Lower Bounds for Subgraph Isomorphism and Consequences in First-Order Logic

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Outline

- The Subgraph Isomorphism Problem
- AC^0 and First-Order Logic
- Upper and Lower Bounds for SUB(G):

AC^0 circuit size ≈ FO variable width ≈ tree-width(G)

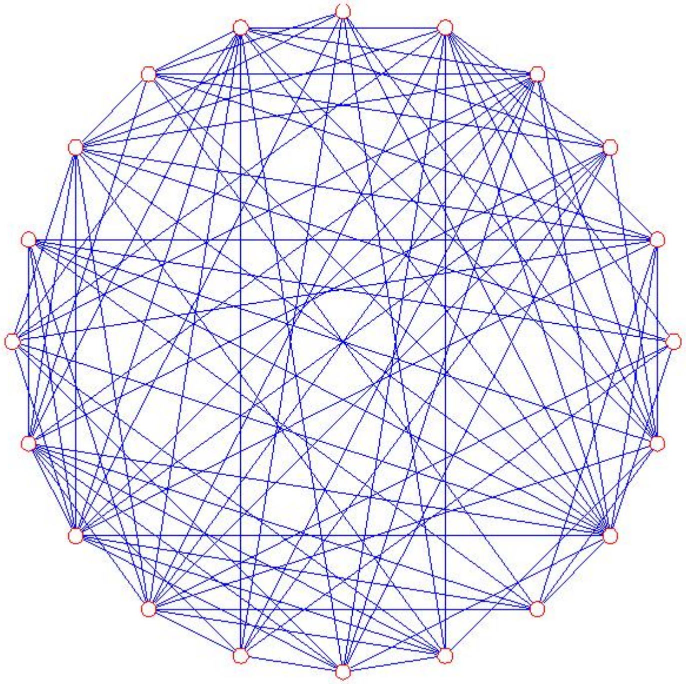
AC^0 formula size ≈ FO quantifier rank ≈ tree-depth(G)

- "Poly-rank" Homomorphism Preservation Theorem

Subgraph Isomorphism Problem

- k-CLIQUE

  Given a graph X, does it contain a k-clique (complete subgraph of size k)?



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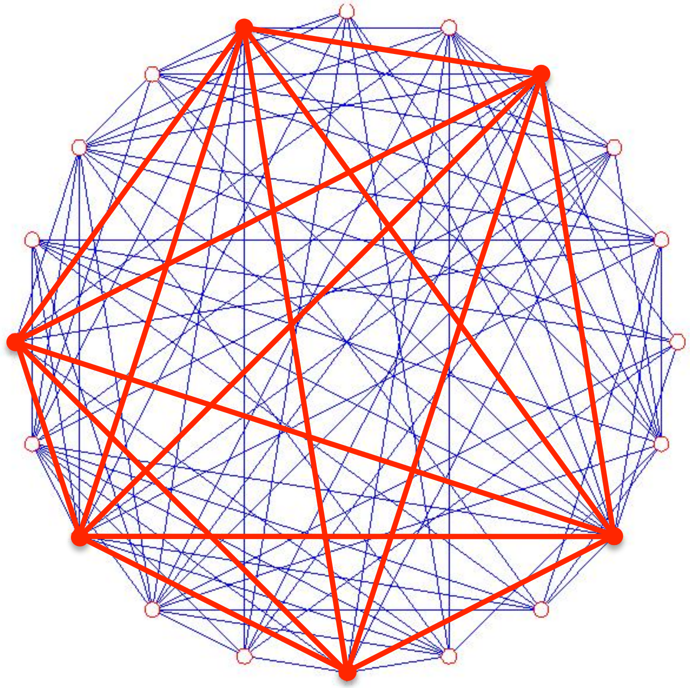


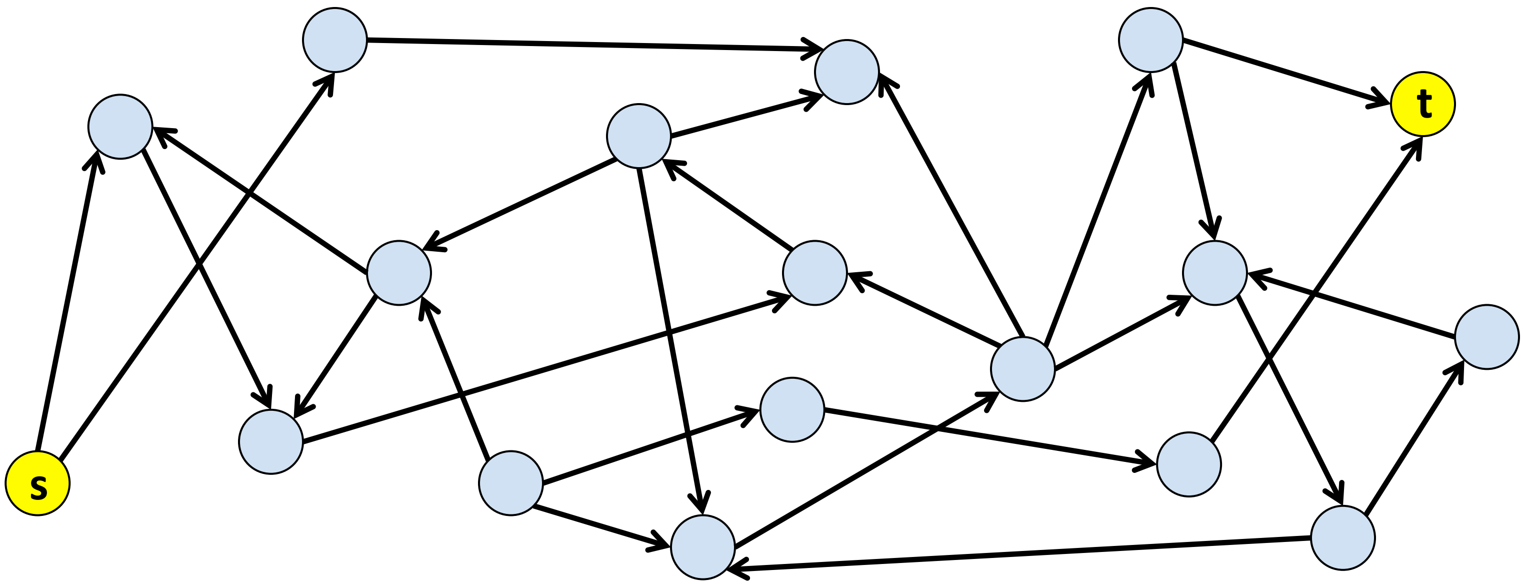
- k-CLIQUE

  Given a graph X, does it contain a k-clique (complete subgraph of size k)?

- **Time complexity of k-CLIQUE**
  - "Brute-force" upper bound: $O(n^k)$
  - Best known upper bound: $O(n^{0.79*k})$
  - Conjectured lower bound: $n^{\Omega(k)}$ $(\Longrightarrow P \neq NP)$

- k-STCONN ("Distance-k Connectivity")

  Given a directed graph X with distinguished vertices s and t, does X contain a st-path of length k?



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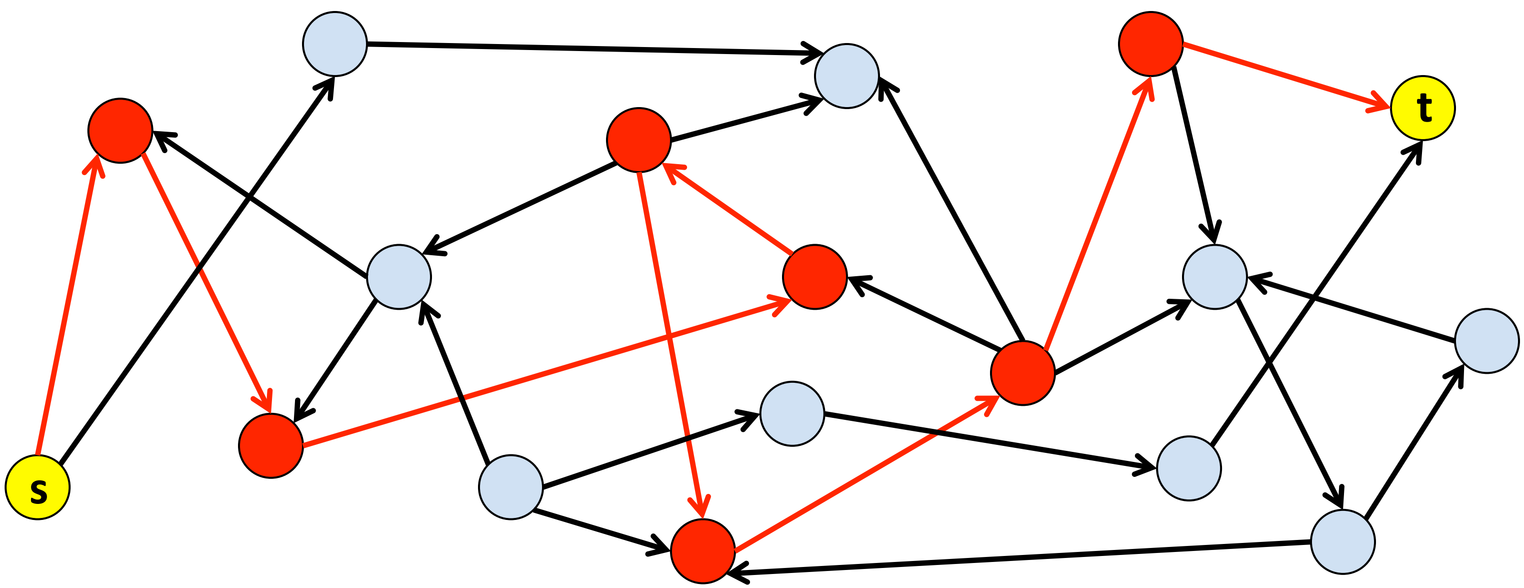
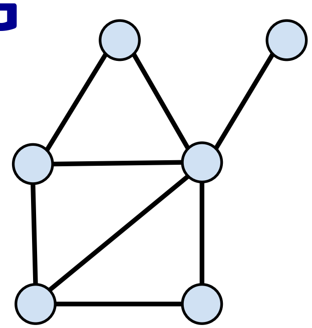


- k-STCONN ("Distance-k Connectivity")

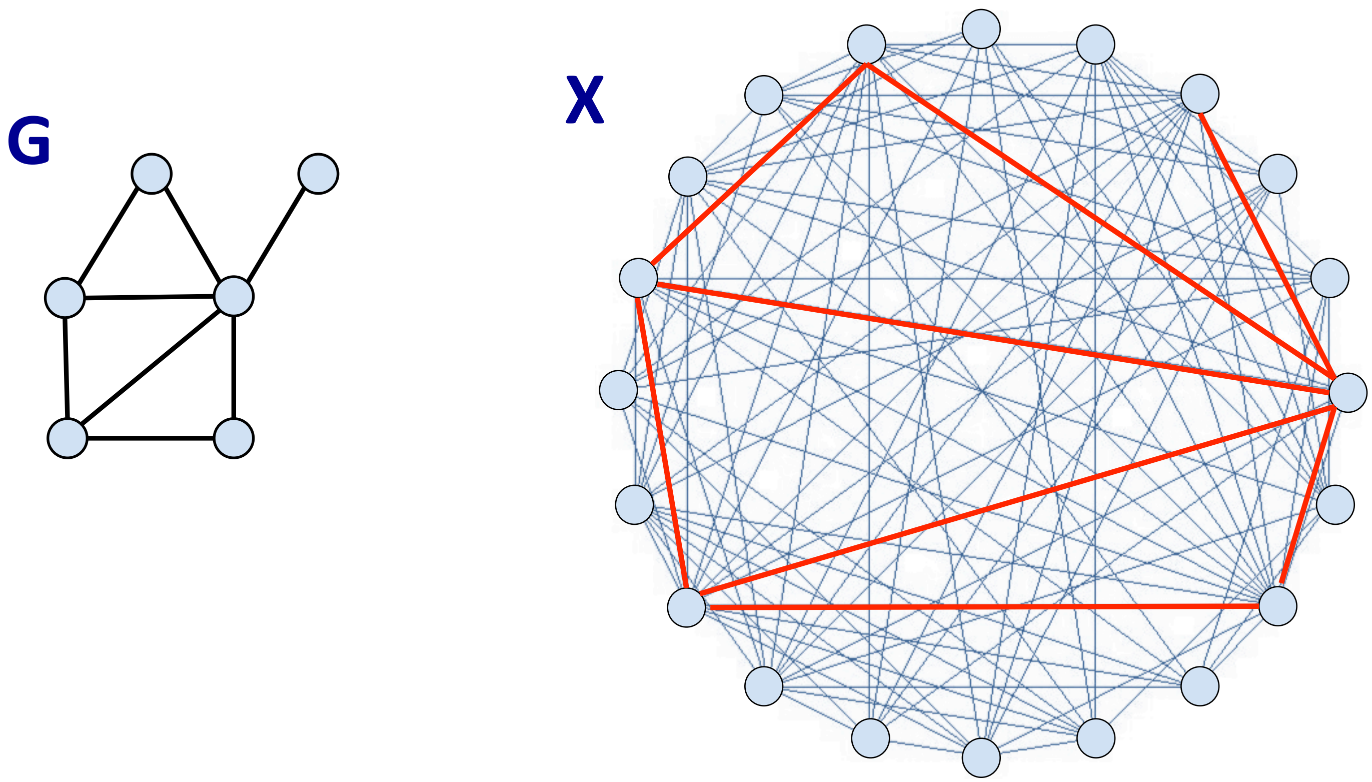
  Given a directed graph X with distinguished vertices s and t, does X contain a st-path of length k?

- **Space complexity of k-STCONN**
  - Best known upper bound: $O(\log k \cdot \log n)$
  - Conjectured lower bound: $\Omega(\log k \cdot \log n)$ $(\Longrightarrow L \neq NL)$

G

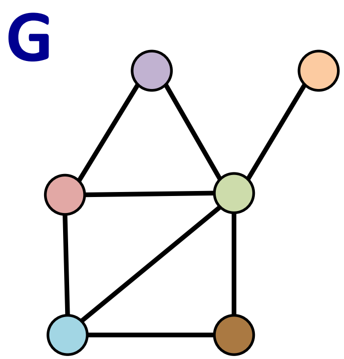


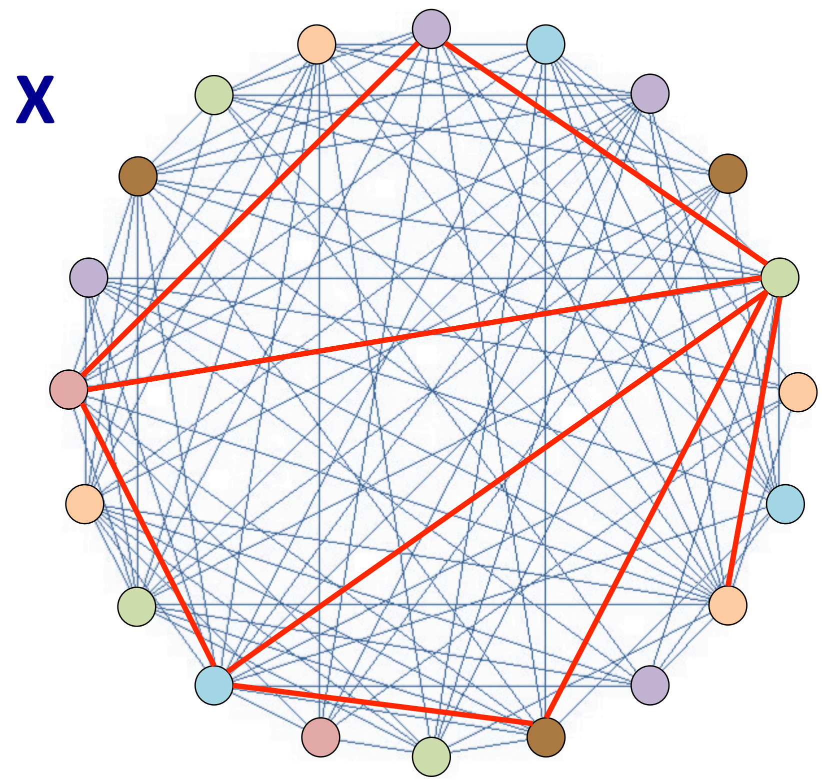
- $SUB_{uncolored}(G)$
  Given a graph X, does it contain a subgraph isomorphic to G?



- SUB(G)

  Given a graph X and a coloring π : V(X) → V(G), does X contain a subgraph G' such that G' ≅ G and π(G') = G?





- SUB(G)

  Given a graph X and a coloring $\pi : V(X) \rightarrow V(G)$, does X contain a subgraph G' such that $G' \cong G$ and $\pi(G') = G$?

- Special cases:

  $SUB(K_k)$ = k-CLIQUE

  $SUB(P_k)$ = k-STCONN

Reductions

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  (i.e. every homomorphism $G \to G$ is one-to-one)

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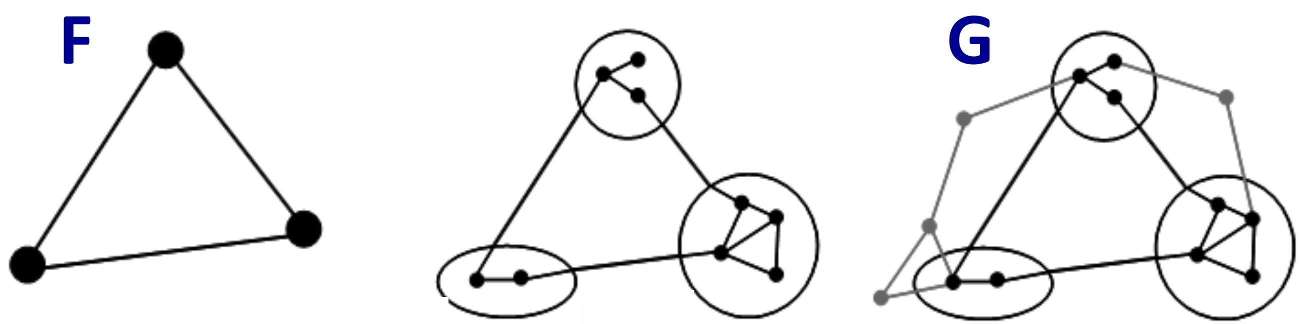
- $SUB_{uncolored}(G) = SUB(G)$ when G is a *core*
  (i.e. every homomorphism $G \rightarrow G$ is one-to-one)
- $SUB(F) \leq SUB(G)$ when F is a *minor* of G



Credit: Wikipedia (NikelsonH)

Summary

- SUB(G) are an important and well-structured family of problems.
- (As we will see,) complexity of SUB(G) tied to natural structural parameters of G.
- Determining the complexity of SUB(G) w.r.t. to different computational resources (time, space, …) would separate various classes ($P \neq NP$, $L \neq NL$, …)

Summary

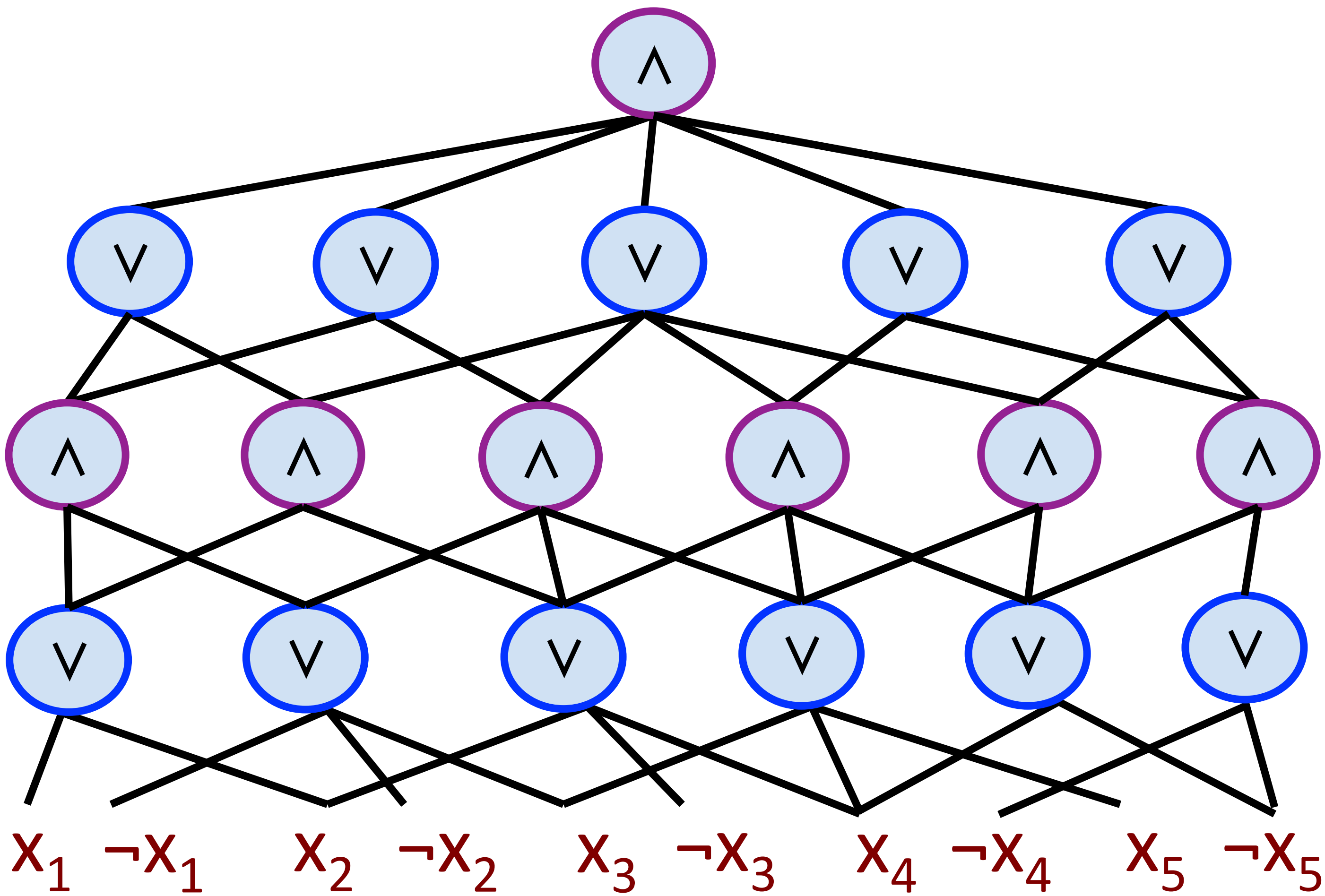
- SUB(… family …
- (… to natural structural paramet…

We will focus on **circuit size** and **formula size**

- Determining the complexity of SUB(G) w.r.t. to different **computational resources** (time, space, …) would separate various classes (P ≠ NP, L ≠ NL, …)

Boolean Circuits and Formulas

Boolean Circuits



P vs. NP

Boolean circuit size =* Turing machine time

(* up to a polynomial factor, ignoring uniformity)

P = { problems solvable by polynomial-size circuits }

NP = { problems whose solutions are verifiable
by polynomial-size circuits }

P vs. NP

- Holy Grail ($P \neq NP$)

  Show that any NP problem (e.g. MAXIMUM CLIQUE) requires **super-polynomial** circuit size

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  It suffices to show that k-CLIQUE requires circuits of size $n^{\Omega(k)}$ for any $k(n) \longrightarrow \infty$

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- ***Circuit lower bounds are hard!***

  Best circuit lower bound for a function in NP:

  2n (1965), 3n (1984), 4n (1991), **5n** (2002)

P vs. NP

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  size n…

To prove **super-linear** lower bounds, need to focus on weaker models of computation (restricted classes of circuits)

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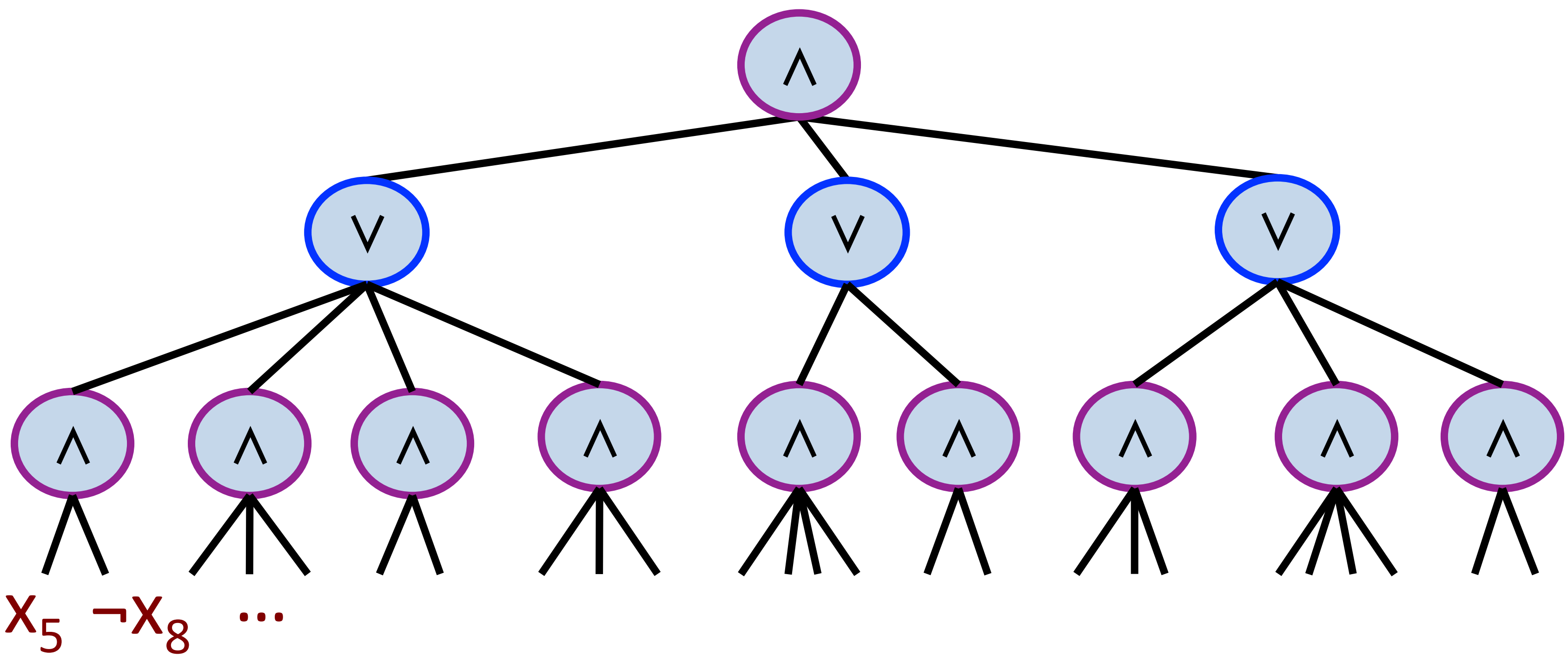
Boolean Formulas

- **Formulas** = tree-like circuits
- “Memoryless”: each sub-computation is used once



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- ***Formula lower bounds are hard!***

  Best formula-size lower bound for a function in P:

  $n^{1.5}$ (1961), n^2 (1966), $n^{2.5}$ (1987), $\mathbf{n^3}$ (1998)

Boolean Formulas

- Another Holy Gr...

To prove **super-polynomial** lower bounds, again must focus on restricted classes

- ***Formula lower bounds are hard!***

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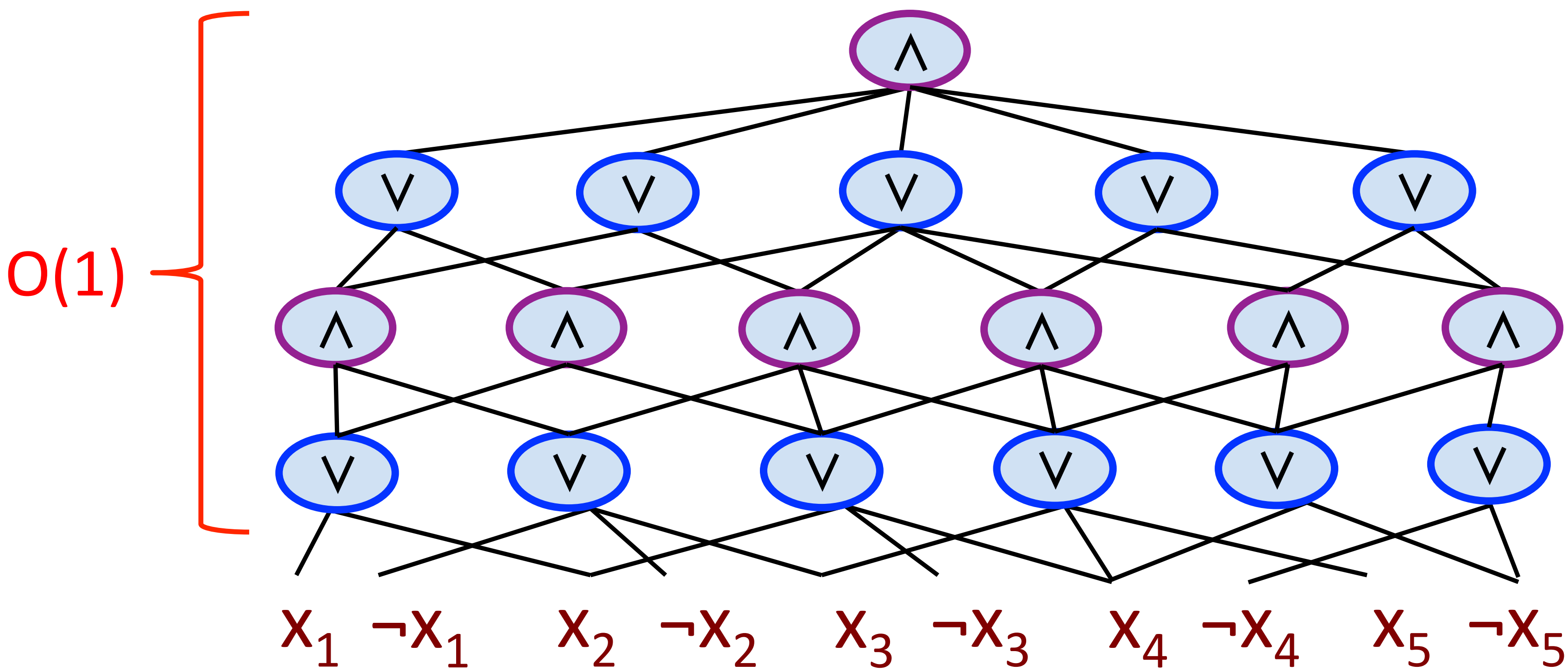
AC^0 Circuit and Formulas

- We restrict attention to circuits and formulas of ***constant depth*** (a.k.a. **AC^0 circuits and formulas**)



AC^0 & First-Order Logic

Hierarchies Within FO

- **Variable-width** (max # of free vars in a subformula)

$$FO^1 \subseteq FO^2 \subseteq FO^3 \subseteq \ldots$$

- **Quantifier-rank** (nesting depth of quantifiers)

$$FO_1 \subseteq FO_2 \subseteq FO_3 \subseteq \ldots$$

- Theorem

  The **model-checking problem** for a FO sentence $\boldsymbol{\varphi}$

  Given a structure A with universe {1,…,n}, is A a model $\boldsymbol{\varphi}$*?*

  is solvable by:

  - AC^0 circuits of size $O(n^{\textbf{variable-width}(\varphi)})$

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layers of <u>fan-in n</u> gates

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(formula size ≤ depth × fan-in)

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Hierarchies Within FO

- Variable-width

$$FO^1 \subseteq FO^2 \subseteq FO^3 \subseteq \ldots$$

- Quantifier-rank

$$FO_1 \subseteq FO_2 \subseteq FO_3 \subseteq \ldots$$

- **Background relations**

$$FO \subseteq FO[<] \subseteq FO[BIT] \subseteq FO[Arb]$$

Hierarchies Within FO

- Variable-width

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[Barrington-Immerman-Straubing 1990]

uniform-AC^0

… AC^0

- **Background relations**

$FO \subseteq FO[<] \subseteq FO[BIT] \subseteq FO[Arb]$

Implications

lower bounds for **AC^0 circuit size**

⇓

lower bounds for **FO[Arb] variable-width**

lower bounds for **AC^0 formula size**

⇓

lower bounds for **FO[Arb] quantifier-rank**

Complexity of SUB(G): Upper Bounds

Upper Bounds

- <u>Theorem (folklore)</u>

  SUB(G) is definable in:

    - FO[**tree-width**(G) + 1 variables]
    - FO[**tree-depth**(G) quantifier rank]

Upper Bounds

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moreover, existential & positive

Upper Bounds

- Theorem (folklore)

  SUB(G) is definable in:

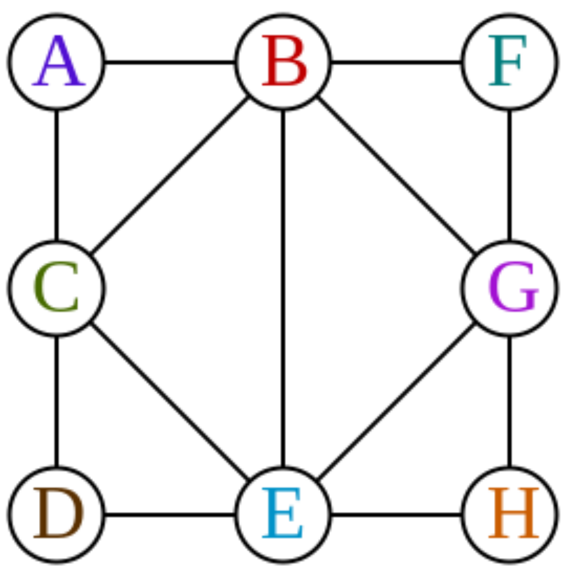
  - FO[**tree-width**(G) + 1 variables]
  - FO[**tree-depth**(G) quantifier rank]

  SUB(G) is solvable by:

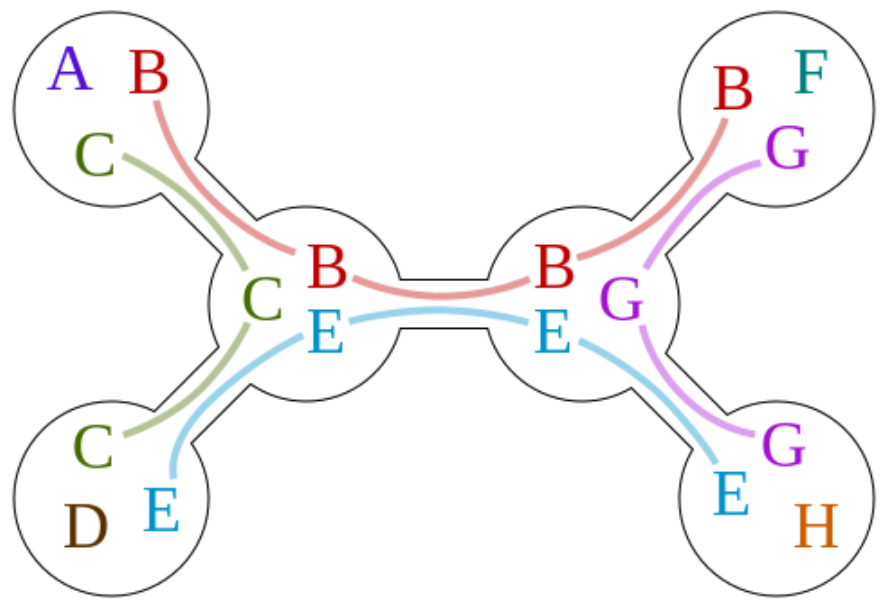
  - AC^0 circuits of size $n^{O(\textbf{tree-width}(G))}$
  - AC^0 formulas of size $n^{O(\textbf{tree-depth}(G))}$

Tree-width: **tw**(G)

graph G



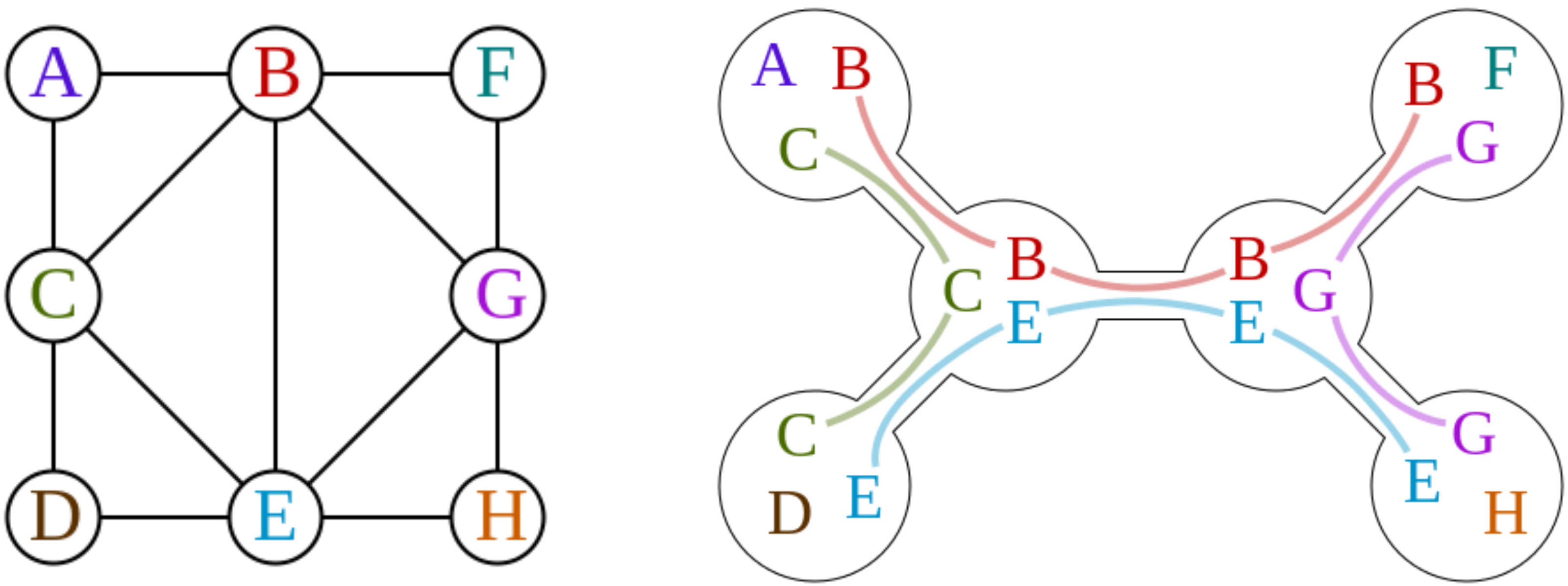
tree decomposition of G



Credit: Wikipedia (David Eppstein)

Tree-width: **tw**(G)

- **tw**(any tree) = 1, **tw**(K_k) = k – 1



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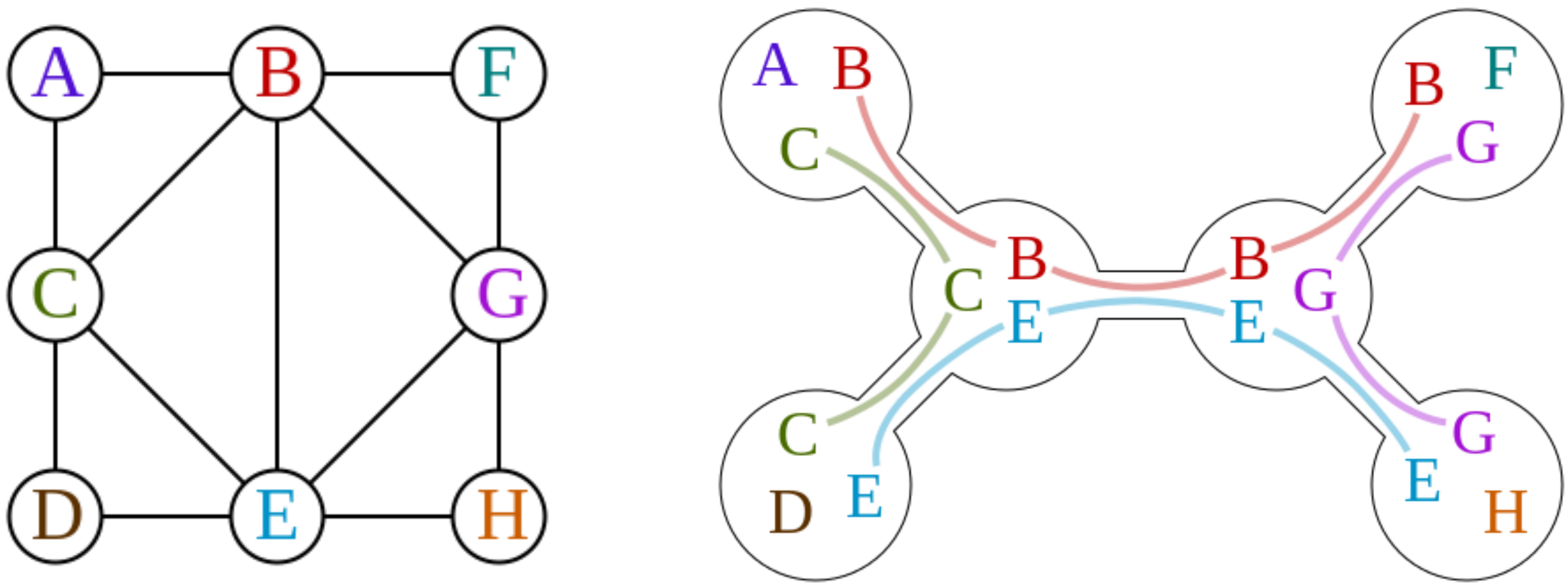
Tree-width: **tw**(G)

- ***Width-k tree decomposition*** of G: blueprint for a (k+1)-variable first-order sentence defining SUB(G)



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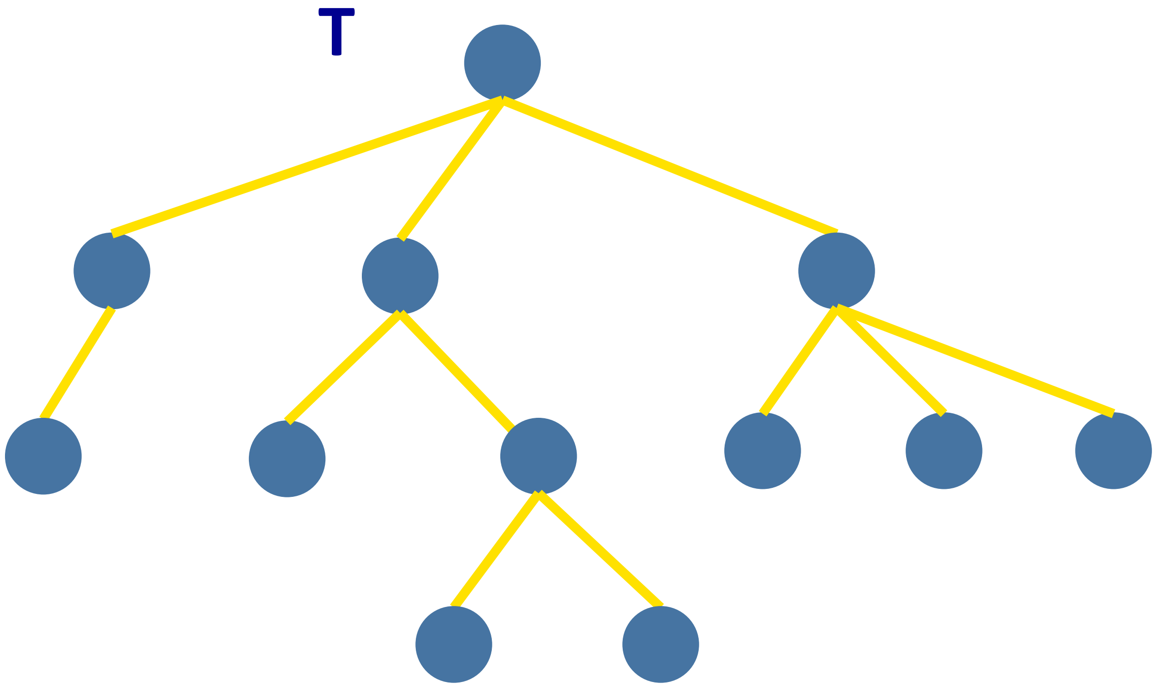
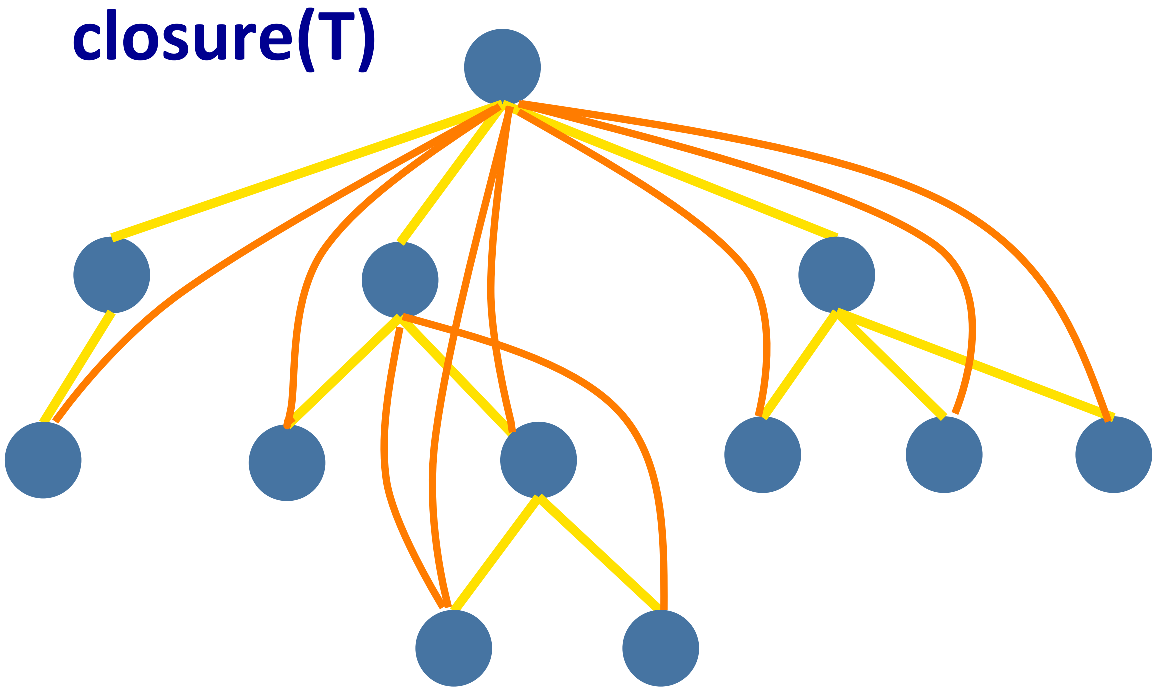
Tree-depth: **td**(G)



Tree-depth: **td**(G)

- Def. The **closure** of a tree T is graph formed by adding edges between all ancestor-descendant pairs



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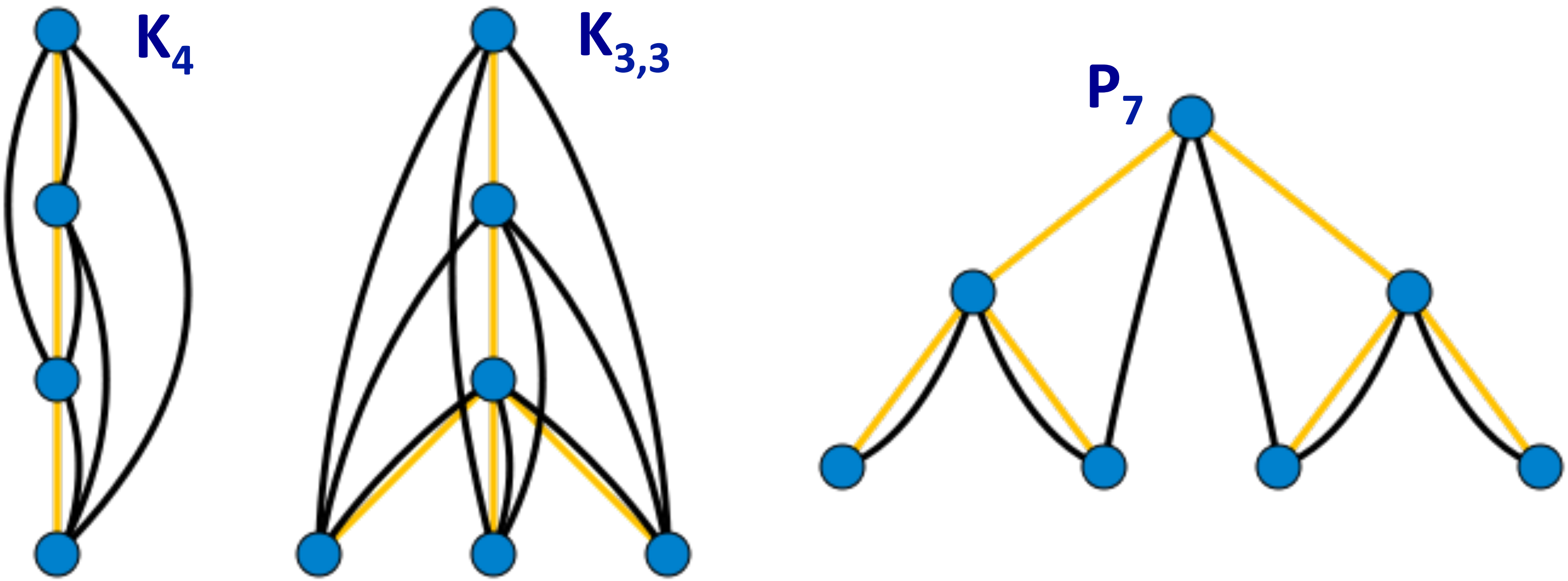

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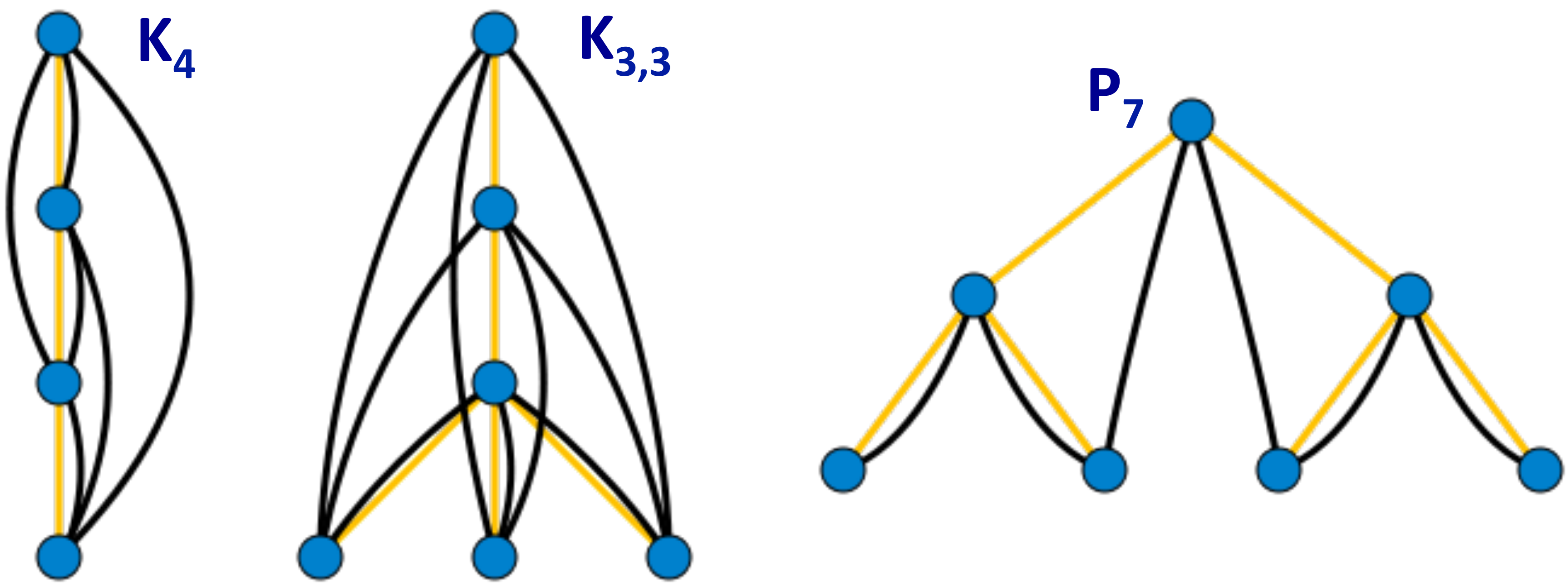
Tree-depth: **td**(G)

- **tw**(G) ≤ **td**(G) ≤ **tw**(G)*log|V(G)|
- log(**longest-path**(G)) ≤ **td**(G) ≤ **longest-path**(G)



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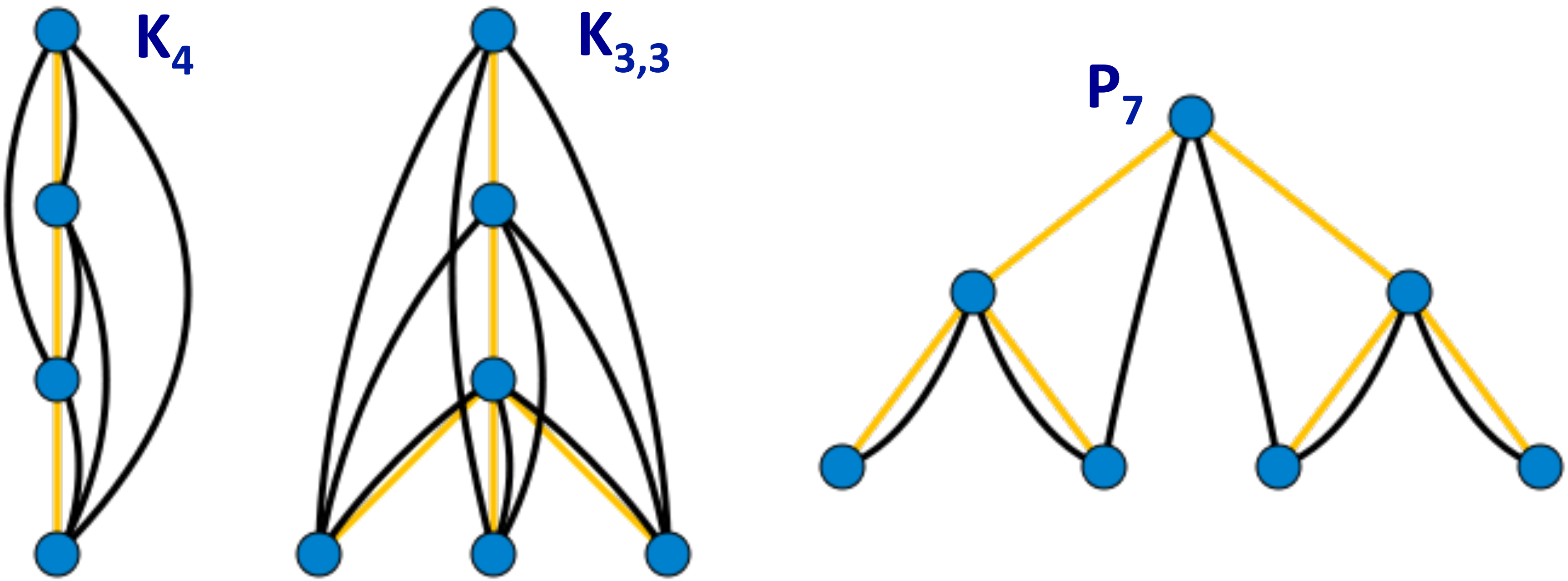
Tree-depth: **td**(G)

- ***Height-k tree*** T with G ⊆ closure(T): blueprint for a quantifier rank-k first-order sentence defining SUB(G)



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AC^0 Complexity of SUB(G): Lower Bounds

Lower Bounds

- Theorem [Li-Razborov-R. 2014]

  The AC^0 *circuit size* of SUB(G) is $n^{\tilde{\Omega}(\mathbf{tw}(G))}$

- Theorem [Kawarabayashi-R. 2016, R. 2016]

  The AC^0 *formula size* of SUB(G) is $n^{\Omega(\mathbf{td}(G)^{\varepsilon})}$

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  [R. 2008]

  k-CLIQUE has AC^0 circuit size $n^{\Omega(k)}$

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  k-STCONN has AC^0 formula size $n^{\Omega(\log k)}$

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  "The variable hierarchy is strict over ordered graphs"

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  "Poly-rank homomorphism preservation theorem"

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k-CLIQUE is definable in FO^k
but not in $FO^{k/4}[\leq]$

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  The [Arb] *vari* ... (G))

  "T ... "

  ... $)^{\varepsilon}$)

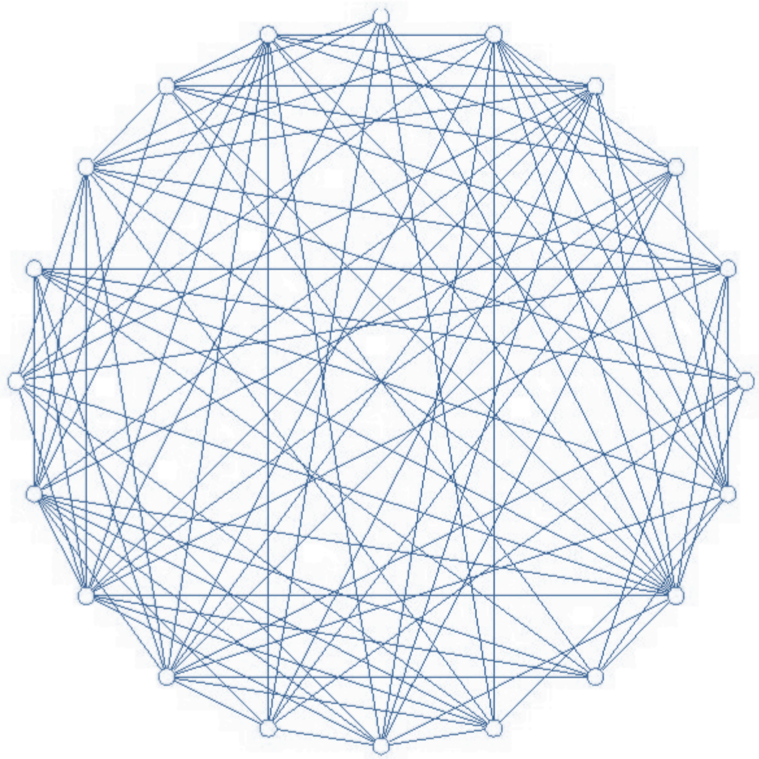
  "Poly- ... vation theorem"

Proof uses probabilistic method: ***average-case*** lower bounds w.r.t. particular random input graphs (generalizations of G(n,p))

Hard-On-Average Input Distributions for SUB(G)

Average-Case for $SUB_{uncolored}(G)$

- Natural input distribution: **ErdosRenyi**(n,p) where p = p(n) is the “threshold” for G-subgraphs



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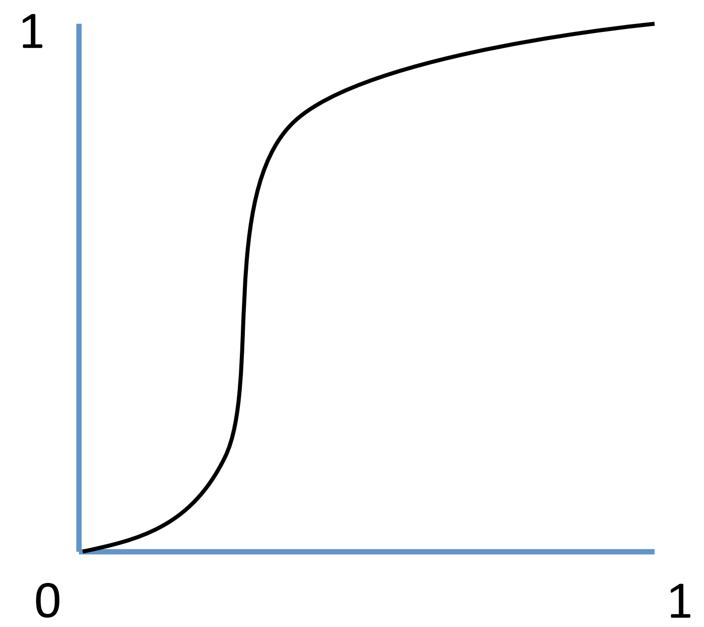

edge probability p

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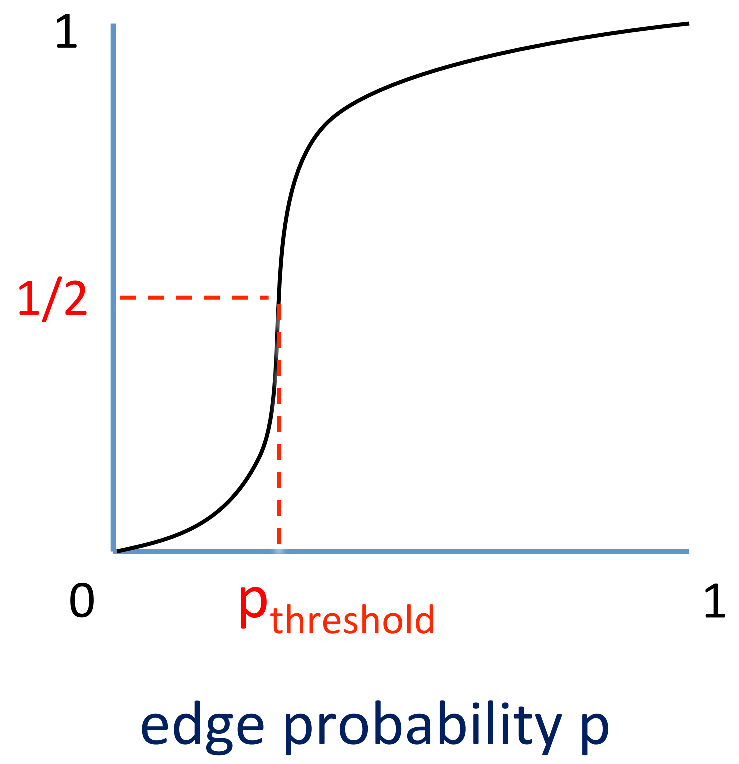

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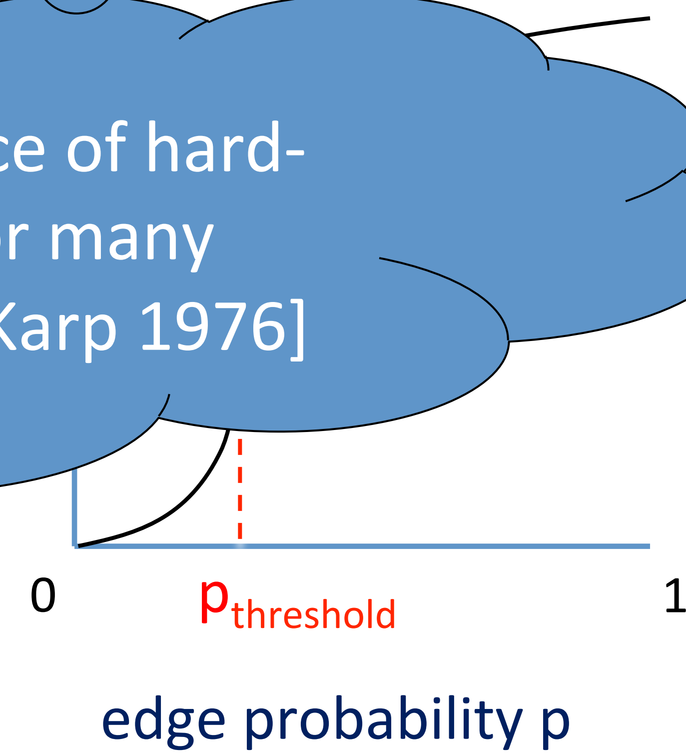
Conjectured to be source of hard-on-average instances for many graphs G, including K_k [Karp 1976]



Average-Case for SUB(G)

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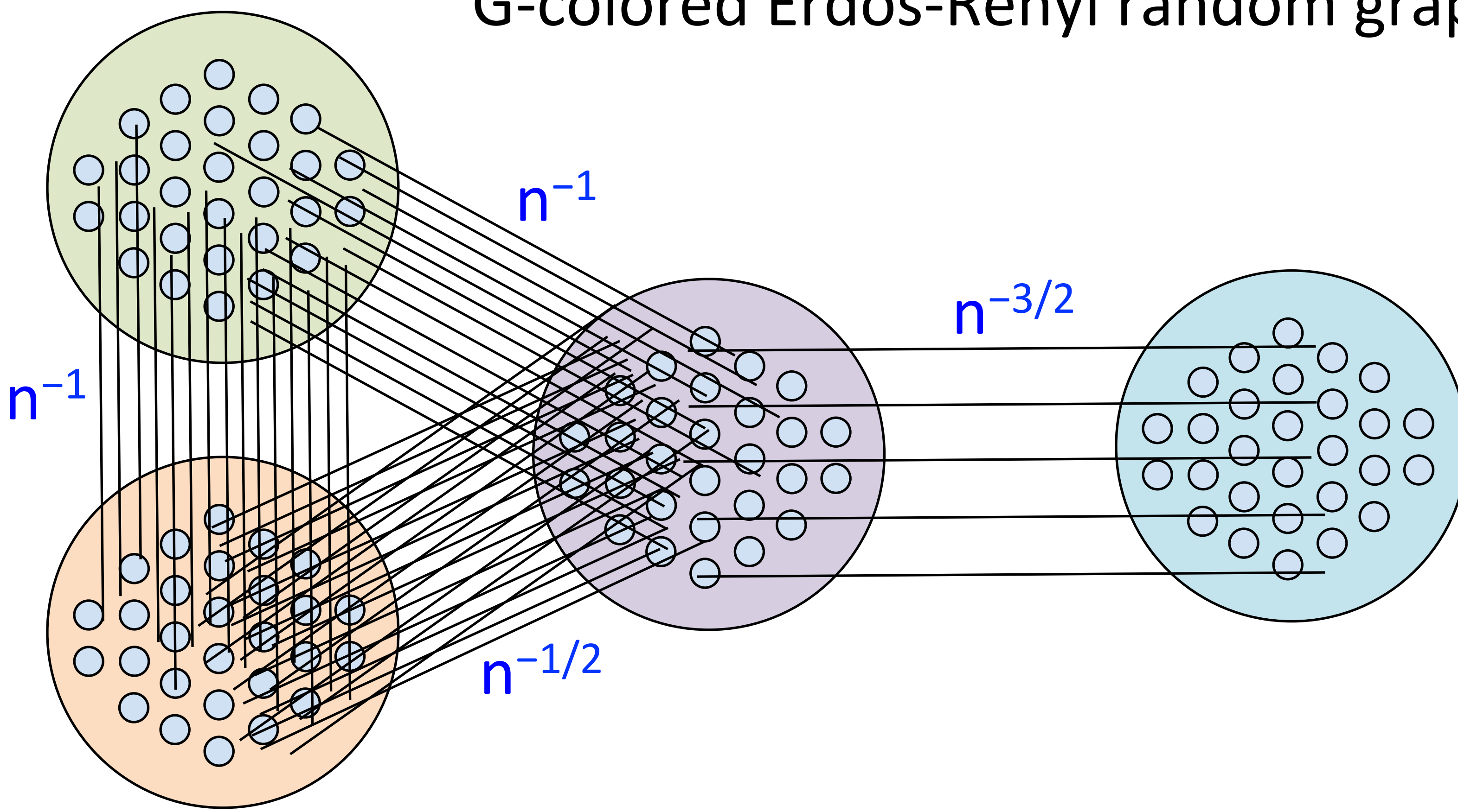


Average-Case for SUB(G)

- Natural *family* of input distributions:

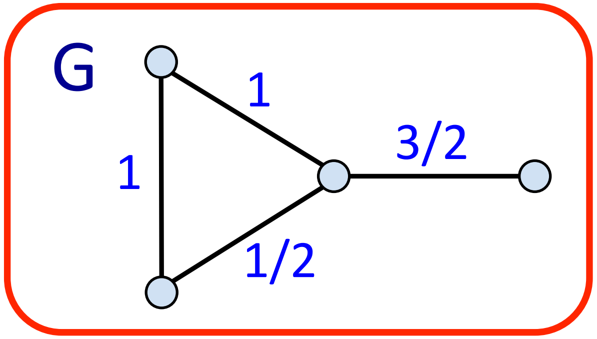
  "G-colored Erdos-Renyi random graphs"

- Different edge density p_e for each $e \in E(G)$ (i.e. each pair of color classes)

- What is a "threshold" family of densities $\{p_e\}_{e \in E(G)}$?

Average-Case for SUB(G)

- Def: $\beta : E(G) \to [0,2]$ is a **threshold weighting** for G if
  1. $\beta(F) := \sum_{e \in E(F)} \beta(e) \leq |V(F)|$ for every $F \subseteq G$
  2. $\beta(G) = |V(G)|$



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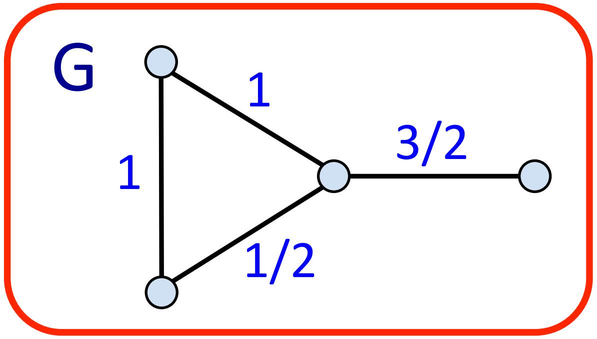
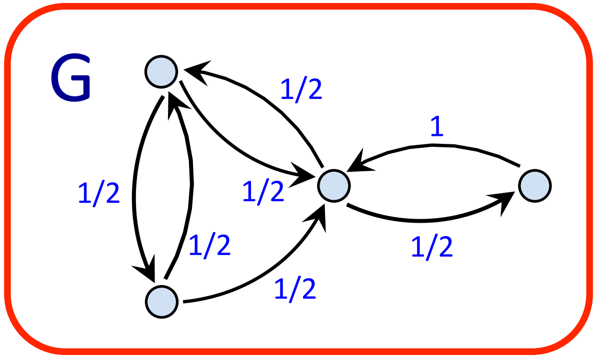


- Obs: Every **Markov chain** on G

  $M : V(G) \times V(G) \to [0,1]$

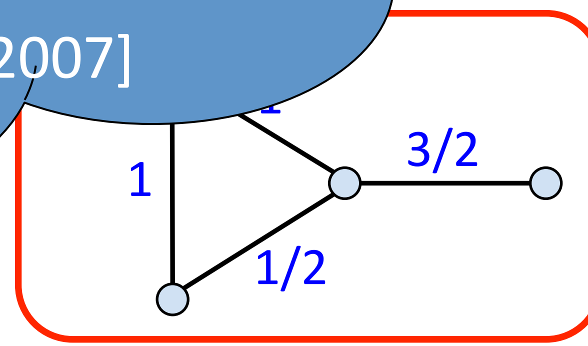
  induces a threshold weighting

  $\beta_M(\{v,w\}) := M(v,w) + M(w,v)$



If G has tree-width k, then there exists a set of S ⊆ V(G) of size Ω(k) and a Markov chain M on G that concurrently routes 1 / k*log k flow between all pairs of vertices in S

[Arora-Rao-Vazirani 2004, Marx 2007]

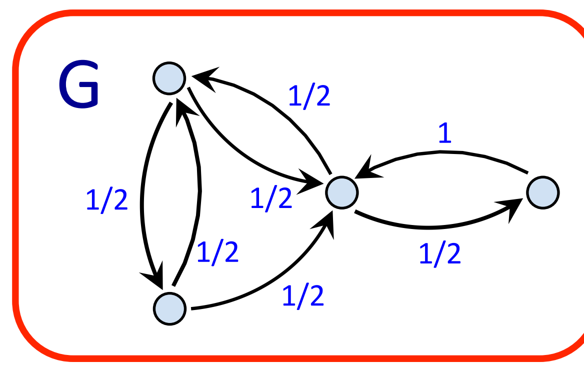


- <u>Obs</u>: Every **Markov chain** on G

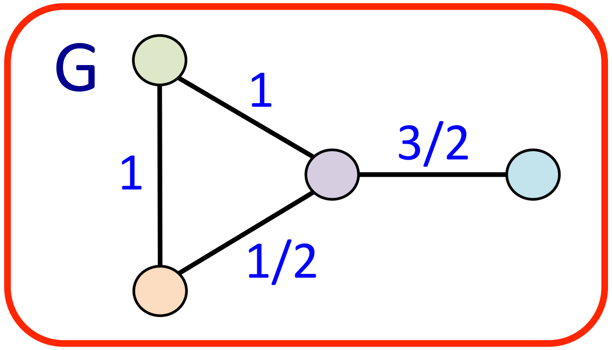
  $M : V(G) \times V(G) \rightarrow [0,1]$

  induces a threshold weighting

  $\beta_M(\{v,w\}) := M(v,w) + M(w,v)$



G-colored random graph $\mathbf{X}_\beta$



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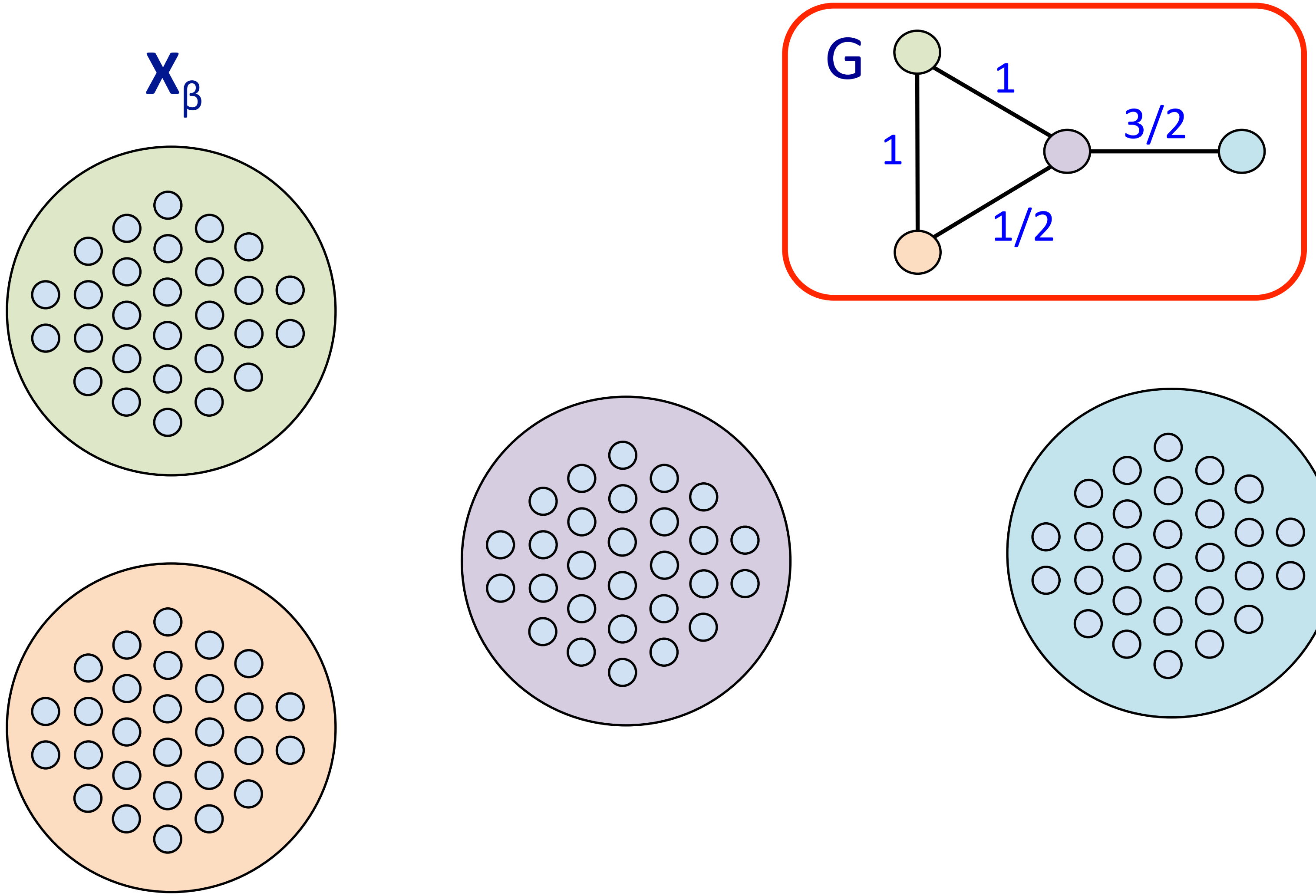

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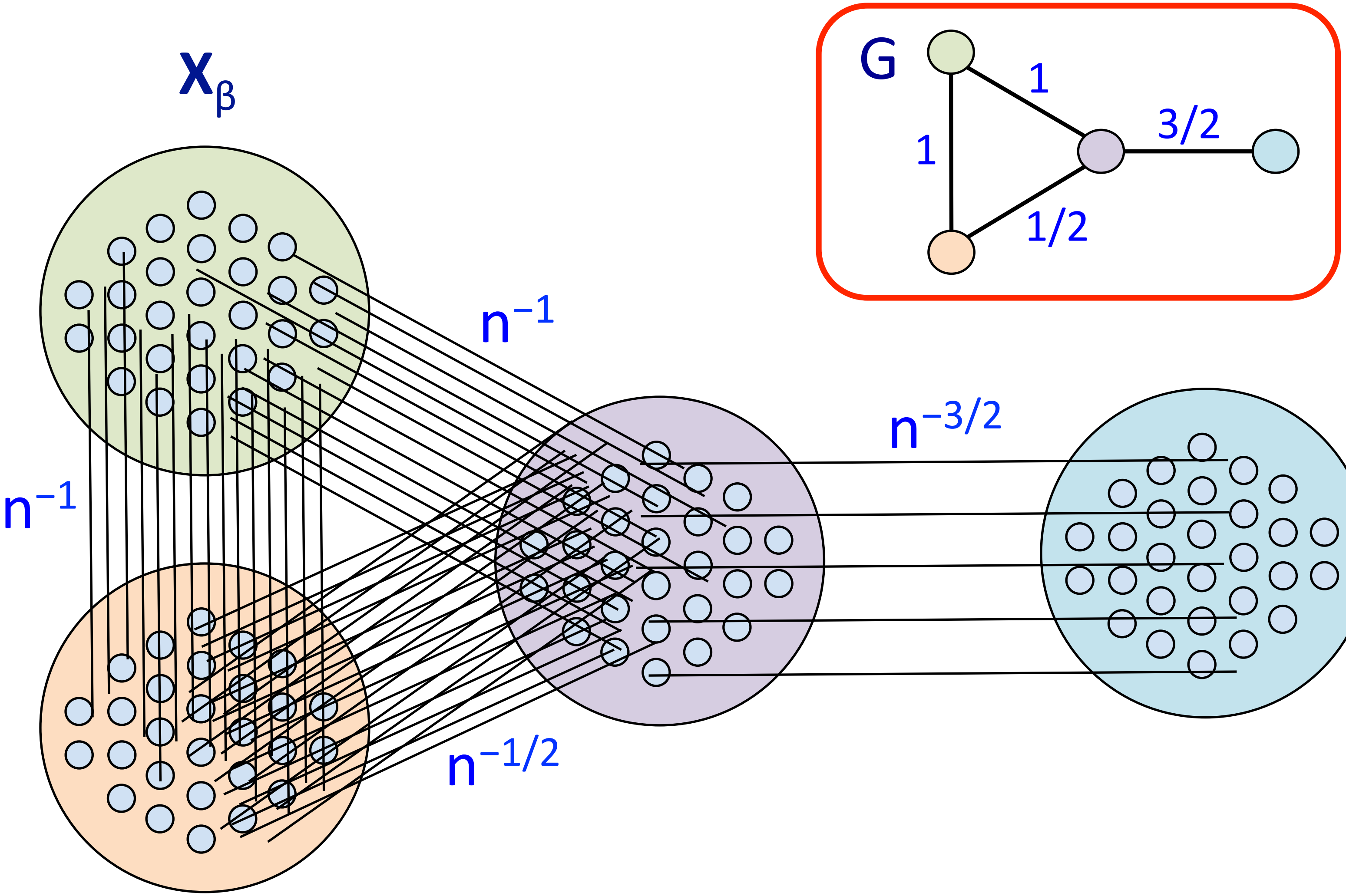

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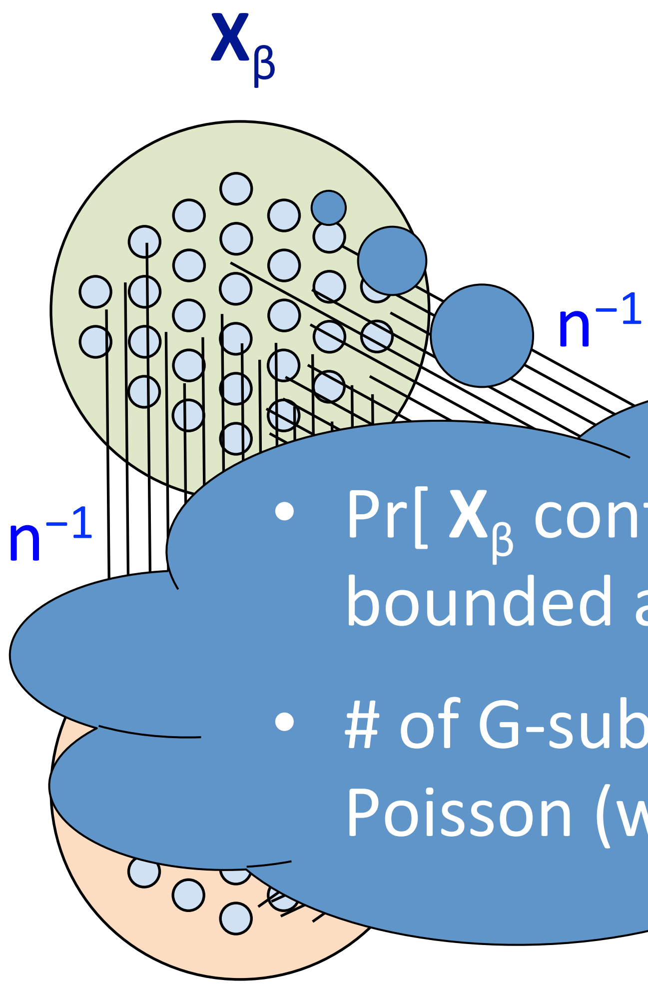
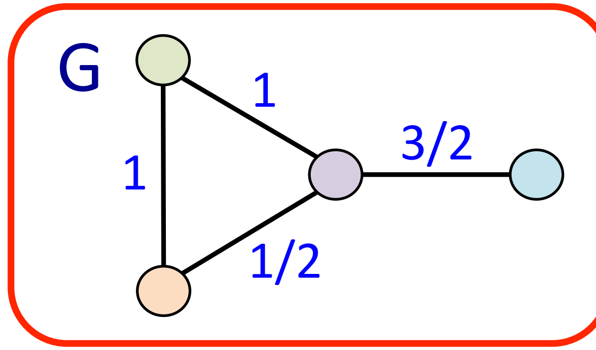
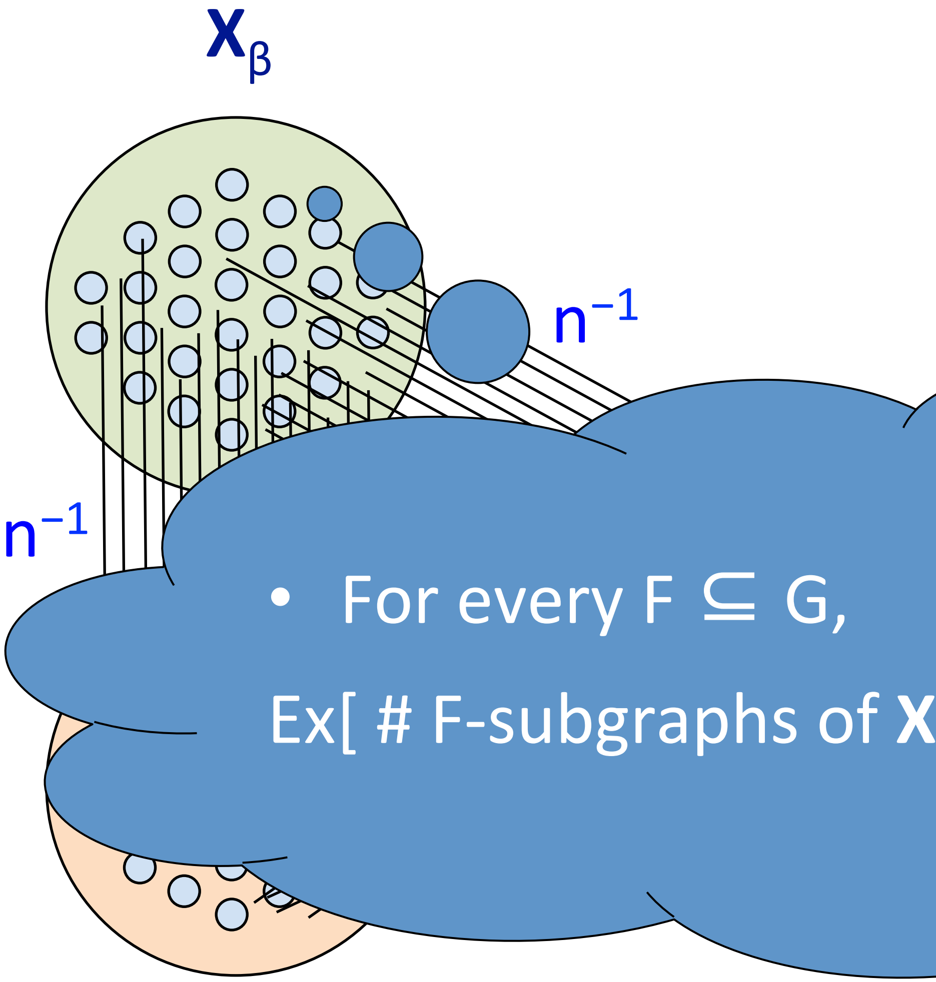
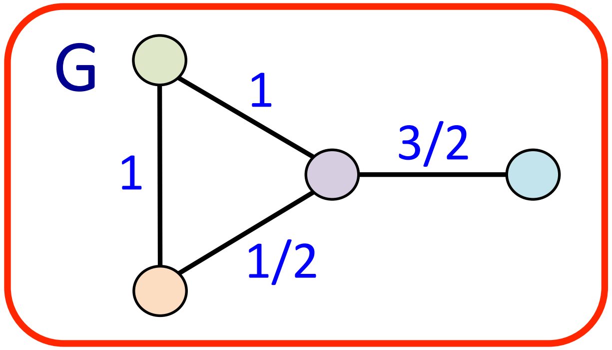




- Pr[$\mathbf{X}_\beta$ contains a G-subgraph] bounded away from 0 and 1
- # of G-subgraphs asymptotically Poisson (when G connected…)

G-colored random graph $\mathbf{X}_\beta$





- For every $F \subseteq G$,

Ex[# F-subgraphs of $\mathbf{X}_\beta$] $\leq n^{|V(F)| - \beta(F)}$

Proof Sketch

- <u>Theorem</u> [Li, Razborov, R. 2014]

  AC^0 circuits for SUB(G) require size $n^{\Omega(\mathbf{tw}(G)/\log \mathbf{tw}(G))}$

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between $n^{c(\beta)}$ and $n^{2c(\beta)}$

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3. For every graph G, there exists β such that $c(\beta) \geq \Omega(\mathbf{tw}(G) / \log \mathbf{tw}(G))$

Proof Sketch

- Theorem [Li, Razborov, R. 2014]

AC^0

This β from the Markov chain of [Arora-Rao-Vazirani 2004], [Marx 2007]

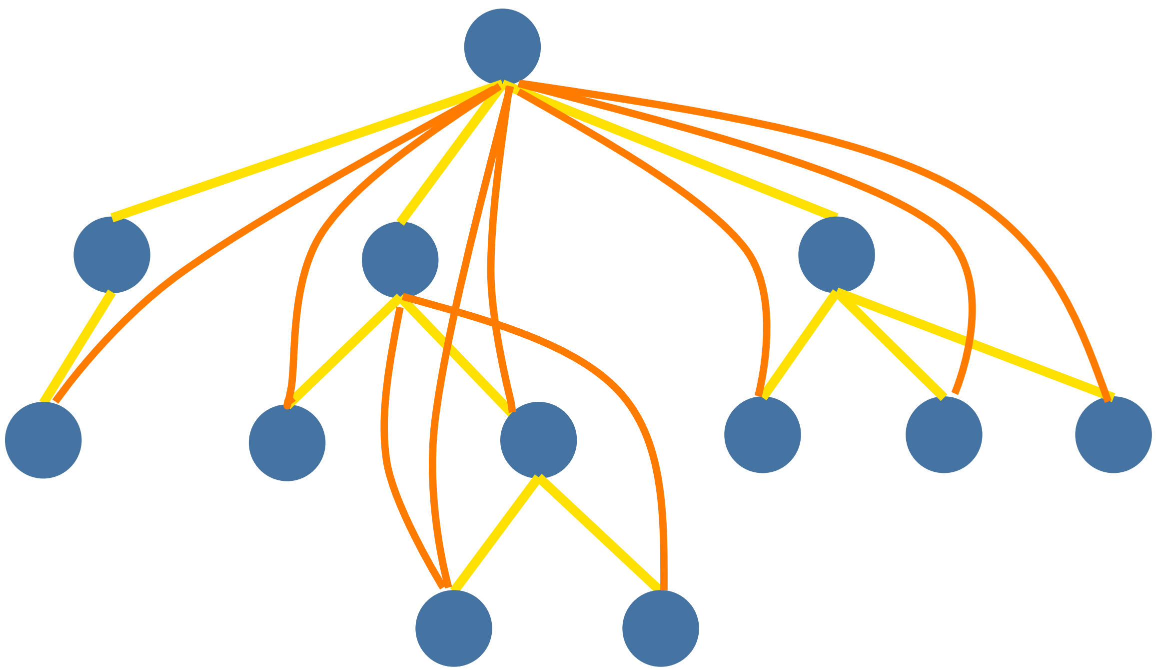
2. The [obscured] complexity of SUB(G) on $\mathbf{X}_\beta$ is $n^{\Theta(c(\beta))}$

3. For every graph G, there exists β such that $c(\beta) \geq \Omega(\mathbf{tw}(G) / \log \mathbf{tw}(G))$

Excluded-Minor Approximation of Tree-Width & Tree-Depth

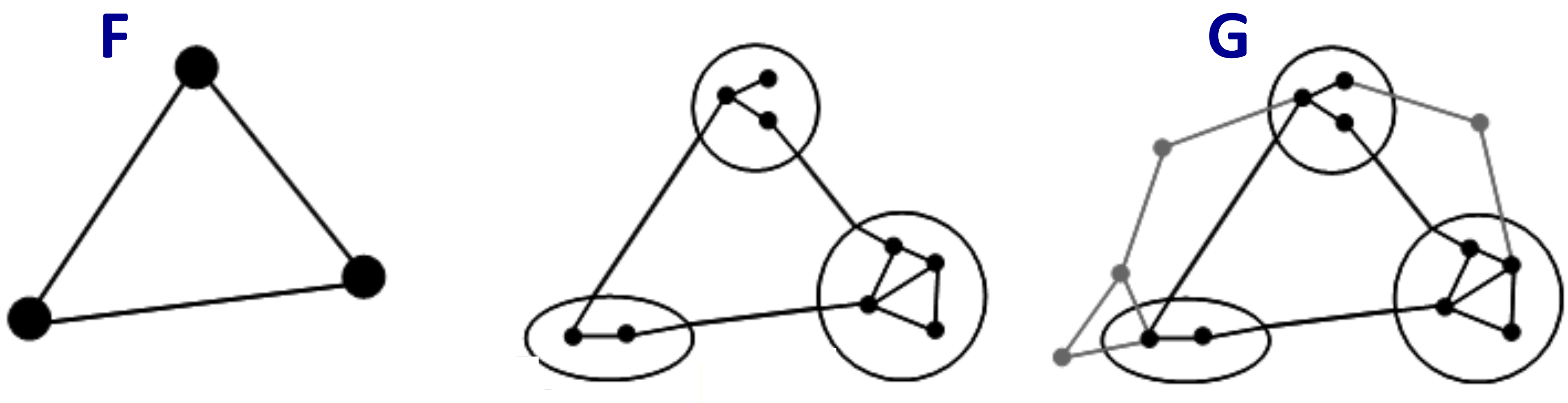
Recall

- <u>Def</u>. The **tree-depth** of a graph G is the minimum height of a tree T such that G ⊆ closure(T)



Recall

- If F is a minor of G, then SUB(F) ≤ SUB(G)
  (there is a linear AC^0 reduction from SUB(F) to SUB(G))



Credit: Wikipedia (NikelsonH)

Minor-Monotonicity

- **tw**(·) and **td**(·) are *minor-monotone*:

  F is a minor of G $\Longrightarrow$ **tw**(F) ≤ **tw**(G) & **td**(F) ≤ **td**(G)

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longest-path(G) (i.e. 1 + largest excluded path minor) gives an *exponential* approximation of **td**(G)

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We seek a *polynomial* approximation of **td**(G)

- Grid Minor Theorem [Chekuri, Chuzhoy 2014]
  Every graph of **tree-width** $\geq k^c$ has a $k \times k$ grid minor.

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That is, grid minors give a *polynomial* approximation of **tw**(G)

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- COROLLARY

  If SUB($Grid_{k \times k}$) has circuit size $n^{\Omega(k)}$ for all k, then SUB(G) has circuit size $n^{\Omega(\mathbf{tw}(G)^\varepsilon)}$ for all graphs G.

- “Grid/Tree/Path Minor Thm” [Kawarabayashi, R. 2016]

  Every graph of **tree-depth** $\geq k^c$ has one of the following minors:

  - $k \times k$ grid
  - complete binary tree of height k
  - path of length 2^k

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These three obstructions give a *polynomial* approximation of **td**(G)

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- COROLLARY
  If $SUB(Grid_{k \times k})$ and $SUB(Tree_k)$ and $SUB(Path_{2^k})$ have AC^0 formula size $n^{\Omega(k)}$ for all k, then SUB(G) has AC^0 formula size $n^{\Omega(\mathbf{td}(G)^\varepsilon)}$ for all graphs G.

- [LRR 2014] $SUB(Grid_{k \times k})$ has AC^0 formula size $n^{\Omega(k)}$
- [R 2014] $SUB(Path_{2^k})$ has AC^0 formula size $n^{\Omega(k)}$
- [R 2016] $SUB(Tree_k)$ has AC^0 formula size $n^{\Omega(k)}$

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- [LRR 2014] SUB($\text{Grid}_{k \times k}$) has AC^0 formula size $n^{\Omega(k)}$
- [R 2014] SUB($\text{Path}_{2^\wedge k}$) has AC^0 formula size $n^{\Omega(k)}$
- [R 2016] SUB(Tree_k) has AC^0 formula size $n^{\Omega(k)}$

The AC^0 formula size of SUB(G) is $n^{\Omega(\mathbf{td}(G)^\wedge \varepsilon)}$

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“Poly-rank” homomorphism preservation theorem

Classical Preservation Theorems

- Los-Tarski / Lyndon / Hom. Preservation Theorem

  A first-order formula $\boldsymbol{\varphi}$ is preserved under

  injective / surjective / all

  homomorphisms if, and only if, it is equivalent to a first-order formula $\boldsymbol{\psi}$ that is

  existential / positive / existential-positive.

Failure on Finite Structures

- Los-Tarski / Lyndon False on Finite Structures

  [Tait 1959], [Ajtai-Gurevich 1997]

  There exists a first-order formula that is preserved under injective (resp. surjective) homomorphisms *on finite structures*, yet is not equivalent *on finite structures* to any existential (resp. positive) formula.

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  There exists a first-order formula that is preserved under injective (resp. surjective) homomorphisms *on finite structures*, yet is not equivalent *on finite structures* to any existential (resp. positive) formula.

- Non-uniform circuit version:

$$\text{Monotone-AC}^0 \neq \text{Monotone} \cap \text{AC}^0$$

Survival on Finite Structures

- Hom. Preservation Theorem on Finite Structures

  [R. 2005]

  If a first-order formula φ of quantifier-rank k is preserved under homomorphisms *on finite structures*, then it is equivalent *on finite structures* to an existential-positive formula ψ of quantifier-rank f(k), where f : N → N is a computable function.

Survival on Finite Structures

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If a first-order formula $\boldsymbol{\varphi}$ of quantifier-rank k is preserved under homomorphisms *on finite structures*, then it is equivalent *on finite structures* to an existential-positive formula $\boldsymbol{\psi}$ of quantifier-rank f(k), where f : N → N is a computable function.

- Proof gives a **non-elementary** upper bound on f(k).

f(k) = k on *Infinite* Structures

- <u>"Equi-rank" Hom. Preservation Theorem</u>

[R. 2005]

If a first-order formula $\boldsymbol{\varphi}$ of quantifier-rank k is preserved under homomorphisms ~~*on finite structures*~~, then it is equivalent ~~*on finite structures*~~ to an existential-positive formula $\boldsymbol{\psi}$ of quantifier-rank **k**.

f(k) ≤ poly(k)

- <u>“Poly-rank” Hom. Pres. Theorem on Finite Structures</u>

  [R. 2016]

  If a first-order formula $\boldsymbol{\varphi}$ of quantifier-rank k is preserved under homomorphisms *on finite structures*, then it is equivalent *on finite structures* to an existential-positive formula $\boldsymbol{\psi}$ of quantifier-rank f(k), where **f(k) ≤ poly(k)**.

f(k) ≤ poly(k)

Proof gives reduction to $n^{\Omega(\mathbf{td}(G)^\wedge\varepsilon)}$ AC^0 formula size lower bound for SUB(G)

- “Po… …res … …] … …ctures, then it … …res to an existential-po…ve formula ψ of quantifier-rank f(k), where **f(k) ≤ poly(k)**.

f(k) ≤ poly(k)

f(k) ≤ $2^{O(k)}$

follows from lower bound for k-STCONN of [R. 2014]

- “Po… …es

  …]

  …*ctures*,

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f(k) ≤ poly(k)

- <u>“Po</u> … <u>res</u>

  … 5]

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f(k) ≤ non-elementary(k)

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k-STCONN of [Ajtai 1989]

$f(k) \le poly(k)$

- "Poly-rank" Hom. Pres. Theorem on Finite Structures

[R. 2016]

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- Non-uniform circuit version:

$\text{HomPres} \cap \text{AC}^0 = \exists^{+}\text{FO} \subseteq$ {poly-size monotone DNFs}

Summary (Last Slide!)

- Complexity of SUB(G) is tied to natural structural parameters of G and to fundamental questions in complexity (P vs. NP, L vs. NL, NC^1 vs. P)

- Connection between AC^0 & FO & tw(G)/td(G):

AC^0 circuit size ≈ FO variable width ≈ tree-width(G)

AC^0 formula size ≈ FO quantifier rank ≈ tree-depth(G)

- Natural family of input distributions $\mathbf{X}_\beta$: hard-on-average for optimal choice of β

Thank you!