Near-Optimal Lower Bounds on Quantifier Depth and Weisfeiler-Leman Refinement Steps

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{Symmetry, Logic, Computation}
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Joint work with Jakob Nordström (KTH Stockholm).

k-variable fragments of first-order logic

Two vertices are connected by a path of length 4:

$$\varphi_{\text{dist-4}}(x, y) = \exists z_1 \exists z_2 \exists z_3 \left(Exz_1 \wedge Ez_1 z_2 \wedge Ez_2 z_3 \wedge Ez_3 y \right)$$

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Equivalent $\mathcal{C}^2$ formula:

$$\varphi'_{\text{deg-7}}(x) = \exists^{\geq 7} y \, Exy$$

Quantifier depth of $\mathcal{C}^k$

Definition

$\mathsf{D}^k(\mathcal{A},\mathcal{B})$ is the minimal quantifier depth of a $\mathcal{C}^k$ sentence that distinguishes two n-element structures $\mathcal{A}$ and $\mathcal{B}$ (with $\mathcal{A} \not\equiv_{\mathcal{C}^k} \mathcal{B}$).

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$$\exists x_1 \cdots \exists x_n \left(\bigwedge_{i \neq j} x_i \neq x_j \;\wedge \bigwedge_{\substack{R \in \sigma, \\ (v_{i_1},\ldots,v_{i_r}) \in R^{\mathcal{A}}}} R x_{i_1} \cdots x_{i_r} \;\wedge \bigwedge_{\substack{R \in \sigma, \\ (v_{i_1},\ldots,v_{i_r}) \notin R^{\mathcal{A}}}} \neg R x_{i_1} \cdots x_{i_r} \right)$$

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Theorem [B., Nordström 2016a]

For every $k \leq n^{0.01}$ there are n-element relational structures $\mathcal{A}$, $\mathcal{B}$ of arity $k-1$ such that $\mathrm{D}^k(\mathcal{A}, \mathcal{B}) \geq n^{\Omega(k/\log k)}$.

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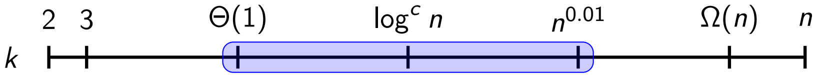
$\mathrm{D}^k(\mathcal{A}, \mathcal{B})$ agrees with the number of refinement steps $(k-1)$-dimensional Weisfeiler-Leman needs to distinguish $\mathcal{A}$ and $\mathcal{B}$.

$\mathcal{C}^k$ and Weisfeiler-Leman



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An application for non-constant k

- Babai's quasipolynomial graph isomorphism test uses $k = \log^c n$ on $(k-1)$-ary relational structures. [Babai 2016]
- Our result implies an $\Omega\big(n^{\log^{c-1} n}\big)$ lower bound in this setting.

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In one sentence, a novel combination of methods from

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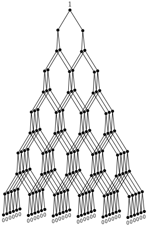

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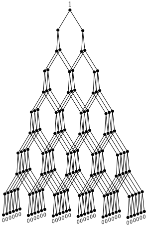

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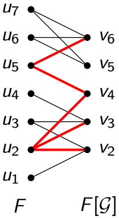

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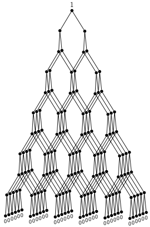
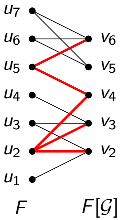


$\bigoplus$



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The link between both areas is made via XOR formulas as a source of hard instances.

XOR formulas

An s-XOR formula F over Boolean variables $x_1, \ldots, x_n$ is a set of parity constraints $x_{i_1} \oplus \cdots \oplus x_{i_r} = a$, $r \leq s$, $a \in \{0, 1\}$.

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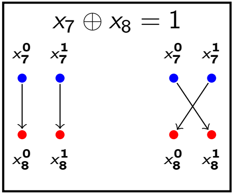
Let $\mathcal{A}(F)$ and $\mathcal{B}(F)$ relational structures with two vertices x_i^0, x_i^1 for every $x_i \in \mathrm{Vars}(F)$ and relations

$$X_i^{\mathcal{A}} = X_i^{\mathcal{B}} = \{x_i^0, x_i^1\}$$
$$R_r^{\mathcal{A}} = \{(x_{i_1}^{a_1}, \ldots, x_{i_r}^{a_r}) \mid (x_{i_1} \oplus \cdots \oplus x_{i_r} = a) \in F,\ \bigoplus_i a_i = 0\}$$
$$R_r^{\mathcal{B}} = \{(x_{i_1}^{a_1}, \ldots, x_{i_r}^{a_r}) \mid (x_{i_1} \oplus \cdots \oplus x_{i_r} = a) \in F,\ \bigoplus_i a_i = a\}$$

Isomorphism $I : \mathcal{A}(F) \to \mathcal{B}(F)$ corresponds to satisfying assignment α for F via

$$\alpha(x_i) = 0 \iff I(x_i^0) = x_i^0 \Leftrightarrow I(x_i^1) = x_i^1$$
$$\alpha(x_i) = 1 \iff I(x_i^0) = x_i^1 \Leftrightarrow I(x_i^1) = x_i^0$$



A pebble game on XOR formulas

The k-pebble game on F is played by two players.

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Player 1 wins the game, if α_i falsifies an XOR-constraint.

Equivalent characterizations of the pebble game

Let F be an s-XOR formula and $R > 0, k > s$ be integers. Then the following statements are equivalent:

(a) Player 1 wins the R-round k-pebble game on F.

(b) There is a k-variable sentence $\varphi \in \mathcal{C}^k$ of quantifier rank R such that $\mathcal{A}(F) \models \varphi$ and $\mathcal{B}(F) \not\models \varphi$.

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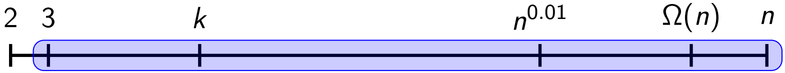
(c) The s-CNF-formula $\mathrm{cnf}(F)$ has a resolution refutation of width $k - 1$ and depth R.

Outline of the proof



[Immerman 1981]:

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Part I (pyramid construction):

For every k there are n-variable 3-XOR formulas such that Player 1 needs $n^{\Omega(\frac{1}{\log k})}$ rounds to win the ℓ-pebble game for $3 \leq \ell \leq k$.

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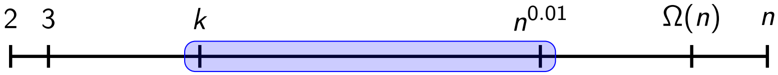

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Part II (hardness condensation):

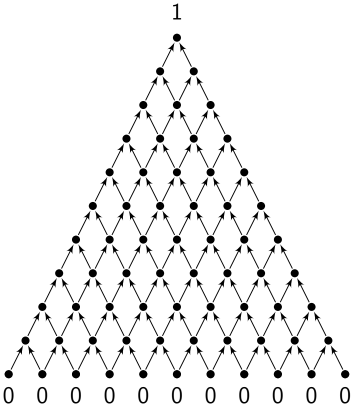
Reduce the number of variables without destroying the lower bound.

Transform n-variable 3-XOR into m-variable k-XOR, for $m \approx n^{\frac{1}{k}}$.

Lower bound remains $n^{\Omega(\frac{1}{\log k})} = m^{\Omega(\frac{k}{\log k})}$.

PART I: An $n^{\Omega(\frac{1}{\log k})}$ lower bound

A two-dimensional pyramid



XORs from DAGs

Let $\mathcal{G}$ be an acyclic directed graph with a unique sink z. The XOR-formula $\mathrm{xor}(\mathcal{G})$ over the variables $v \in V(\mathcal{G})$ contains the following constraints:

(i) $v \oplus \bigoplus_{w \in N^-(v)} w = 0$

(ii) $s = 0$ for every source s,

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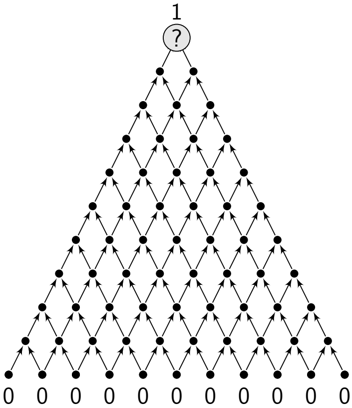

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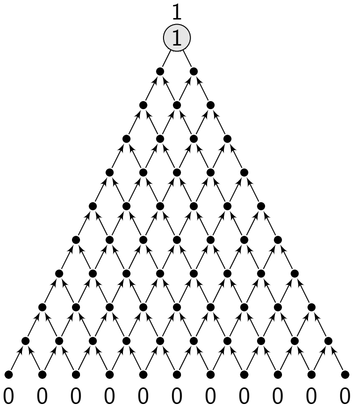

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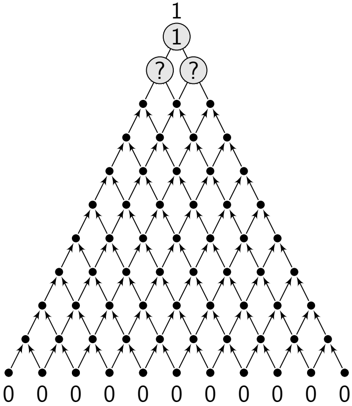

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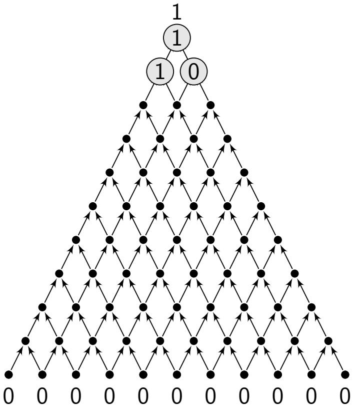

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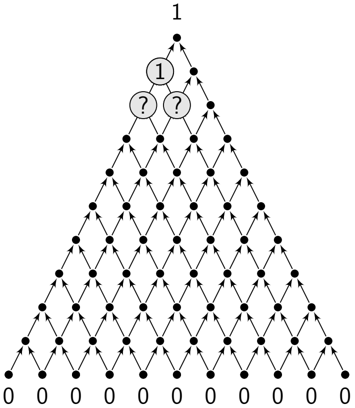

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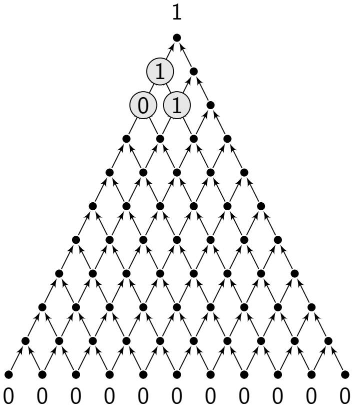

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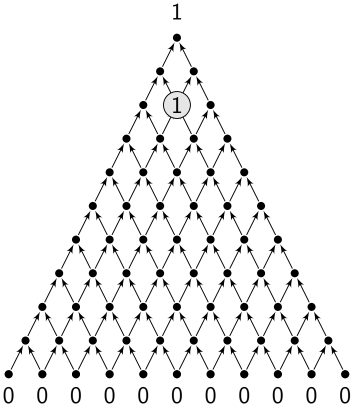

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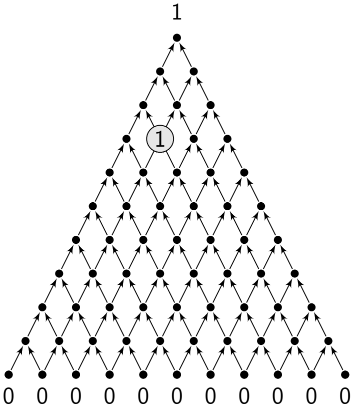

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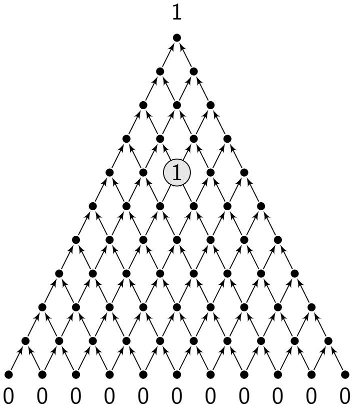

XORs from DAGs

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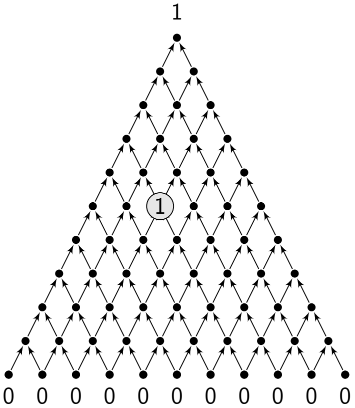

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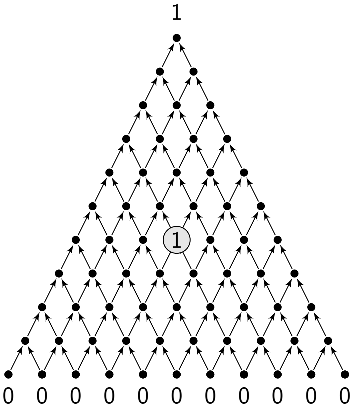

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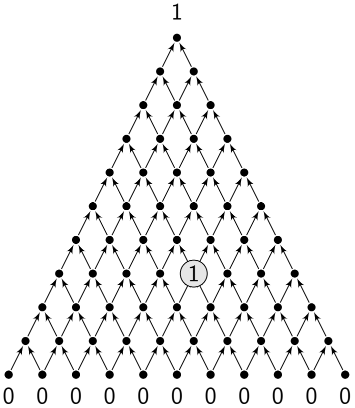

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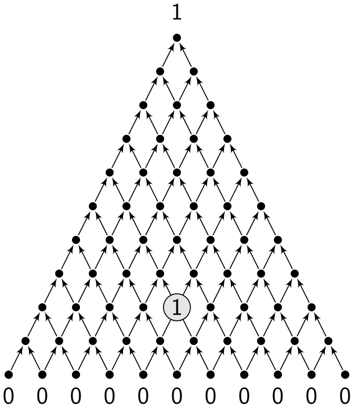

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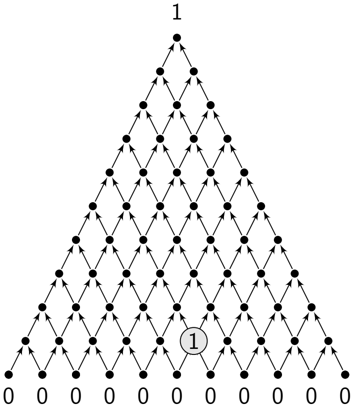

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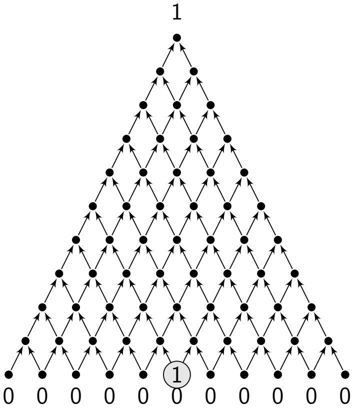

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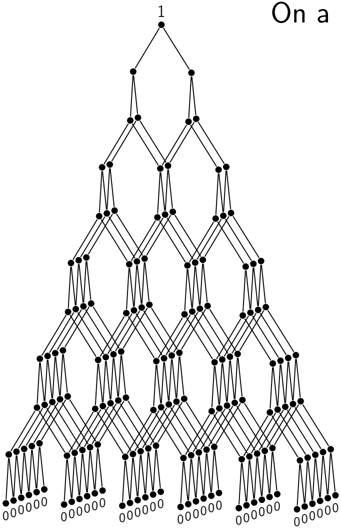
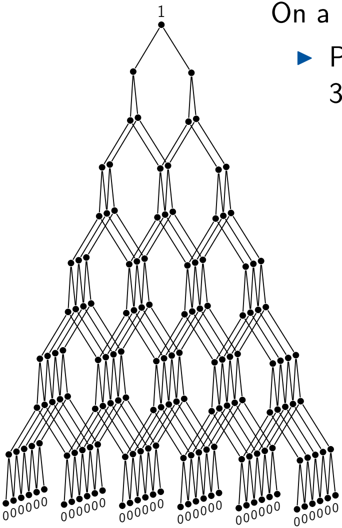


On a d-dimensional pyramid of height h

A three-dimensional pyramid



On a d-dimensional pyramid of height h

- Player 1 wins the k-pebble game, $3 \leq k \leq 2^{d-1}$, in $\Theta(h)$ rounds

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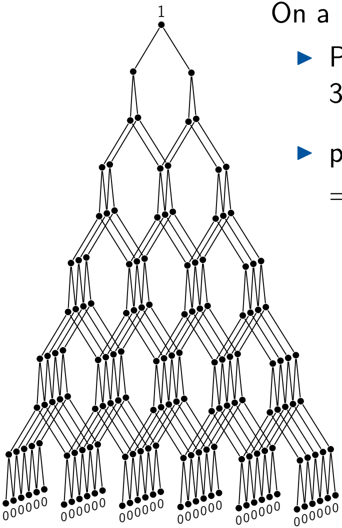

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- Player 1 wins the k-pebble game, $3 \leq k \leq 2^{d-1}$, in $\Theta(h)$ rounds
- pyramid has $n \approx h^d$ vertices $\Rightarrow n^{\Theta(\frac{1}{\log k})}$ rounds in k-pebble game

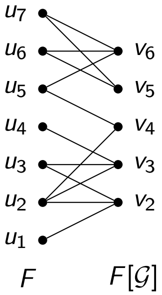
PART II: Hardness condensation

Substitution with recycling

Let F be an XOR-formula over variables U and $\mathcal{G} = (U \,\dot\cup\, V, E)$ a bipartite graph.
The formula $F[\mathcal{G}]$ is defined over variables V with constraints

$$\Big(\bigoplus_{v \in N(u_1)} v\Big) \oplus \cdots \oplus \Big(\bigoplus_{v \in N(u_\ell)} v\Big) = a$$

for all constraints $u_1 \oplus \cdots \oplus u_\ell = a$ from F.



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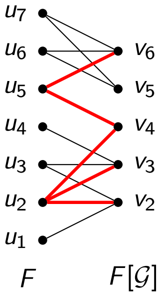

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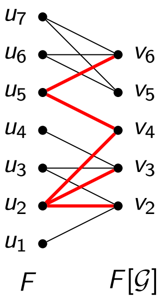

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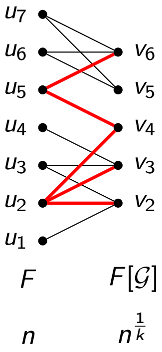

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- We use boundary expander $\mathcal{G}$ with left-degree $< k/3$, $|U| = n$, and $|V| \approx n^{\frac{1}{k}}$.

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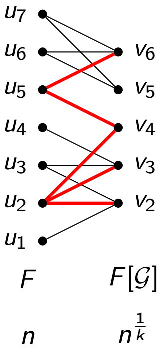

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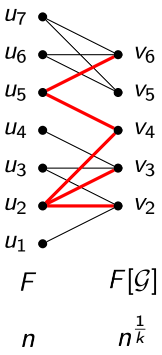

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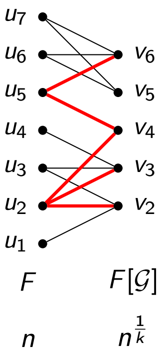

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This method was introduced in [Razborov 2016a] to show that treelike resolution refutations of bounded width k require doubly exponential length $2^{n^{\Omega(k)}}$.

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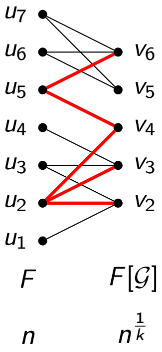

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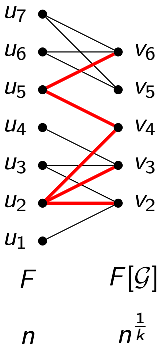

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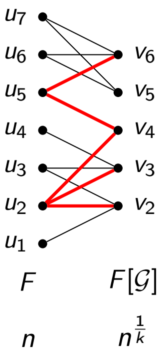

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Our examples are k-ary relational structures. Are there similar lower bounds for graphs?

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Thank you for your attention!