Testing for Affine Invariant Properties of Algebraic Functions

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Based on:

- Bhattacharyya, Fischer, and Lovett, Testing low complexity affine-invariant properties. SODA 2013.
- Bhattacharyya, Fischer, HH, P. Hatami, and Lovett, Every locally characterized affine-invariant property is testable, STOC 2013.
- HH and Lovett, Estimating the distance from testable affine-invariant properties, FOCS 2013.
- HH, P. Hatami, and Lovett, in preparation.

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Common Theme

Extending the property testing results in graph theory to the algebraic setting.

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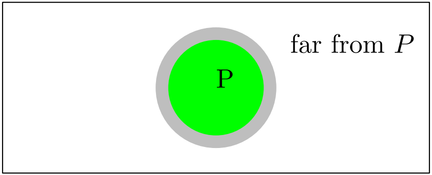
- Evaluate it on a small number of points.
- Decide whether
  - it satisfies a given property (e.g. triangle-freeness),
  - or is “far” from satisfying that property.



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- Formally defined by [Rubinfeld, Sudan 96], [Goldreich, Goldwasser, Rubinfeld 98].
- Closely related to limit theories of combinatorial objects [Lovász-Szegedy 2010].

Our setting

Functions of the form $f : \mathbb{F}_p^n \to \{0, \ldots, R\}$ where

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Two important cases:

- $R = 1$: Functions $f : \mathbb{F}_p^n \to \{0, 1\}$.
- $R = p - 1$: Functions $f : \mathbb{F}_p^n \to \mathbb{F}_p$.

Definition

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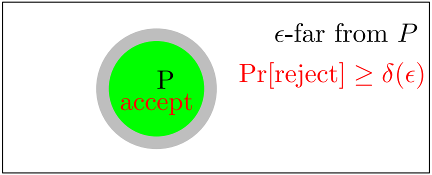
A (Proximity Oblivious) property tester for P must

- Make a constant number of queries to f.
- Accepts if $f \in P$.
- Rejects with probability $\geq \delta(\epsilon) > 0$ if $\mathrm{dist}(f, P) > \epsilon > 0$.



Example

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$$P = \{\text{functions } f : \mathbb{F}_p^n \to \{0,1\} \text{ where } f \equiv 0\}.$$

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- If $f(x) = 0$ accept
  otherwise reject.

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Analysis

- If $f \equiv 0$, then $\Pr[\text{accept}] = 1$.
- If $\mathrm{dist}(f, P) > \epsilon$, then $\Pr[\text{reject}] \geq \epsilon$.

- What conditions should we impose on P?

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P is called affine-invariant if

$$f \in P \Rightarrow f \circ A \in P$$

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$$P = \{\text{Polynomials } f : \mathbb{F}_p^n \to \mathbb{F}_p \text{ of degree} \leq d\}.$$

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Local Characterization of P

- $f \in P \iff$
- $f|_V \in P$ for all affine subspace $V \subseteq \mathbb{F}_p^n$ with $\dim(V) = d + 1$.

Test for deg $\leq d$.

- Pick a $d+1$-dimensional random affine subspace $V \subseteq \mathbb{F}_p^n$.
- Accept if $\deg(f|_V) \leq d$, and reject otherwise.

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We have

- if $f \in P$ then $\Pr[\text{accept}] = 1$.
- if $\text{dist}(f, P) \geq \epsilon$ then $\Pr[\text{reject}] > \delta(\epsilon) > 0$. [Alon, Kaufman, Krivelevich, Litsyn, Ron 2005].

Locally characterizable

P is locally characterizable if there exists $k > 0$ such that

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Theorem (Bhattacharyya, Fischer, HH, P. Hatami, and Lovett)

Every locally characterizable property is (PO)-testable.

Proof Sketch

A classical example

The graph property of triangle-freeness.

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The test

- Pick three vertices at random.
- If they form a triangle reject.
- Otherwise accept.

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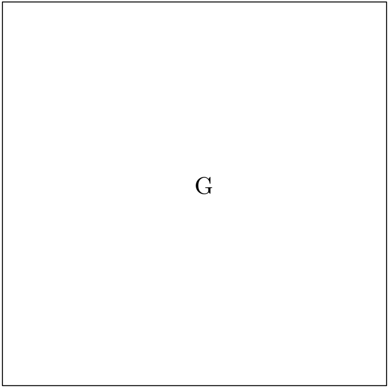
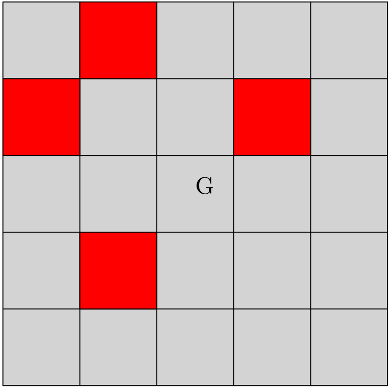
Analysis

- If $\triangle$-free, we always accept. (trivial)
- If ϵ-far from $\triangle$-free, then $\Pr[\text{reject}] > \delta(\epsilon) > 0$. (non-trivial)

- Suppose G is ϵ-far from being $\triangle$-free.



- Suppose G is ϵ-far from being $\triangle$-free.
- Regularize: Partition vertices into almost equal parts, so that almost all cells are uniform.



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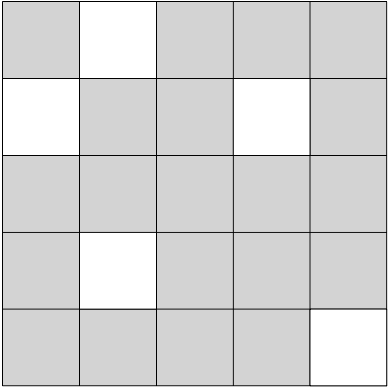
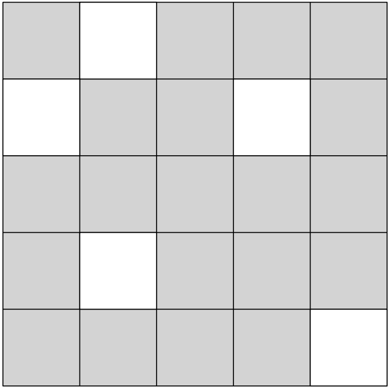


- The new graph H is close to $G \Rightarrow$ it is far from being $\triangle$-free.



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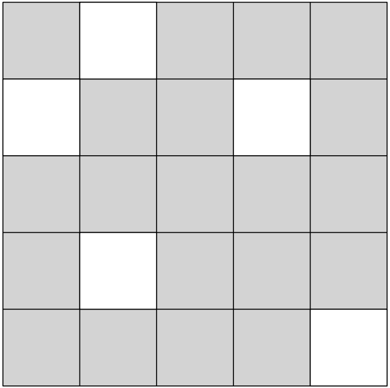

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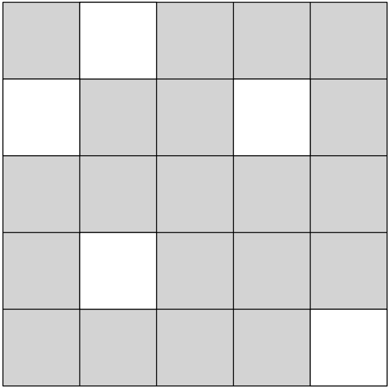
- $\Rightarrow G$ has many $\triangle$'s (we only removed edges from G).



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- If induced-C_5-free, we always accept. (trivial)
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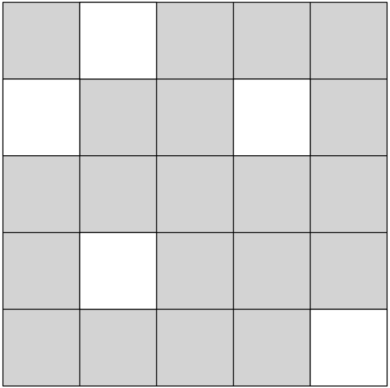

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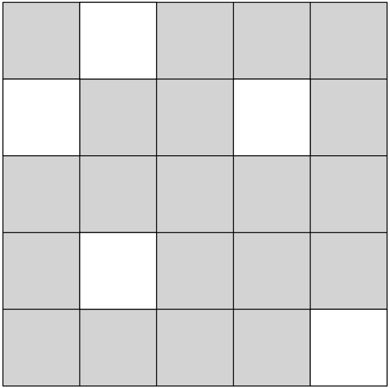
  - Might create many C_5's, and so
  - H has many C_5's $\not\Rightarrow$ G has many C_5's.



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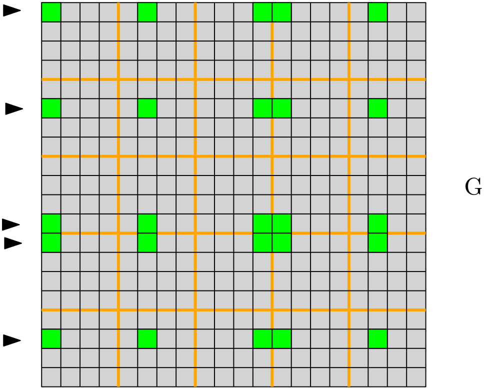

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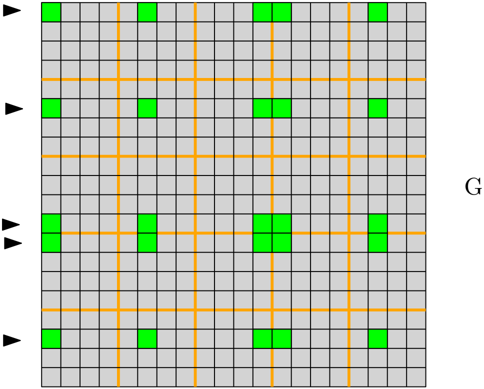

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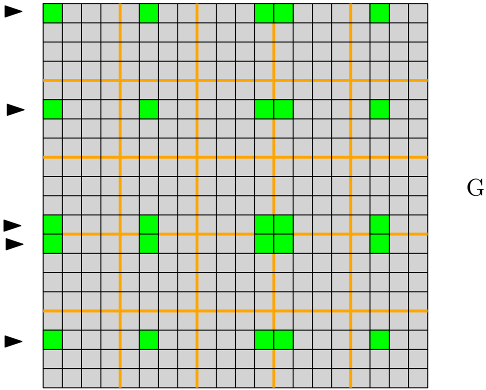

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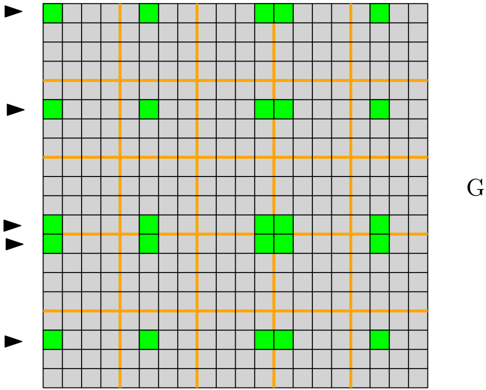
- all pairs of sub-parts are uniform.
- For most cells: density $\approx$ subcell density.



The algebraic setting $\mathbb{F}_p^n$

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The general approach

- Consider f that is ϵ-far from P.
- Regularize f.
- Clean-up the regularization of f to obtain g close to f.
- Then $g \notin P$ and thus violates some local condition.
- Exploit the nice structure of g to show that the test works for f.

Regularization

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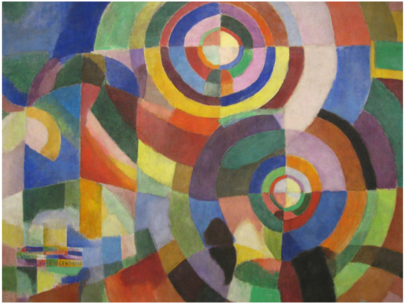
Partition $\mathbb{F}_p^n$ such that f is uniform on almost all parts.

- Consider polynomials $Q_1, \ldots, Q_c : \mathbb{F}_p^n \to \mathbb{F}_p$ of degree $\leq d$.
- Partition $\mathbb{F}_p^n$ according to $(Q_1(x), \ldots, Q_c(x))$



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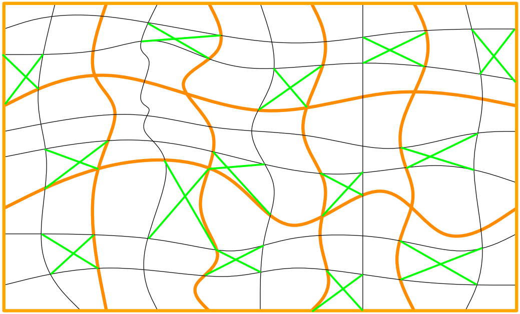

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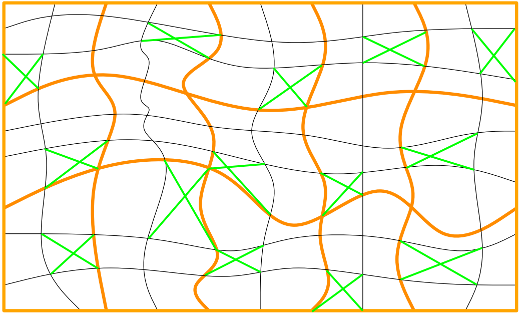

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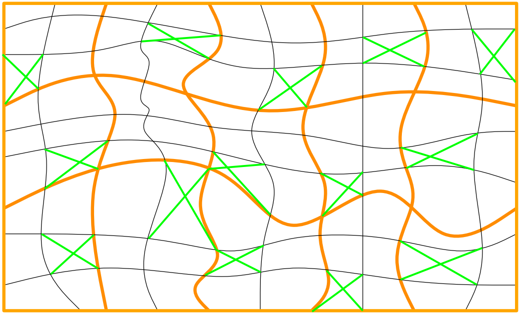

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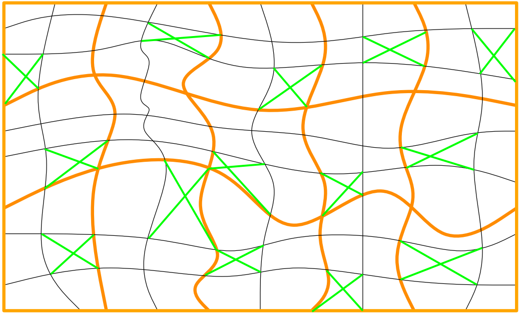
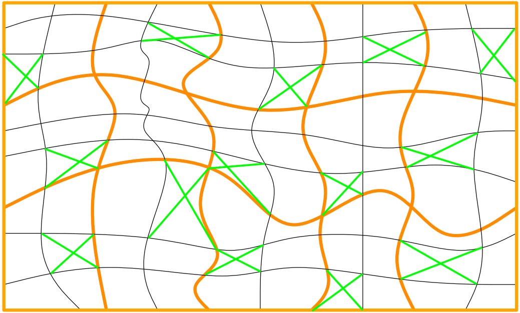
BFL Subatoms are chosen by setting $(Q_1(x), \ldots, Q_b(x)) = \vec{c}_0$.



clean-up

- Modify f to remove all irregularities:



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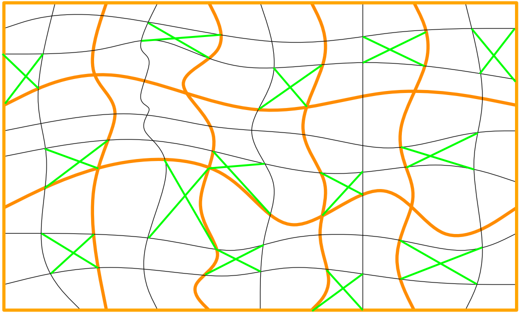

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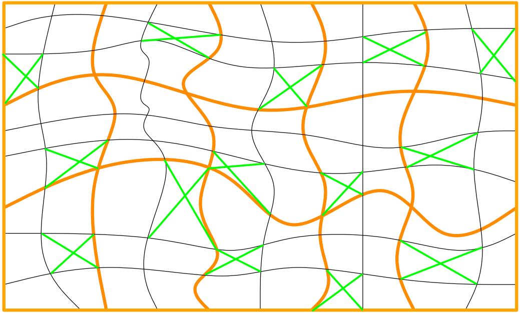
  - Change the unpopular values on every atom c to t_c.



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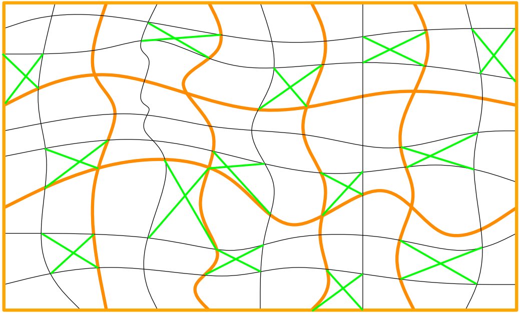

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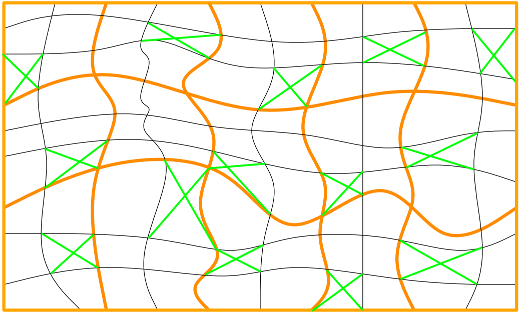

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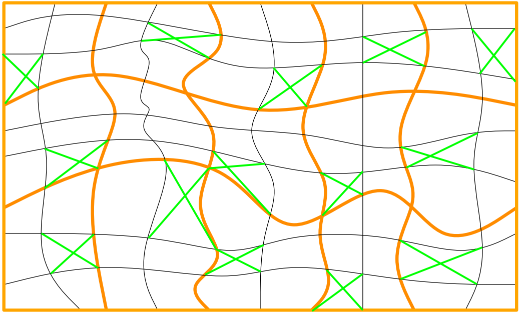
- There are many W's for which $f|_W \notin P$.



Equidistribution for Polynomial factors

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- Need to analyze the distribution of $f|_V$ for a random V.
- Let $L_1, \ldots, L_{p^k}$ be the points of a random V.
- We need to understand the distribution of

$$\begin{pmatrix} Q_1(L_1) & \ldots & Q_c(L_1) \\ Q_1(L_2) & \ldots & Q_c(L_2) \\ \vdots & & \\ Q_1(L_{p^k}) & \ldots & Q_c(L_{p^k}) \end{pmatrix}.$$

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- Green-Tao, Kaufman-Lovett: If $Q_1, \dots, Q_c$ are of "high rank", then

$$Q_1(X), \dots, Q_c(X),$$

are almost independent (entries in each row are almost independent).

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  are almost independent (entries in each row are almost independent).
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  - Note that if $\deg(Q) = 1$, then $Q(L_1) + Q(L_2) = Q(L_3) + Q(L_4)$ if $L_1 + L_2 = L_3 + L_4$.

$$\begin{pmatrix} Q_1(L_1) & \dots & Q_c(L_1) \\ Q_1(L_2) & \dots & Q_c(L_2) \\ \vdots & & \\ Q_1(L_{p^k}) & \dots & Q_c(L_{p^k}) \end{pmatrix}$$

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  - If $\deg(Q) = 2$, then $\sum_{S \subseteq \{1,2,3\}} (-1)^{|S|} Q(e_0 + \sum_{i \in S} e_i) = 0$.

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- General case: [H, P. Hatami, and Lovett in preparation].

Examples of locally characterizable properties

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Testable

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- Fix $d_1, \ldots, d_c$ and $\Gamma : \mathbb{F}_p^c \to [R]$.
- The property of being expressible as $\Gamma(P_1, \ldots, P_c)$ where $\deg(P_i) \leq d_i$.

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Example

- Polynomials $f : \mathbb{F}_p^n \to \mathbb{F}_p$ that are products of two quadratics.
- Polynomials $f : \mathbb{F}_p^n \to \mathbb{F}_p$ that are squares of a quadratics.
- Polynomials $f : \mathbb{F}_p^n \to \mathbb{F}_p$ of the form $f = ab + cd$ where a, b, c, d are cubics.

Theorem (Bhattacharyya, Fischer, HH, P. Hatami, and Lovett)

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- Our proof uses regularity $f \approx \Gamma(Q_1, \ldots, Q_c)$.
- Consequently does not provide any reasonable bound on the dimension.

A stronger notion of testing

Definition (Recall)

A (Proximity Oblivious) property tester for P must

- Make a constant number q of queries.
- Accepts if $f \in P$.
- Rejects with probability $\geq \delta(\epsilon) > 0$ if $\mathrm{dist}(f, P) > \epsilon > 0$.

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A property tester for P must

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Theorem (Alon-Shapira 2005)

Every hereditary graph property is testable with one-sided error.

Definition

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Theorem (Bhattacharyya, Fischer, HH, P. Hatami, and Lovett)

Every affine subspace hereditary property of "bounded complexity" is testable with one-sided error.

Estimating the distance from a property

Definition

For a property *P*, and $\alpha > 0$, let P_α be the set of functions with distance at most α from *P*.

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For every testable affine-invariant property P and every $\alpha > 0$, the property P_α is testable with two-sided error.

- One can estimate the distance from every testable property.
- Was unknown even for simple properties such as cubic polynomials.

How does the test work?

- Let $f : \mathbb{F}_p^n \to [R]$ be a given function.

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- Let $f : \mathbb{F}_p^n \to [R]$ be a given function.
- Let W be a random affine subspace of large dimension.
- With high probability $\mathrm{dist}(f|_W, P) \approx \mathrm{dist}(f, P)$:
    - Completeness: If f is α-close to P then $f|_W$ is $(\alpha + \epsilon/2)$-close to P.
    - Soundness: If f is $\alpha + \epsilon$-far from P then $f|_W$ is $(\alpha + \epsilon/2)$-far from P.

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- Since f and $f|_W$ have similar structures, we can lift h to some g:

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  - g is $(\alpha + \epsilon/2)$-close to P.
  - The test cannot distinguish g and h, so g is close to P.
- We conclude that f is $(\alpha + \epsilon)$-close to P

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For $f : \mathbb{F}_2^n \to \mathbb{F}_2$, the Gowers U^4 norm (16 queries) can be used to distinguish:

- $\mathrm{Corr}(f,$ non-classical cubics) is non-negligible.
- $\mathrm{Corr}(f,$ non-classical cubics) is negligible.

Is there such a test (constant number queries) for cubic polynomials?

Thank you!