The Logic of Counting Query Answers

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Simons Institute, Berkeley – November 2016

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Act: Overview

Query evaluation

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Basic problem in logic/database theory:

Evaluate a formula $\phi(V)$ on a finite structure $\mathbf{B}$,
that is, determine $\phi(\mathbf{B}) = \{h : V \to B \mid \mathbf{B}, h \models \phi\}$

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Example: $\phi(u, v) = \exists x(E(u, x) \wedge E(x, v))$

Generalizes model checking — determine if $\mathbf{B} \models \phi$
where ϕ is a sentence

- Here, we have $|\phi(\mathbf{B})| = 1 \quad \Leftrightarrow \quad \mathbf{B} \models \phi$

♯-logic

$\sharp$-logic

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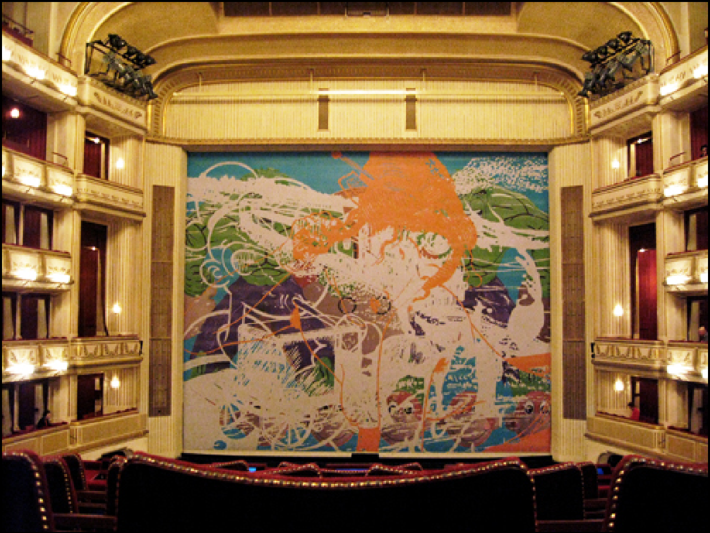
$\sharp$-logic enjoys & balances:

- Expressivity: in a precise sense, can express known efficient algorithms for counting query answers in $\sharp$-logic
- Optimizability: minimizing width can be done computably (in an expressive fragment of $\sharp$-logic)



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Let Φ be a class of first-order formulas

Def: p-count(Φ) is the problem...

Given $\phi(V) \in \Phi$ and a finite struct $\mathbf{B}$,
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  Here: the query/formula is the parameter

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Here: p-count(Φ) is tractable if

$\exists$ an algorithm f and a poly-time algorithm A such that

given $(\phi, \mathbf{B})$, the value $|\phi(\mathbf{B})|$ is computed by $A(f(\phi), \mathbf{B})$

("fixed-parameter tractable")

Classification

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- If (X), then p-count(Φ) is tractable (in FPT).
- Else, p-count(Φ) is not tractable, unless W[1] = FPT.

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$\exists k \geqslant 1$ and an alg f that computes, for each $\phi \in \Phi$,
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(By model checking on a class of sentences Φ,
we refer to p-count(Φ))

Model checking

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Conceptual point: FO logic is a useful model of computation!

- If we have tractability at all, we have tractability via putting the sentences in the right "format"
- FO logic contains the computational primitives needed to express the algorithm witnessing tractability

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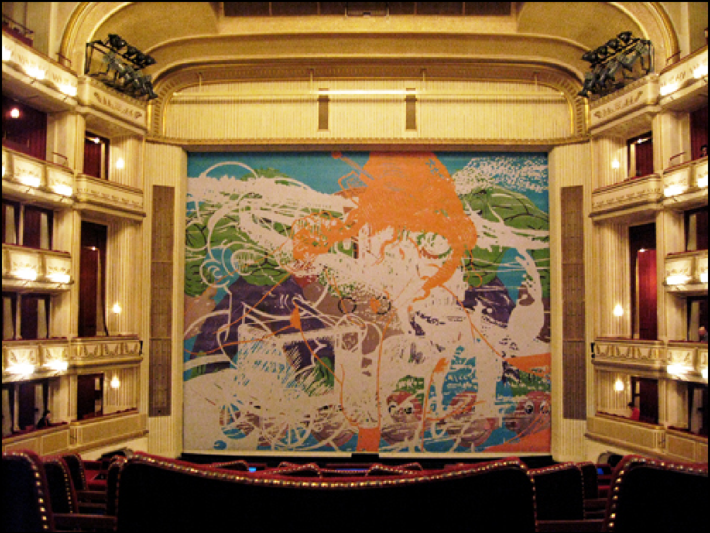
Will give a logic: $\sharp$-logic



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A Main Thm: On classes Φ of $\{\exists, \wedge, \vee\}$-formulas, this condition is exclusive explanation for FPT!

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Consider the formula $\phi(v, y, z) = E(v, y) \wedge F(v, z)$

Set $\psi_E = C(E(v, y))$, $\psi_F = C(F(v, z))$ ("casting")

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- So if we take the previous $\sharp$-formula and project v, get representation $Pv(Py\psi_E \times Pz\psi_F)$ of ϕ

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- Constants

  Each $n \in \mathbb{Z}$ is a $\sharp$-formula

  $\text{free}(n) = \text{closed}(n) = \varnothing$

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Note: follows that there is an algorithm that decides counting equivalence

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- But to justify this...

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Thm (independence): For any "linear combination" $\sum_i a_i|\psi_i|$ where each $a_i \neq 0$ and the ψ_i are counting inequivalent, $\{\exists, \wedge\}$, there exists a structure **D** such that $\sum_i a_i|\psi_i(\mathbf{D})| \neq 0$

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- Would like to control the values of components independently

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Say the components $\theta_1, \ldots, \theta_n$ ($\{\exists, \wedge\}$-queries) are in play

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- To get the struct $\mathbf{D}$ as described, take a poly p such that $|\theta_i(\mathbf{C})| \mapsto t_i$, and set $\mathbf{D} = p(\mathbf{C})$

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Thm: If $\mathbf{A}_1, \ldots, \mathbf{A}_k$ pairwise non-isomorphic structs,
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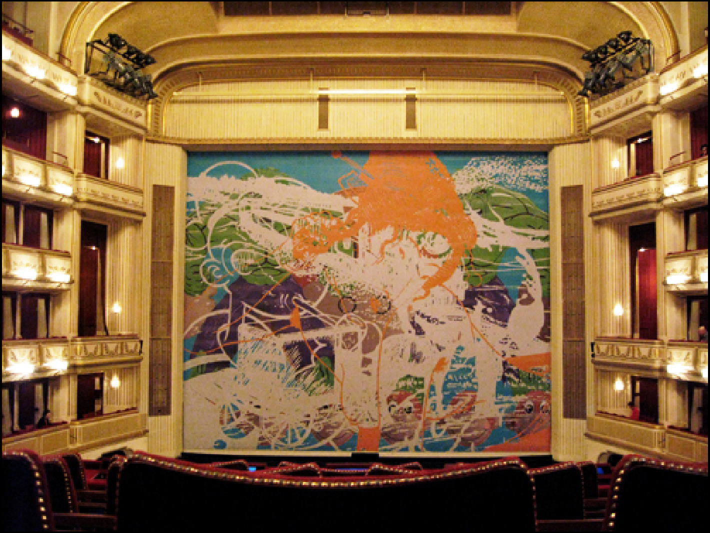

Act: Reflection

Discussion

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Our $\sharp$-logic balances...

- Expressivity
- Computability: there is an algorithm for width minimization, so width is well-characterized (in some sense) (Width minimization not computable in positive FO [Bova & Chen '14])

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Open: We focused on $\{\exists, \wedge, \vee\}$-formulas;
what can one say about FO logic in general?

What can one say about other logics?