA_1}{dt} = -r_1\, A_1 A_2 + I_1(t)$$

$$\frac{dA_2}{dt} = -r_1\, A_1 A_2 + I_2(t) + I_3(t)$$

$$\frac{dA_3}{dt} = 2r_1\, A_1 A_2 - O_1(t)$$

So: open Petri nets with rates serve as a 'syntax', with open dynamical systems providing one possible 'semantics'.

Let's understand this using functorial semantics! We'll get a symmetric monoidal functor

$$\square \colon \mathbf{Petri} \to \mathbf{Dynam}$$

Other choices of semantics correspond to other symmetric monoidal functors.

There is a symmetric monoidal category **Dynam** where:

- an object is a finite set;
- a morphism $f\colon X \to Y$ is an **open dynamical system**, meaning a cospan of finite sets

$$X \xrightarrow{i} S \xleftarrow{o} Y$$

equipped with a smooth vector field v on $\mathbb{R}^S$.

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Given input and output flows $I(t) \in \mathbb{R}^X$, $O(t) \in R^Y$, an open dynamical system describes how a point $A(t) \in \mathbb{R}^S$ changes with time:

$$\frac{d}{dt}A(t) = v(A(t)) + i_*(I(t)) - o_*(O(t))$$

where i_*, o_* push forward $\mathbb{R}$-valued functions from X, Y to S.

Theorem (JB–Blake Pollard)

There is a symmetric monoidal functor $\square \colon \mathbf{Petri} \to \mathbf{Dynam}$ *sending any open Petri net with rates to its open dynamical system.*

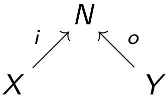
This is a statement of *compositionality*: we can determine the rate equation of a Petri net with rates by breaking it down into a composite and/or tensor product of simpler *open* Petri nets with rates, and repeatedly using:

$$\square(fg) = \square(f)\,\square(g)$$

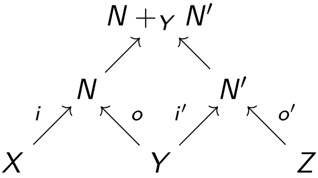
$$\square(f \otimes g) = \square(f) \otimes \square(g).$$

How do we prove this theorem? We use Fong's theory of decorated cospans.

Say we start with a category **C** with finite colimits: in our example, **C** = **FinSet**. We can build a bicategory where morphisms are cospans in **C**:



and composition is done by pushout:



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Next, if we choose a functor $F\colon \mathbf{C} \to \mathbf{Set}$, we can try to build a category where a morphism is an isomorphism class of cospans

$$X \xrightarrow{i} S \xleftarrow{o} Y$$

with S 'decorated' by an element of $F(S)$.

In the case of **Petri**, $F(S)$ is the set of all ways of making S into the set of places in some Petri net with rates.

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But how do we 'compose the decorations' when we compose cospans?

Given composable morphisms

$$X \xrightarrow{i} S \xleftarrow{o} Y \qquad d \in F(S)$$

$$Y \xrightarrow{i} S' \xleftarrow{o} Z \qquad d' \in F(S')$$

we compose the cospans by taking a pushout. We compose the decorations by taking $(d, d') \in F(S) \times F(S')$ and applying the composite function

$$F(S) \times F(S') \longrightarrow F(S + S') \longrightarrow F(S +_Y S')$$

where the first step comes from F being a *lax* monoidal functor.

Theorem (Brendan Fong)

Suppose that **C** *has finite colimits and*

$$F\colon (\mathbf{C}, +) \longrightarrow (\mathbf{Set}, \times)$$

is a lax symmetric monoidal functor. Then there is a symmetric monoidal category of ***F*-decorated cospans**, F**Cospan**, *where:*

- *an object is an object of* **C***;*
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Corollary (JB–Blake Pollard)

There is a symmetric monoidal category **Petri** *where:*

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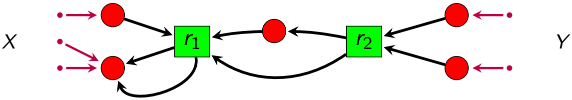
So, a morphism looks like this:



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Suppose $\mathbf{C}$ *has finite colimits,*

$$F, G \colon (\mathbf{C}, +) \longrightarrow (\mathbf{Set}, \times)$$

are lax symmetric monoidal functors, and

$$\theta \colon F \Rightarrow G$$

is a monoidal natural transformation. Then we obtain a symmetric monoidal functor

$$T_\theta \colon F\mathbf{Cospan} \to G\mathbf{Cospan}.$$

Corollary (JB–Blake Pollard)

There is a symmetric monoidal functor $\square\colon \mathbf{Petri} \to \mathbf{Dynam}$ *sending any open Petri net with rates to the corresponding open dynamical system.*

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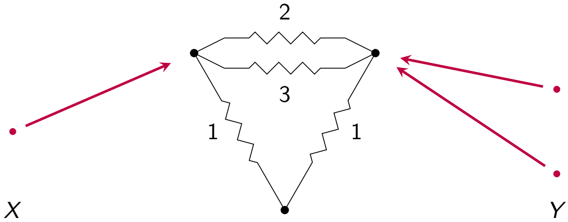
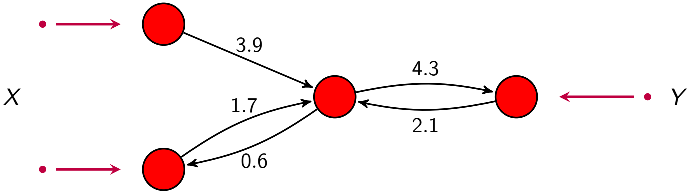
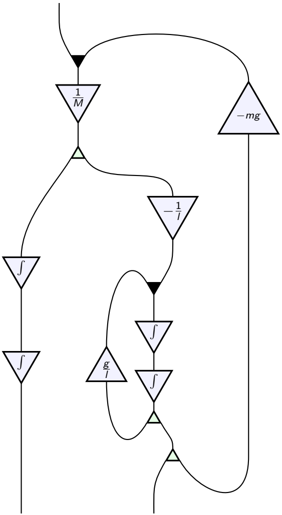
- **Electrical circuits**: JB and B. Fong, A compositional framework for passive linear networks.



- **Markov processes**: JB, B. Fong and B. Pollard, A compositional framework for Markov processes.



- **Signal-flow graphs in control theory**:
  Jason Erbele, *Categories in Control: Applied PROPs*.
  B. Fong, *The Algebra of Open and Interconnected Systems*.



We find a 'network of network languages': interesting symmetric monoidal categories connected by symmetric monoidal functors:

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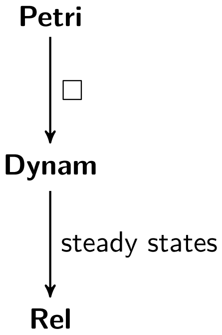

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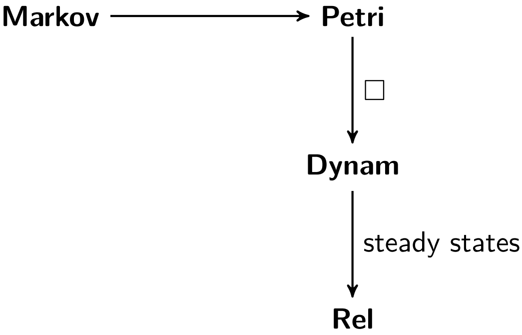

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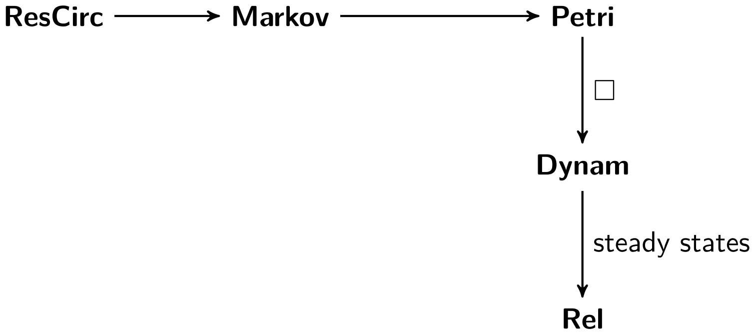

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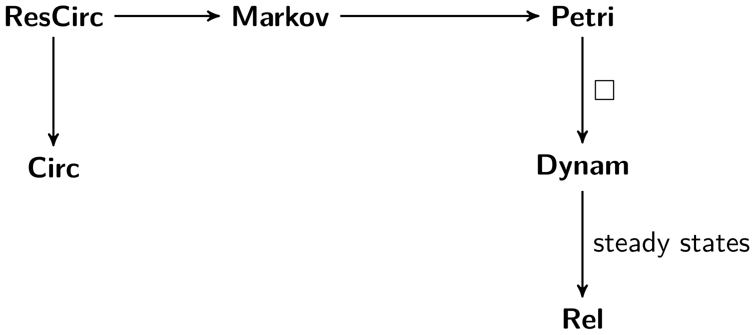

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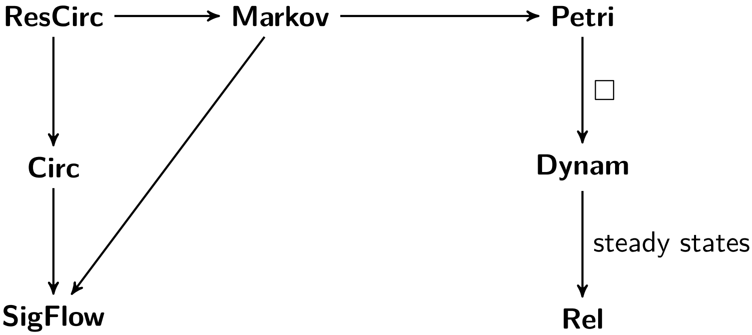

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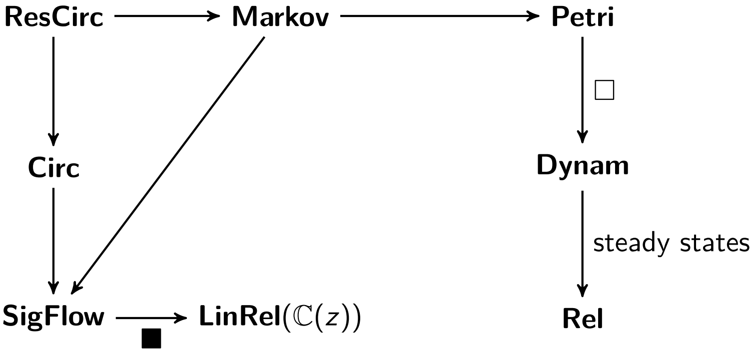

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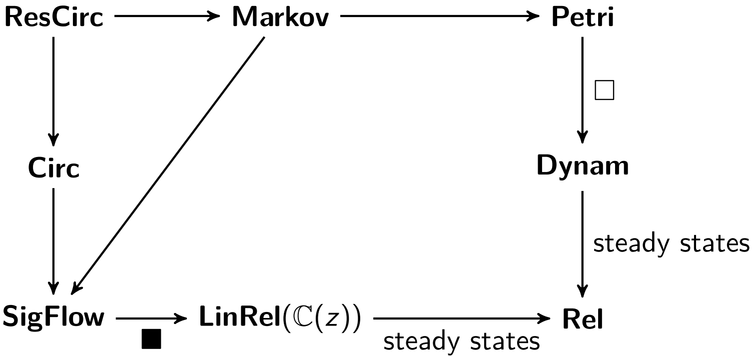

There is also more to say about decorated cospans! For example:

Theorem (Courser)

Suppose that $\mathbf{C}$ *has finite colimits and* $F\colon (\mathbf{C},+) \longrightarrow (\mathbf{Set},\times)$ *is a lax symmetric monoidal functor. Then there is a symmetric monoidal bicategory where:*

- *an object is an object of* $\mathbf{C}$*;*
- *a morphism from* X *to* Y *is a cospan* $X \xrightarrow{i} S \xleftarrow{o} Y$ *together with a decoration* $d \in F(S)$*;*
- *a 2-morphism is a map of cospans* $\begin{array}{ccccc} & & S & & \\ & {}^{i}\nearrow & \downarrow f & \nwarrow^{o} & \\ X & & & & Y \\ & {}_{i'}\searrow & & \swarrow_{o'} & \\ & & S' & & \end{array}$ *such that* $F(f)$ *maps* $d \in F(S)$ *to the decoration* $d' \in F(S')$*.*