Towards Capturing Order-Independent P

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Descriptive Complexity

Query $q_1\ q_2\ \cdots\ q_n$ $\mapsto$ **Computation** $\mapsto$ **Answer** $a_1\ a_2\ \cdots\ a_i\ \cdots\ a_m$

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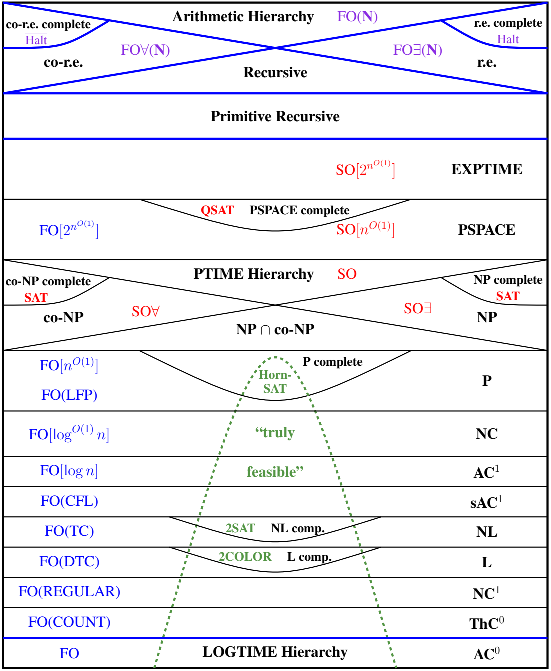
There is a **computable isomorphism** between these two approaches.



Arithmetic Hierarchy
FO(N)
co-r.e. complete
Halt
r.e. complete
Halt
FO∀(N)
FO∃(N)
co-r.e.
r.e.
Recursive
Primitive Recursive
SO[$2^{n^{O(1)}}$]
EXPTIME
QSAT
PSPACE complete
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PSPACE
PTIME Hierarchy
SO
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NP
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FO[$n^{O(1)}$]
P complete
Horn-SAT
P
FO(LFP)
FO[$\log^{O(1)} n$]
"truly
NC
FO[$\log n$]
feasible"
AC^1
FO(CFL)
sAC^1
FO(TC)
2SAT
NL comp.
NL
FO(DTC)
2COLOR
L comp.
L
FO(REGULAR)
NC^1
FO(COUNT)
ThC^0
FO
LOGTIME Hierarchy
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Inductive Definitions and Least Fixed Point

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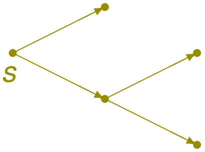

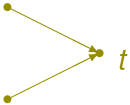

Inductive Definitions and Least Fixed Point

$$E^\star(x,y) \stackrel{\text{def}}{=} x = y \;\vee\; E(x,y) \;\vee\; \exists z(E^\star(x,z) \wedge E^\star(z,y))$$

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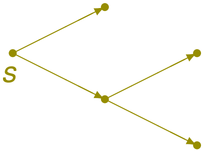

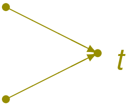

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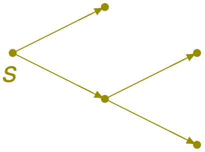

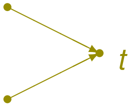

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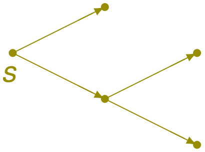

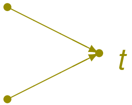

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LFP is a Polynomial Iteration Operator

Thm. $\mathrm{P} = \mathrm{FO(LFP)} = \mathrm{FO}[n^{O(1)}]$

$\mathrm{FO}[n^{O(1)}]$ means for graphs with n vertices, the formula φ_n expressing the property has $n^{O(1)}$ quantifiers, but only a **fixed number** of requantified **variables**, $x_1, \ldots, x_k$, i.e, $\varphi_n \in \mathcal{L}^k$.

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Wanted: a language capturing Order-Independent P (**OIP**).

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$\mathrm{FO}_{(\mathrm{wo}\leq)}\mathrm{(LFP)} \subseteq$ **OIP**

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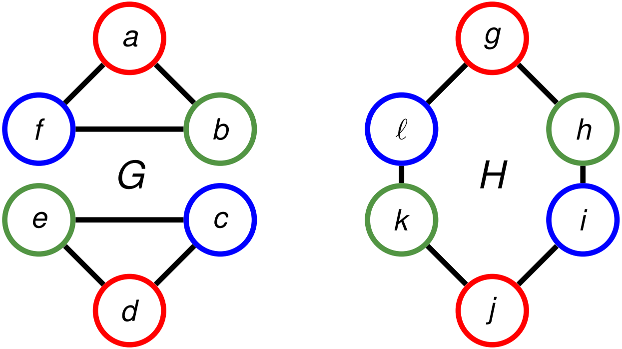
How do we prove EVEN $\notin \mathrm{FO(wo}{\leq}\mathrm{)(LFP)}$?

Ehrenfeucht-Fraïssé Game

$\mathcal{G}_m^k(G, H)$ m moves, k pebbles, 2 players



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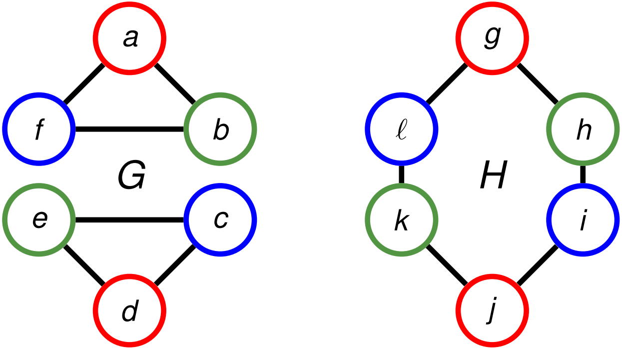

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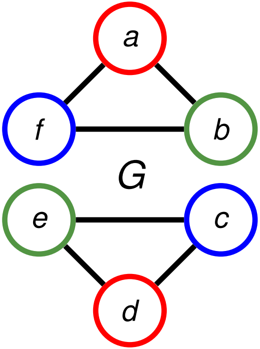

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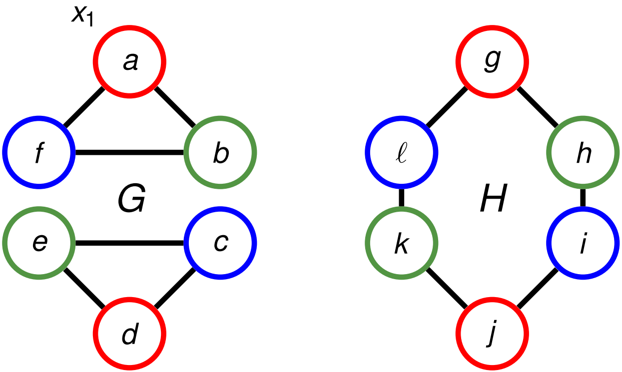

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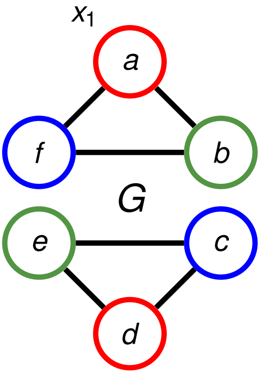

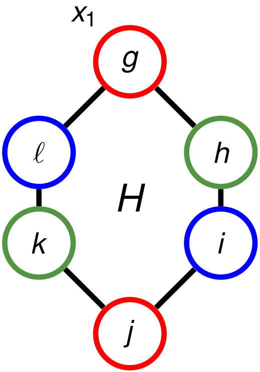

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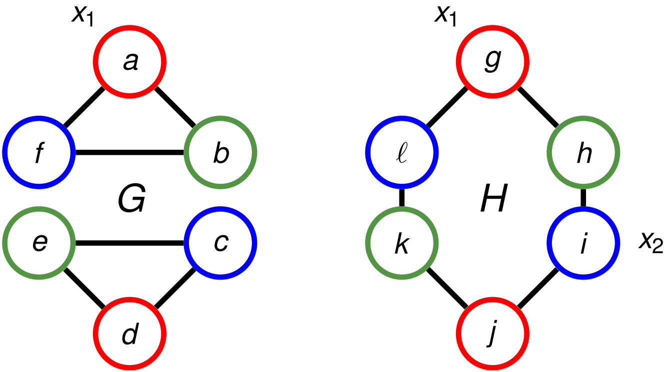

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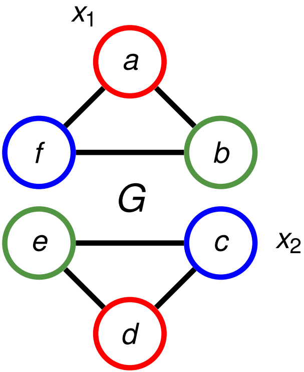

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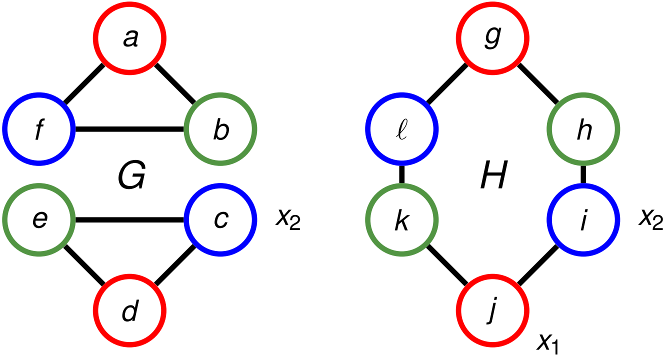

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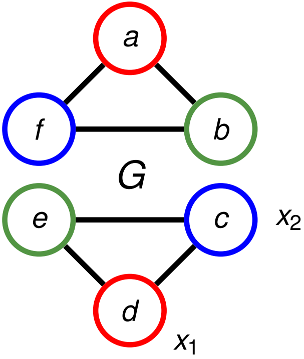

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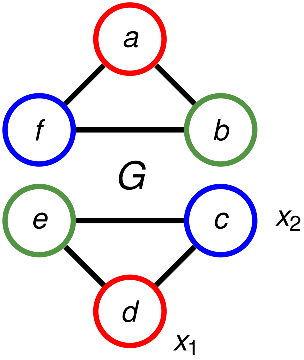

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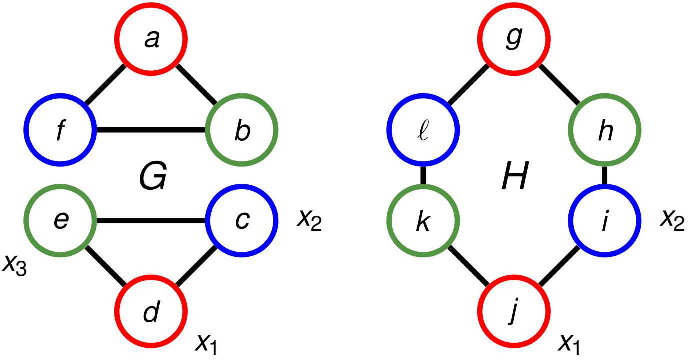

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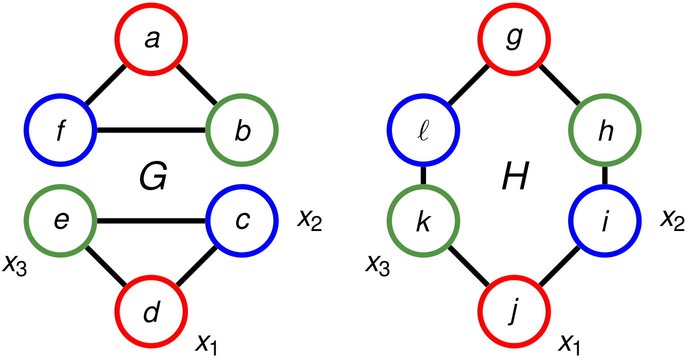

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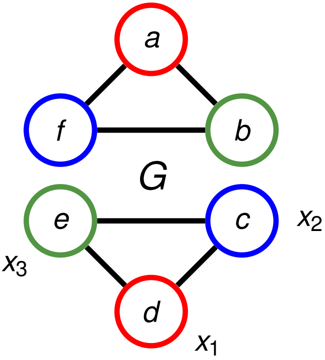

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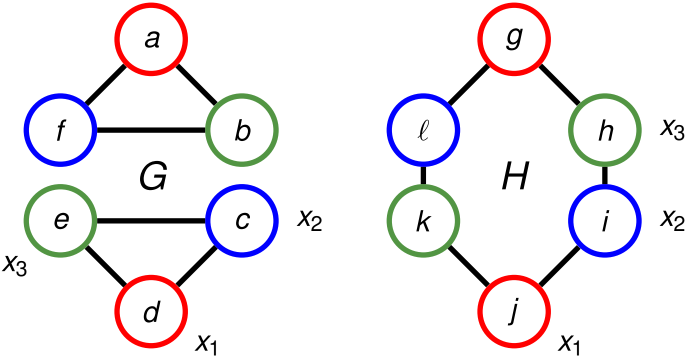
For all m, **D** wins $\mathcal{G}_m^2(G, H)$; but **S** wins $\mathcal{G}_3^3(G, H)$.



Fundamental Thm of Ehrenfeucht-Fraïssé Games

Notation: $G \sim_m^k H$ means that **Delilah** has a winning strategy for $\mathcal{G}_m^k(G, H)$.

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Thm. **D** has a winning strategy on the m-move, k-pebble game on G, H iff G and H agree on all formulas using k variables and quantifier depth m.

$$G \sim^k_m H \quad \Leftrightarrow \quad G \equiv^k_m H$$

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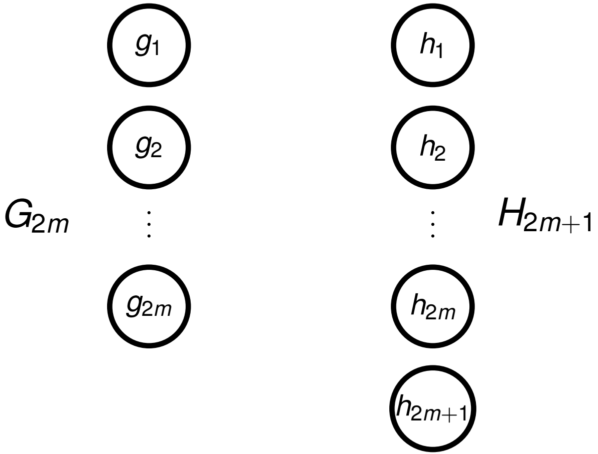

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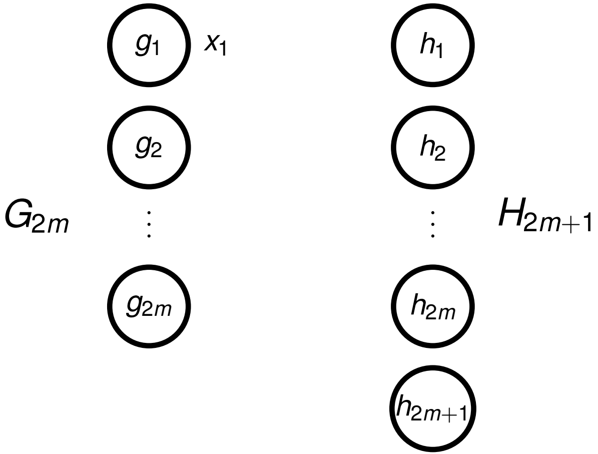

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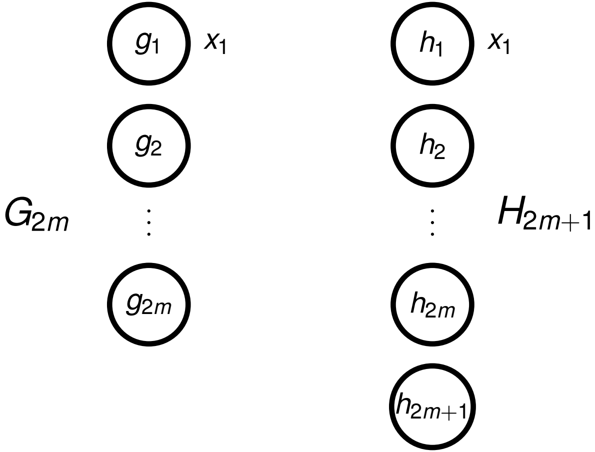

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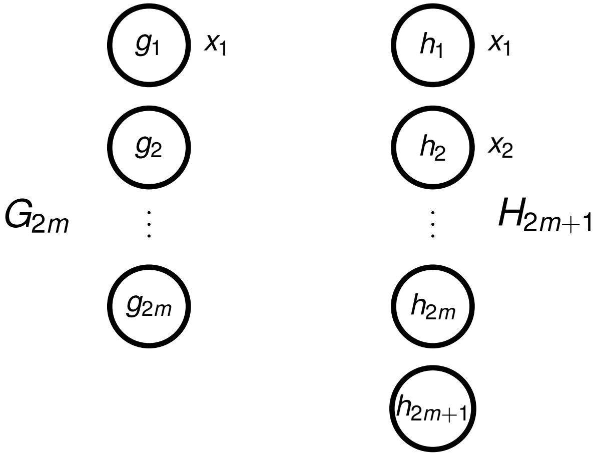

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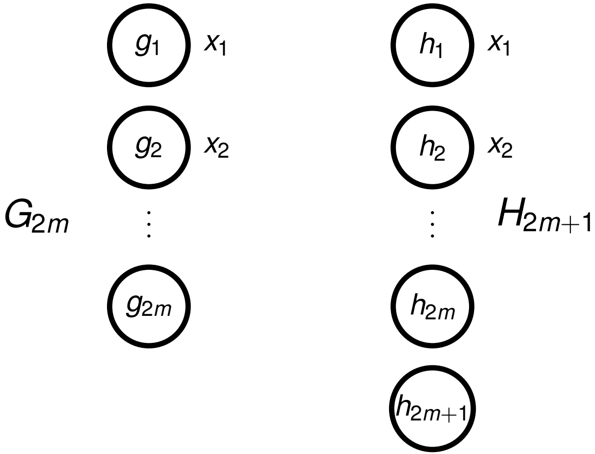

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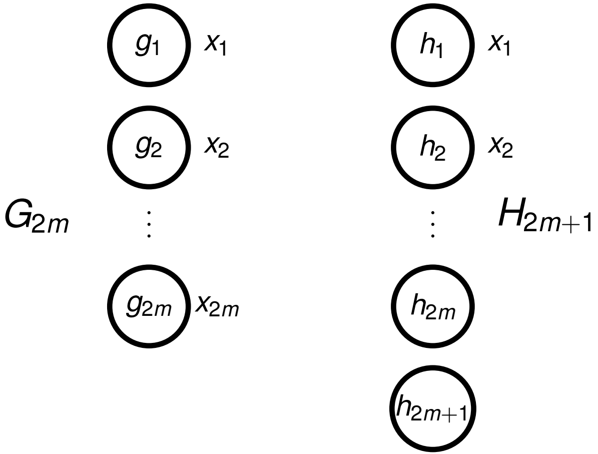

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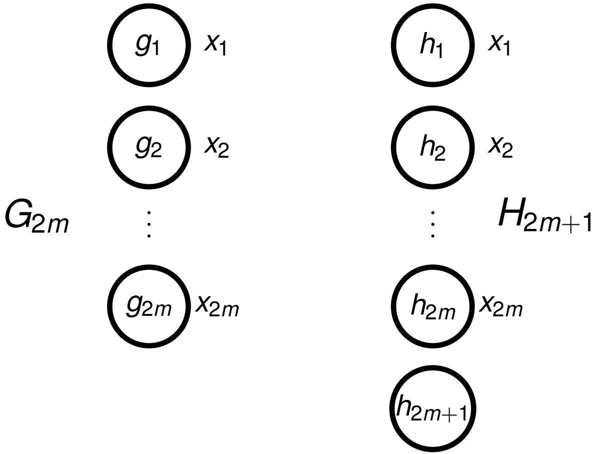

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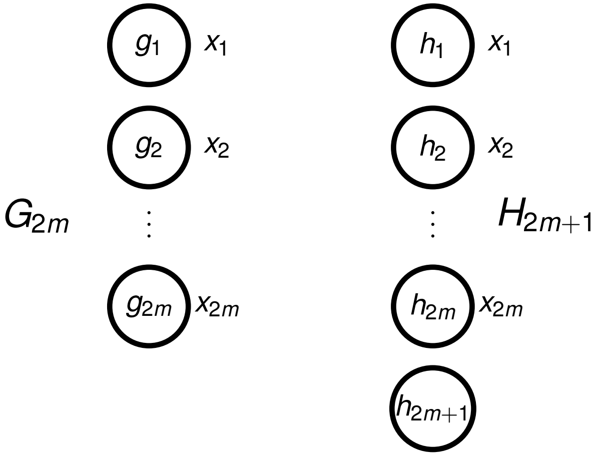

$$G_{2m} \sim^{2m} H_{2m+1}$$

□

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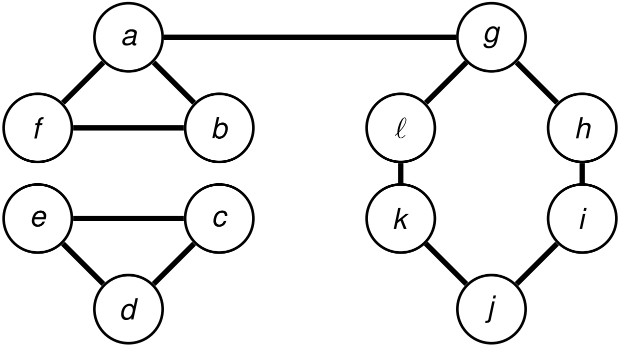

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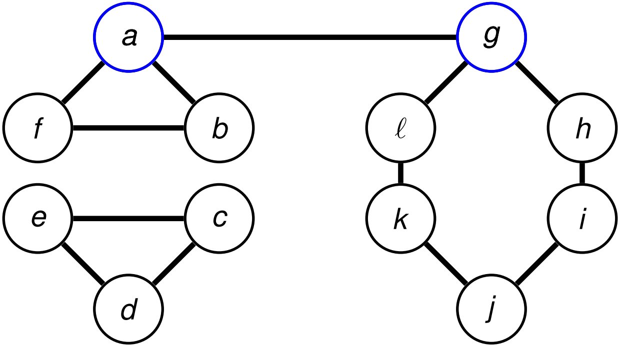

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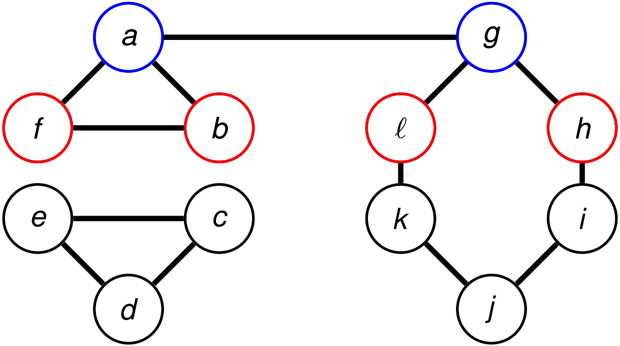

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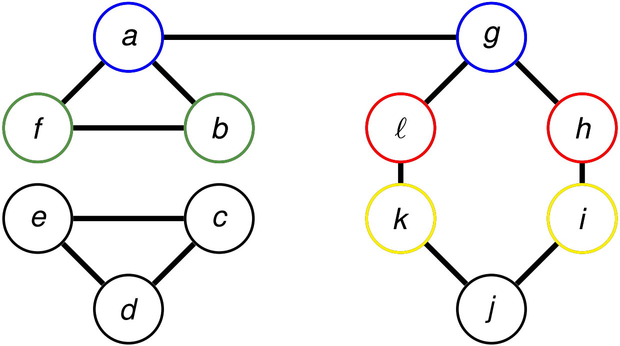

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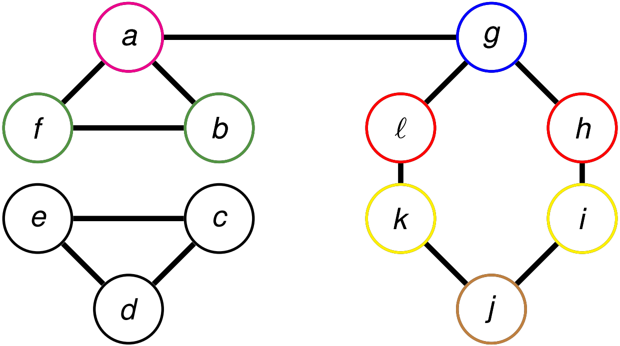

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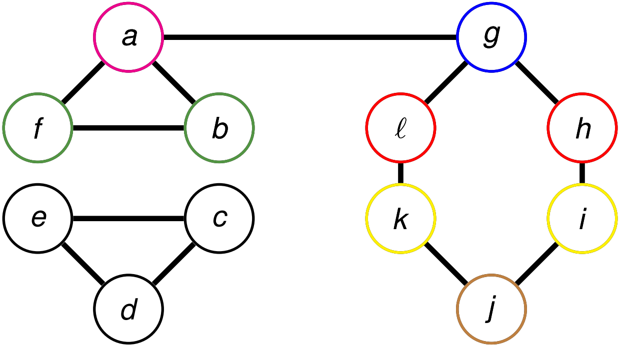

Thm. Stable Coloring of Vertices $=$ C^2 type.

Round m of stable coloring is quantifier depth of C^2 formula.

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Thm. [Babai, 2015] $\quad \mathrm{GI} \in \mathrm{DTIME}[n^{\log^7 n}]$. (Before this it was only known that $\mathrm{GI} \in \mathrm{DTIME}[n^{\sqrt{n}}]$.)

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proof: Apply arbitrary $\mathrm{FO(LFP)}$ formula to the canonical form of the input graph. □

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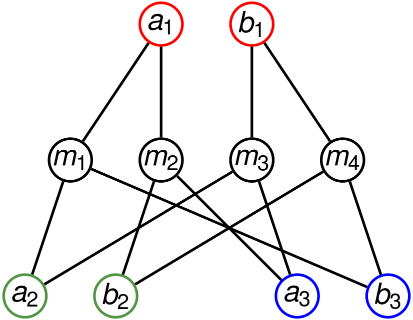
Thm. [CFI] No!

A simple graph property (now called the CFI property) checkable in $\mathrm{DTIME}[n]$, requires $v = \Omega(n)$ variables to express in C^v. Thus, $\mathrm{CFI} \in \mathbf{OIP} - \mathrm{FPC}$

Proof of CFI Thm



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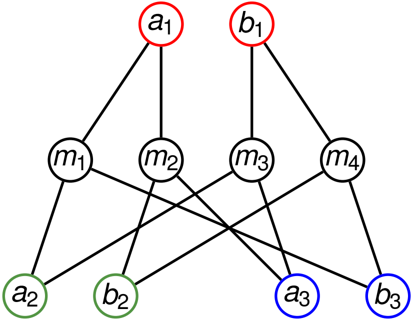
CFI Gadget X: Each m_i adjacent to an even number of a_j's.



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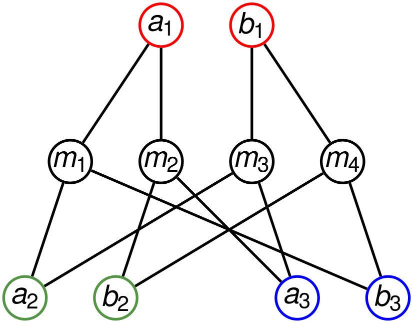

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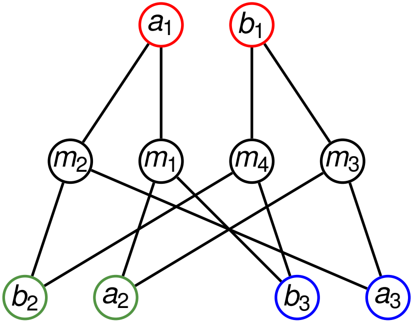

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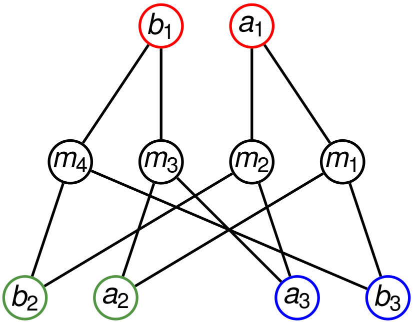

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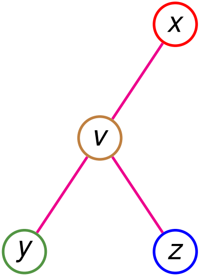
- Thus $X(G_n)$ has color class size 4.



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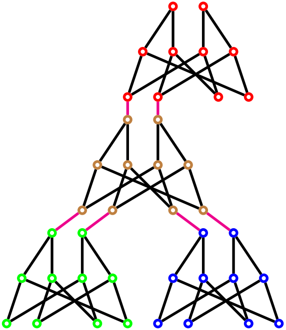

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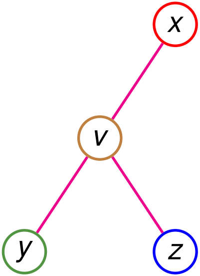

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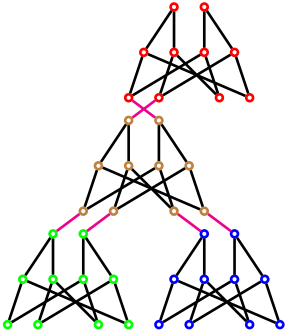

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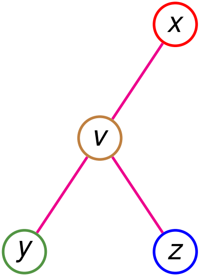

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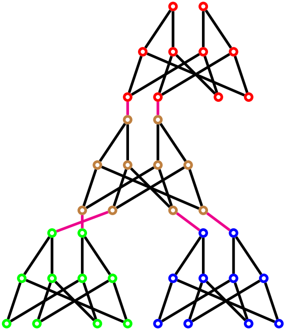

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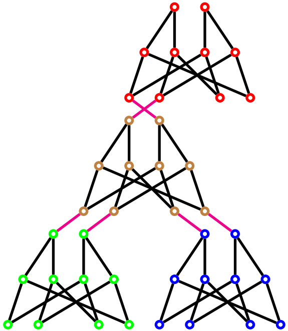

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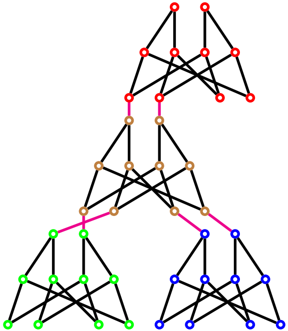

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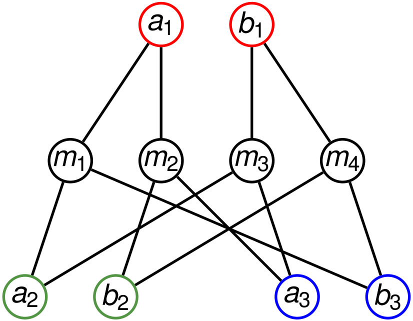

X

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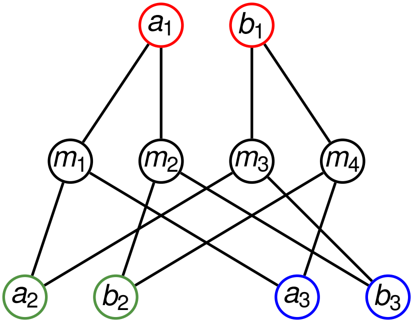

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proof Use the ordering to label boundary pairs a_i, b_i when $a_i \leq b_i$. Then count the number, m, of flips of vertices and edges mod 2. $X'(G) \in$ CFI iff m is even. □

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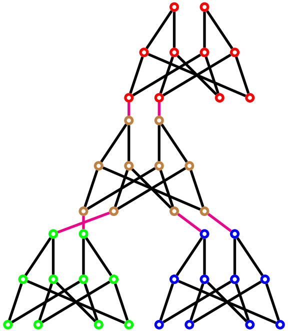

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Thus **Delilah** never loses. □

Recap

We have shown that the linear-time CFI problem is in **OIP** $-$ FPC.

Cor. $\Omega(n)$ variables are needed to characterize graphs.

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Thm. [Anderson, Dawar and Holm] Linear Programming is in FPC.

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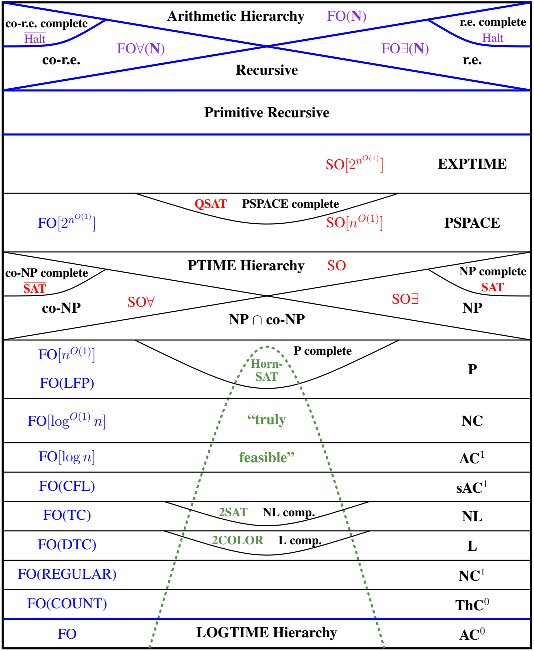
What I want: more natural extension to FPC that adds group theory and characterizes graphs using $O(\log n)$ variables.


Arithmetic Hierarchy
FO(**N**)
co-r.e. complete
$\overline{\text{Halt}}$
r.e. complete
Halt
FO∀(**N**)
FO∃(**N**)
co-r.e.
r.e.
Recursive
Primitive Recursive
SO[$2^{n^{O(1)}}$]
EXPTIME
QSAT
PSPACE complete
FO[$2^{n^{O(1)}}$]
SO[$n^{O(1)}$]
PSPACE
PTIME Hierarchy
SO
co-NP complete
$\overline{\text{SAT}}$
NP complete
SAT
SO∀
SO∃
co-NP
NP
NP ∩ co-NP
FO[$n^{O(1)}$]
FO(LFP)
P complete
Horn-
SAT
P
FO[$\log^{O(1)} n$]
"truly
NC
FO[$\log n$]
feasible"
AC^1
FO(CFL)
sAC^1
FO(TC)
2SAT
NL comp.
NL
FO(DTC)
2COLOR
L comp.
L
FO(REGULAR)
NC^1
FO(COUNT)
ThC^0
FO
LOGTIME Hierarchy
AC^0