Braid Groups, Hecke Algebras, Representations, and Anyons

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Anyons

Anyons are particle-like excitations that can exist in two-dimensional space.

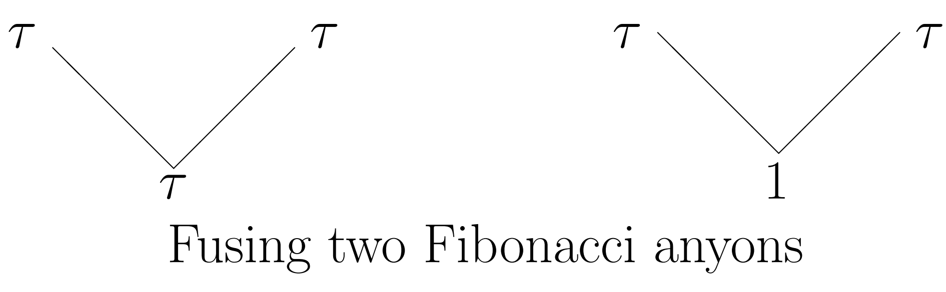
The state of a system of anyons can be changed if one moves the anyons around each other, even if every anyon returns to its original position.

Such changes can include operations useful for quantum computation, and they are less susceptible to errors than other proposed mechanisms for quantum computation.

Anyon models specify **fusion rules**: What can you get by bringing two anyons together?

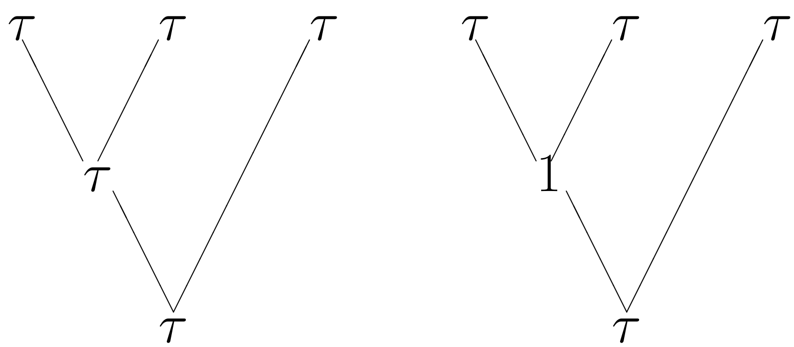
Fibonacci Anyons

Fibonacci fusion rule: $\tau \otimes \tau = \tau \oplus 1$.



Fusing two Fibonacci anyons

Fusing Three Fibonacci Anyons to One



These two fusion processes span a 2-dimensional Hilbert space, the quantum analog of a bit, called a qubit.

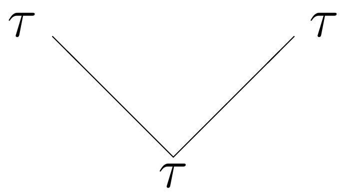
Another basis for the same Hilbert space is given by simlar pictures fusing the right two anyons first.

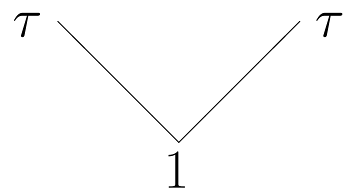
The unitary matrix transforming one basis to the other is completely determined, up to normalizations:

$$\begin{pmatrix} q & \sqrt{q} \\ \sqrt{q} & -q \end{pmatrix} \quad \text{where } q = \frac{\sqrt{5}-1}{2}.$$

Braiding

Braiding of two anyons, interchanging them by passing the first in front of the second, acts on the two fusion processes by nontrivial phase factors.





phase $e^{3\pi i/5} = -\zeta^4$ phase $e^{6\pi i/5} = \zeta^3$

Here $\zeta = e^{2\pi i/5}$.

It's related to q by $q = \zeta + \zeta^4$.

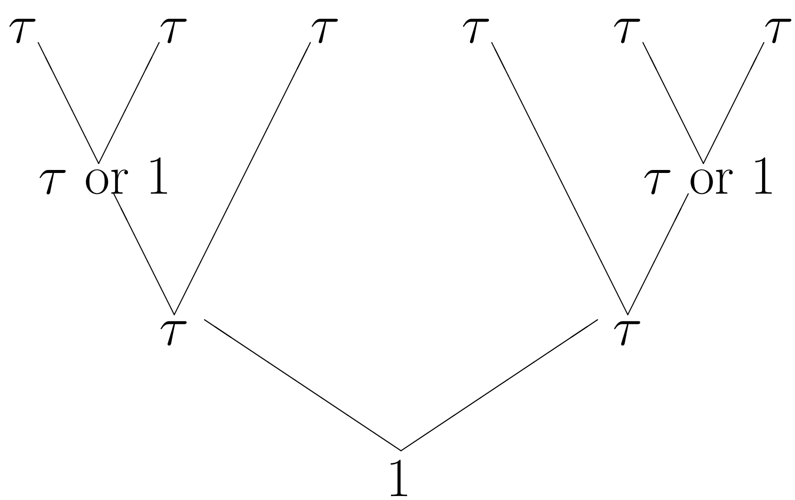
These results imply more complicated formulas for braiding of 3 or more Fibonacci anyons.

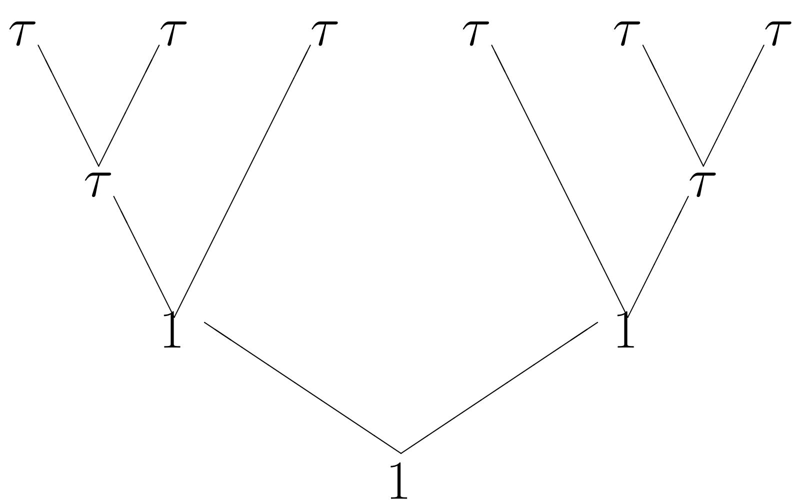
For three anyons fusing to one, the two braid operations generate a dense subgroup of the unitary matrices, modulo phases.

Two Qubits from Six Anyons

Six Fibonacci anyons can fuse to vacuum (1) in five ways.

Four correspond to two qubits each formed from three of the 6 anyons. The fifth is junk.





Braid Group

The braid group B_6 on 6 strands is generated by 5 **Artin generators** $\sigma_1, \ldots \sigma_5$ subject to relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i - j| \geq 2$$

and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.$$

Similarly of other B_n's.

It has one-dimensional representations where all generators are represented by the same scalar factor.

B_n also has a well-studied **Burau** representation of dimension n, which contains the reduced Burau representation, of dimension $n - 1$ as summand.

A theorem of Formanek says that every irreducible representation of B_n of dimension $<$ n is the product of a scalar (one-dimensional representation) and a composition factor of a Burau representation **except** if n is 4, 5, or 6.

Jones-Formanek Representation

The 5-dimensional representation of B_6 for braiding 6 Fibonacci anyons is one of the exceptions. Fortunately, it's explicitly known. The matrices representing the Artin generators are, up to a change of basis and an overall factor of $-\zeta$:

$$\sigma_1 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & -\zeta^4 \\ 0 & -\zeta^4 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\zeta^4 \end{pmatrix} \sigma_2 \mapsto \begin{pmatrix} -\zeta^4 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\zeta^4 & 0 & 0 \\ 0 & 0 & -\zeta^4 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_3 \mapsto \begin{pmatrix} 1 & 0 & 0 & -\zeta^4 & 0 \\ 0 & -\zeta^4 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\zeta^4 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix} \sigma_4 \mapsto \begin{pmatrix} -\zeta^4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\zeta^4 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\zeta^4 \end{pmatrix}$$

$$\sigma_5 \mapsto \begin{pmatrix} 1 & 0 & -\zeta^4 & 0 & 0 \\ 0 & -\zeta^4 & 0 & 0 & 0 \\ 0 & 0 & -\zeta^4 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix}$$

About These Matrices

They aren't unitary. The new basis is not orthonormal.

The entries are in the cyclotomic field $\mathbb{Q}[\zeta]$, in fact in its ring of integers $\mathbb{Z}[\zeta]$. There is no $\sqrt{q}$ here.

Each of these matrices can be completely described in terms of just the locations of $-\zeta^4$ on the diagonal. There's a left location 1 or 2 and a right location 3, 4, or 5. The other three diagonal entries are 1, and each of these 1's has only 0's in the rest of its column. The column of the left $-\zeta^4$ has two -1's in the rows of the 1's on the right. The column of the right $-\zeta^4$ has another $-\zeta^4$ in the row of the 1 on the left.

The locations of the diagonal $-\zeta^4$ progress almost cyclically:

$$\begin{array}{lll} \sigma_1 & 2 & 5 \\ \sigma_2 & 1 & 3 \\ \sigma_3 & 2 & 4 \\ \sigma_4 & 1 & 5 \\ \sigma_5 & 2 & 3 \end{array}$$

A Sixth "Artin" Generator

Symmetry calls for a sixth Artin generator, completing the table with

$$\sigma_6 \;\; 1 \;\; 4$$

Such an element exists in B_6 (with analogs in all other B_n's, for geometric reasons.

Instead of starting with the 6 anyons in a line, arrange them in a circle. That makes a 6-fold symmetry of the situation visible, and σ_6 appears naturally. It has the expected representation matrix.

Which way should you bend the line of anyons into a circle?

There are two **different** versions of σ_6.

But they both have the same Fibonacci representation.

Hecke Algebras

A representation of a group is also a representation of its group algebra.

The Fibonacci representations of braid groups are special in that the matrices representing Artin generators have only two eigenvalues, ζ^3 and $-\zeta^4$.

As representations of the group algebra, they factor through the quotient algebra obtained by imposing the relations

$$(\sigma - \zeta^3)(\sigma + \zeta^4) = 0$$

on all the Artin generators.

That quotient, called the Iwahori-Hecke algebra, is $n!$-dimensional for B_n.

From Hecke Algebra to Fibonacci Representation

For three anyons fusing to one, the Hecke algebra has dimension 6, and the Fibonacci representation (by 2×2 matrices) has dimension 4, so there's a 2-dimensional kernel. We have a basis for the kernel:

$$\{\sigma_1\sigma_2\sigma_1 + \zeta^2\sigma_1 + \zeta^2\sigma_2 + \zeta - 1,$$
$$(\zeta^3 - \zeta^4)\sigma_1\sigma_2\sigma_1 + \zeta^2\sigma_1\sigma_2 + \zeta^2\sigma_2\sigma_1 + \zeta^4\}$$

For six anyons fusing to vacuum, the Hecke algebra has dimension 720, and the Fibonacci representation (by 5×5 matrices) has dimension 25, so there's a 695-dimensional kernel. It remains mysterious to us.