Logics of Finite Hankel Rank

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joint work with

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{Symmetry, Logic, Computation},
Simons Institute, November 2016

To Yuri Gurevich at his 75th birthday.

We met 39 years ago
at the beginning of our reorientation towards
Computer Science,
and stayed friends even in fertile disagreements.

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Thanks to Moritz Müller from the Logic Group
of Vienna University
for pointing out misprints and other sources of confusions.

Reference for the birthday paper

Nadia Labai , Johann A. Makowsky
Logics of Finite Hankel Rank
Fields of Logic and Computation II
Volume 9300 of the series
Lecture Notes in Computer Science pp 237-252
September 2015

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Also in: arXiv preprint arXiv:1512.02507, 2015

Outline

- The Feferman-Vaught theorem for finite structures
- FV-theorems in an abstract setting
- New directions in characterizing logics with FV-theorems

  Generalized Hankel matrices of finite rank

- Open problems (and some ideas)

Generalized sums, products and connections

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- The sum of two τ-structures is the (model theoretic) disjoint union.

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- Generalized products are (first order) transductions of products.
- Generalized sums are scalar (first order) transductions of sums.

- We also look at k-connections: These are disjoint unions of graphs with k distinctly labeled vertices, where vertices with corresponding labels are identified.
- Generalized k-connections are scalar (first order) transductions of k-connections.

The Feferman-Vaught theorem for finite structures

What are the logics on finite structures which satisfy some version of the FV Theorem?

The Feferman-Vaught theorem for first order logic:

Let $\otimes$ be a generalized product on τ-structures.

There is $t \in \mathbb{N}$ and a function $p : \mathrm{FOL} \to (\mathrm{FOL})^\star$ with

$$p(\phi) = (\psi_1^1, \ldots, \psi_{k(\phi)}^1, \psi_1^2 \cdot \psi_{k(\phi)} i^2) \quad \text{where} \quad k(\phi) \in \mathbb{N}$$

and a Boolean function B_ϕ such that for all $\phi \in \mathrm{FOL}(\tau)^q$ and all structures $\mathcal{A} = \mathcal{B}_1 \otimes \mathcal{B}_2$,

$$\mathcal{A} \models \phi \quad \text{iff} \quad B_\phi(\psi_1^{B_1}, \ldots, \psi_{k(\phi)}^{B_1}, \psi_1^{B_2}, \ldots, \psi_{k(\phi)}^{B_2}) = 1$$

where for $1 \leqslant j \leqslant k$ we have $\psi_j^{B_1}, \psi_j^{B_2} \in \mathrm{FOL}(\tau)^{q+t}$, and

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$\psi_1^1, \ldots, \psi_{k(\phi)}^1, \psi_1^2, \ldots, \psi_{k(\phi)}^2$ are called **reduction formulas.**

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- Actually, for sums and products one can even get $t = 0$!

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Let $\oplus$ be a generalized sum on τ-structures.
There is $t \in \mathbb{N}$ and a function $p : MSOL \to (MSOL)^{\star}$ with

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- If a class of τ-structures has a decidable $MSOL$ theory, so does its closure under disjoint union.

$t = 0$ is essential for these applications!
Generalized sums (products) where $t = 0$ will be called **sum-like** (**product-like**).

FV-theorems in an abstract setting

A general notion of logic

A **Lindström logic $\mathcal{L}$** is a tuple $\langle \mathcal{L}(\tau), \mathrm{Str}(\tau), \models_{\mathcal{L}}, \rho \rangle$ where

- $\mathcal{L}(\tau)$ is the set of τ-sentences of $\mathcal{L}$
- $\mathrm{Str}(\tau)$ are the finite τ-structures
- $\models_{\mathcal{L}}$ is the satisfaction relation
- ρ is a (quantifier) rank function attaching some weight (cost) to each formula.

If $\mathcal{L}(\tau)$ and $\models_{\mathcal{L}}$ are uniformly computable, it is a **Gurevich logic.**

ρ is **nice** if it holds that for finite τ, there are, up to logical equivalence, only finitely many $\mathcal{L}(\tau)$-formulas of fixed quantifier rank with a fixed set of free variables.

A logic with nice quantifier rank is called **a nice logic.**

Examples of nice logics

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As the quantifier rank we can take
$\rho_1(D_{k,m}x\ \varphi(x)) = \rho(\varphi(x)) + 1$
or $\rho_2(D_{k,m}x\ \varphi(x)) = \rho(\varphi(x)) + m$.

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ρ_1 is not nice,
since there are infinitely many sentences with the same quantifier rank,
whereas ρ_2 is nice.

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Some properties, nevertheless, may be desirable.

- ρ is weakly monotone with respect to subformulas:
  If ψ is a subformula of ϕ then $\rho(\psi) \leqslant \rho(\phi)$.
- Boolean combinations of formulas do not increase ρ.

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- Boolean combinations of formulas do not increase ρ.

Every abstract Lindström Logic can be given a canonical syntax by adding generalized quantifiers for each definable class of structures.

Translation schemes

Let τ, σ be two relational vocabularies with $\tau = \langle R_1, \ldots, R_m \rangle$, and denote by $r(i)$ the arity of R_i.

A $(\sigma - \tau)$ translation scheme T is a sequence of $\mathcal{L}(\sigma)$-formulas $(\phi; \phi_1, \ldots, \phi_m)$ where

- ϕ has k free variables, and
- each ϕ_i has $k \cdot r(i)$ free variables.

The transduction $T^\star$ induced by T operates on σ-structures $\mathcal{A}$ and maps them to τ-structures $T^\star(\mathcal{A})$ where the vocabulary is interpreted by the formulas given in the translation scheme.

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The translation of a τ-formula is obtained by substituting atomic τ-formulas with their definition through σ-formulas given by the translation scheme.

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A translation scheme (induced transduction) is

- **scalar** if $k = 1$, otherwise it is ***k*-vectorized.**
- It is **quantifier-free** if so are the formulas $\phi; \phi_1, \ldots, \phi_m$.

Binary operations on τ-structures

The **disjoint union $\mathcal{A} \sqcup \mathcal{B}$** of $\mathcal{A}$ and $\mathcal{B}$ is the structure obtained by taking the disjoint union of the universes and of the corresponding relation interpretations in $\mathcal{A}$ and $\mathcal{B}$.

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If τ had no constant symbols, $\mathcal{A} \sqcup \mathcal{B}$ is a τ-structure.

If τ had k constant symbols, it is a structure with $2k$ constant symbols.

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A binary operation $\Box : \mathrm{Str}(\sigma) \times \mathrm{Str}(\sigma) \to \mathrm{Str}(\tau)$ is

- **sum-like (product-like)** if it is obtained from the disjoint union of σ-structures by applying a quantifier-free scalar (vectorized) $(\sigma - \tau)$-transduction.
- **connection-like** if it is obtained from a connection operation on σ-structures by applying a quantifier-free scalar $(\sigma - \tau)$-transduction.

If $\sigma = \tau$, we say $\Box$ is an operation on τ-structures.

The FV-property

Let $\Box$ be a binary operation on τ-structures.
$\mathcal{L}$ has the FV-property for $\Box$ with respect to ρ if for every $\phi \in \mathcal{L}(\tau)^{q}$ there are

- $k = k(\phi) \in \mathbb{N}$
- $\psi_1, \ldots, \psi_k \in \mathcal{L}(\tau)^{q}$
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such that for all τ-structures $\mathcal{A} = \mathcal{B}_1 \Box \mathcal{B}_2$ we have that

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where for $1 \leqslant j \leqslant k$ we have

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Note that ϕ and the reduction formulas are required to have the same quantifier rank q.

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Can one characterize the logics which have the FV-property for some $\Box$?

$\mathcal{L}$-smooth operation $\Box$

Let $\mathcal{L}$ be a nice logic and $\Box$ be a binary operation on τ-structures.

$\Box$ is $\mathcal{L}$-smooth if whenever $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{B}_1$, $\mathcal{B}_2$ satisfy pairwise the same $\mathcal{L}^q$-sentences then $\mathcal{A}_1 \Box \mathcal{B}_1$ and $\mathcal{A}_2 \Box \mathcal{B}_2$ also satisfy the same $\mathcal{L}^q$-sentences.

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Observation: If $\mathcal{L}$ has the FV-property for $\Box$ then $\Box$ is $\mathcal{L}$-smooth.

$\mathcal{L}$-smooth operation $\square$

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Proposition:(M.-Shelah, 1982) For compact logics $\mathcal{L}$ with a quantifier rank ρ the converse is true:
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Problem: Find a natural example of a nice logic $\mathcal{L}$ and an operation $\square$, such that $\square$ is $\mathcal{L}$-smooth, but $\mathcal{L}$ does not have the FV-property for $\square$.

The topic today:
New directions in attacking this problem

Hankel matrices

Hankel matrices for τ-properties

A τ-property Φ is a class of finite τ-structures closed under τ-isomorphisms.

The Boolean Hankel matrix $\mathrm{H}(\Phi, \Box)$ for a τ-property Φ and a binary operation $\Box : \mathrm{Str}(\sigma) \times \mathrm{Str}(\sigma) \to \mathrm{Str}(\tau)$ is the infinite $(0,1)$-matrix where the rows and columns are labeled by all the finite σ-structures $\mathcal{A}_0, \mathcal{A}_1, \ldots$, and

$$\mathrm{H}(\Phi, \Box)_{\mathcal{A}_i, \mathcal{A}_j} = 1 \text{ iff } \mathcal{A}_i \Box \mathcal{A}_j \in \Phi$$

Φ has finite $\Box$-rank if the rank of $\mathrm{H}(\Phi, \Box)$ over $\mathbb{Z}_2$ is finite.

Equivalence relations for τ-properties

The rows of $H(\Phi, \Box)$ naturally define an equivalence relation:
Two σ-structures are $(\Phi, \Box)$-equivalent $\mathcal{A} \equiv_{\Phi,\Box} \mathcal{B}$, if they have identical rows in $H(\Phi, \Box)$.
In other words, $\mathcal{A} \equiv_{\Phi,\Box} \mathcal{B}$ if for all σ-structures $\mathcal{C}$ we have

$$\mathcal{A}\Box\mathcal{C} \in \Phi \quad \text{iff} \quad \mathcal{B}\Box\mathcal{C} \in \Phi$$

Φ has finite $\Box$-index iff there are only finitely many $(\Phi, \Box)$-equivalence classes.

It is easy to see that Φ has finite $\Box$-index iff it has finite $\Box$-rank.

A τ-property Φ has finite S-rank (P-rank, C-rank) if it has finite rank for all sum-like (product-like, connection-like) operations.

Lovász' Theorem

Theorem:(Lovász, 2007) Let Φ be a graph property such that $\mathrm{H}(\Phi, \sqcup_k)$ has finite rank. Then Φ can be checked in polynomial time on structures of tree-width at most k.
Here $\sqcup_k$ is the k-connection operation.

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This is a vast improvement of Courcelle's meta-theorem for $\mathcal{CMSOL}$.

Notion of rank finiteness for logics

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Theorem (Godlin, Kotek, M., 2008).
Let $\mathcal{L}$ be a nice fragment of SOL and let $\Box$ be $\mathcal{L}$-smooth.
Then any $\mathcal{L}$-definable property Φ has finite $\Box$ rank.

Notion of rank finiteness for logics

If every definable property in the logic $\mathcal{L}$ has finite S-rank (P-rank, C-rank), we say **$\mathcal{L}$ is of finite S-rank (P-rank, C-rank).**

Theorem (Godlin, Kotek, M., 2008).
Let $\mathcal{L}$ be a nice fragment of SOL and let $\Box$ be $\mathcal{L}$-smooth.
Then any $\mathcal{L}$-definable property Φ has finite $\Box$ rank.

As a consequence, we have that:

$MSOL$ and $CMSOL$ are of finite S-rank and C-rank, and FOL and $CFOL$ are additionaly of finite P-rank.

The FV-property implies finite rank

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By analyzing the proof, we can give a general version:

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Proposition.
Let $\mathcal{L}$ be a nice Lindström logic. If $\mathcal{L}$ has the FV-property for $\Box$, then $\Box$ is $\mathcal{L}$-smooth.

Thus we have:

If $\mathcal{L}$ has the FV-property for all sum-like (product-like, connection-like) operations,
then $\mathcal{L}$ is of finite S-rank (P-rank, C-rank).

What implies the FV-property?

We saw that the FV-property implies finite rank.

Does the converse relationship hold?

When a nice logic $\mathcal{L}$ has the FV-property for $\square$, we can reason about a structure $\mathcal{A} = \mathcal{B}_1 \square \mathcal{B}_2$ and a sentence $\phi \in \mathcal{L}$ by reasoning about a finite number of $\mathcal{L}$-formulas ψ_i and the structures $\mathcal{B}_1$ and $\mathcal{B}_2$.

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We take a closer look at possible conditions and try to reformulate the role they play.

Closed logics

The following statement proves key to a logic having the FV-property:

Proposition.
If $\mathcal{L}$ is nice and has the FV-property for $\square$, then for every $\mathcal{L}^q$-definable property Φ, the equivalence classes of $\equiv_{\Phi,\square}$ are also $\mathcal{L}^q$-definable.

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A logic $\mathcal{L}$ is $\Box$-closed if for every $\mathcal{L}^q$-definable property Φ, the equivalence classes of $\equiv_{\Phi,\Box}$ are also $\mathcal{L}^q$-definable.

$\mathcal{L}$ is S-closed (P-closed, C-closed) if $\mathcal{L}$ is $\Box$-closed for all sum-like (product-like,connection-like) operations $\Box$.

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Theorem.
If $\mathcal{L}$ is nice and has the FV-property for all sum-like (product-like, connection-like) operations, it is S-closed (P-closed, C-closed).

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(correct version of Theorem 16 in the birthday paper)

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The same holds if we replace S-closed and sum-like by P-closed and product-like (C-closed and connection-like).

Open problems
(and some ideas)

Maximal logics

We have seen that there are logics for which any definable property is of finite S-rank (such as $\mathrm{MSOL}, \mathrm{CMSOL}$), P-rank or C-rank (such as $\mathrm{FOL}, \mathrm{CFOL}$).

Is there a logic $\mathcal{L}$ whose definable properties are exactly the ones of finite S-rank or P-rank?

Maximal logics of finite S-rank (P-rank, C-rank).

First approach:
Forget syntax and think of a logic as a collection of properties closed under certain set operations corresponding to the usual Boolean connectors and quantifiers for formulas.

Denote by $\mathcal{S}(\tau)$ and $\mathcal{P}(\tau)$ the collections of all τ-properties of finite S-rank and finite P-rank, respectively, and let $\mathcal{S} = \bigcup_\tau \mathcal{S}(\tau)$ and $\mathcal{P} = \bigcup_\tau \mathcal{P}(\tau)$.

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Is it true that
$\mathcal{S}$ and $\mathcal{P}$ and are (abstract) Lindström (or even nice Gurevich) logics which have finite S-rank and finite P-rank, respectively?

To show that $\mathcal{S}(\tau)$ is a Lindström logic one would have to show several closure properties:

- Closure under boolean operation (true).
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None of this is done in an obvious way.

Some remaining obstacles

If we consider $\mathrm{C}M\mathcal{S}\mathrm{OL}$ to be the collection of $\mathrm{C}M\mathcal{S}\mathrm{OL}$-definable properties, we have $\mathrm{C}M\mathcal{S}\mathrm{OL} \subseteq \mathcal{S}$.

Do we have $\mathrm{C}M\mathcal{S}\mathrm{OL} = \mathcal{S}$?

The following theorems show the diverse obstacles we came across when we tried to answer this question.

Some remaining obstacles

If we consider CMSOL to be the collection of CMSOL-definable properties, we have $\mathrm{CMSOL} \subseteq \mathcal{S}$.

Do we have $\mathrm{CMSOL} = \mathcal{S}$?

The following theorems show the diverse obstacles we came across when we tried to answer this question.

Theorem.
There is a sum-like operation $\square$ for which there are continuum many properties of finite $\square$-rank.

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Such a class can still have infinite rank for other sum-like operations.

Many classes with finite rank for the disjoint union

Proof:
Let $A \subseteq \mathbb{N}$.
Let $Cycle(A)$ be the family of graphs $\{C_n : n \in A\}$.
For each $A \subseteq \mathbb{N}$ the $H(\sqcup, Cycle(A))$ has rank 1.
This is so, because for each G_1, G_2 is connected iff G_1 is connected and $G_2 = \emptyset$ or vice versa.

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This does not imply that $Cycle(A)$ has finite rank for all sum-like operations.

What is left open:

Problem:

- Assume that a τ-property Φ has finite S-rank and C-rank. Does it follow that Φ is definable in $CMSOL$?
- If additionally Φ is P-closed, does it follow that Φ is definable in $CFOL$?

We do not dare to conjecture that the answer is positive, but it might well be.

Summary

We asked whether one can characterize the logics which have a Feferman-Vaught theorem sum-like and product-like operations.

- We related the Feferman-Vaught theorem to Hankel matrices and described their exact relationship.
- We investigated under which conditions one can construct logics satisfying the Feferman-Vaught theorem.
- We could not (yet) show the existence of maximal logics of finite S-rank and P-rank.
  (as we claimed in Theorem 17 in the birthday paper that we could).

Questions?

Thank you.