The Analysis of Partially Symmetric Functions

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Based on joint work with

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Classes of “simple” functions

Classes of “simple” functions

Constant

Classes of “simple” functions

Constant

↓

Juntas

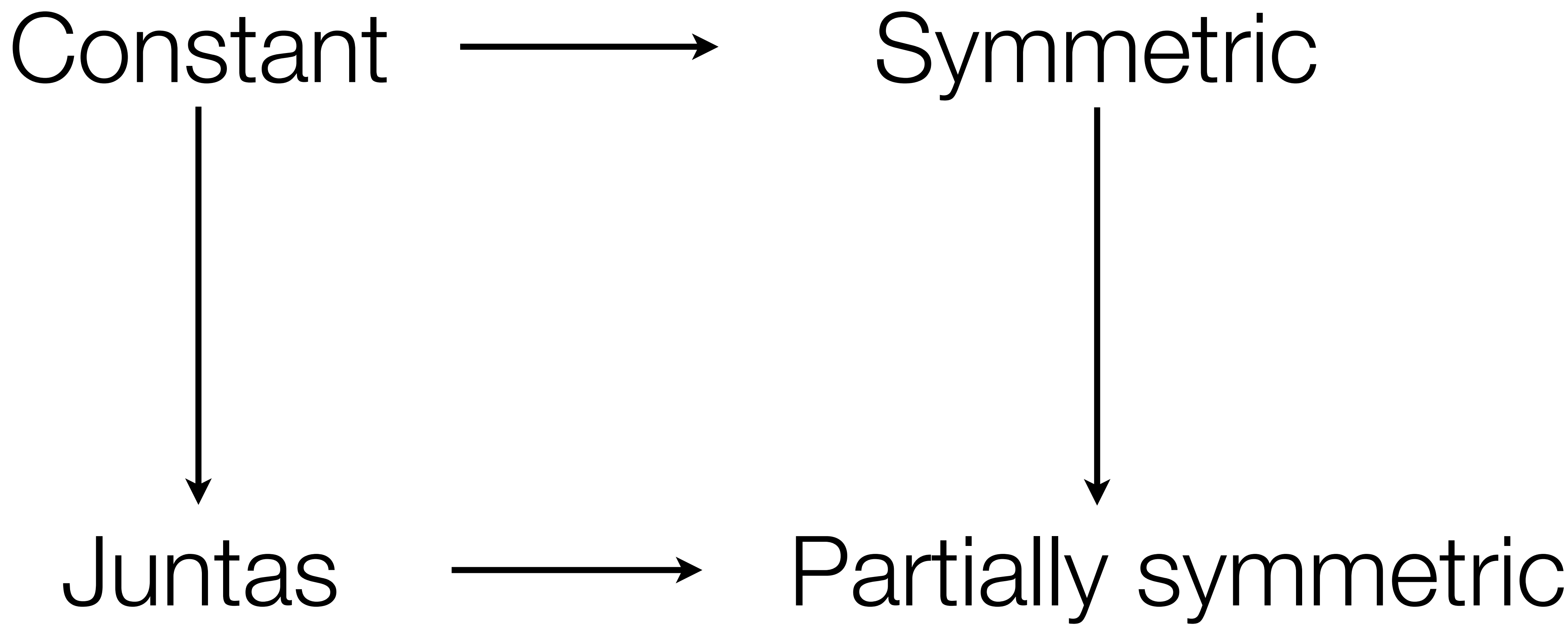
Classes of “simple” functions

Constant → Symmetric

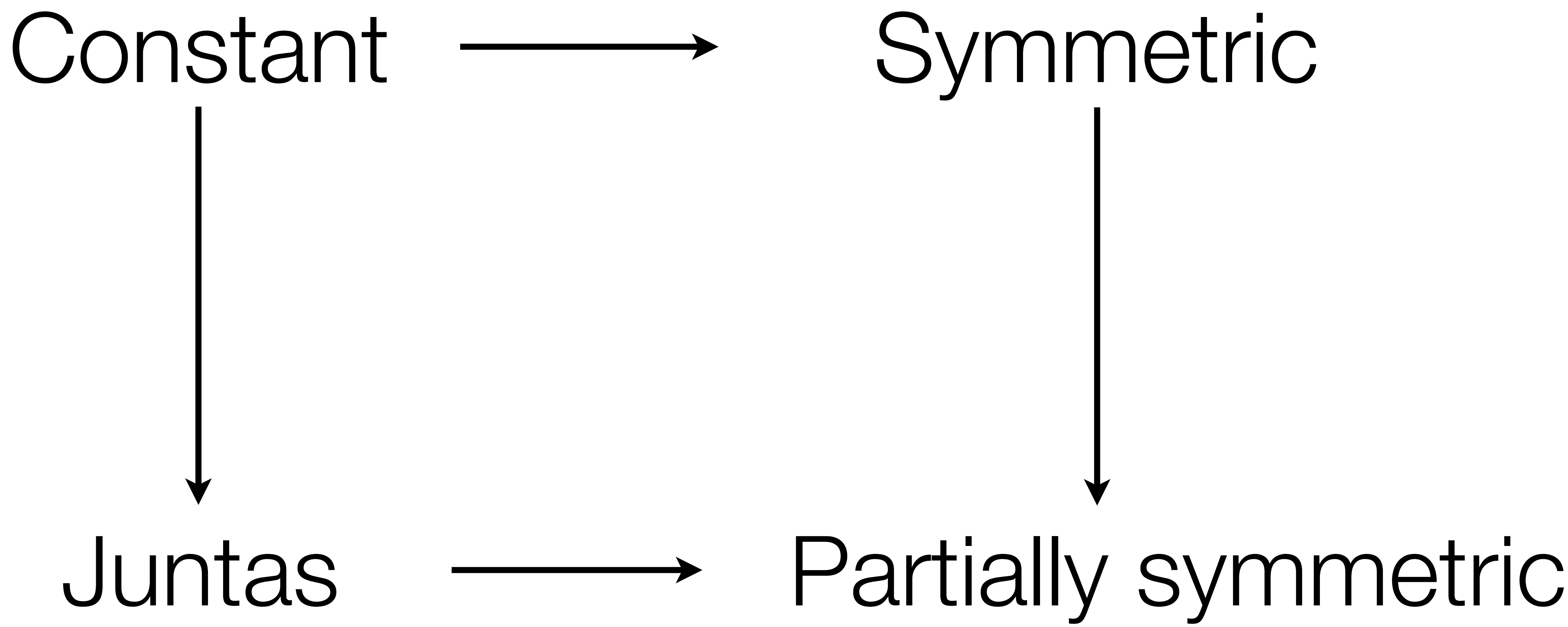
↓

Juntas

Classes of “simple” functions



Classes of "simple" functions



Def'n. $f:\{0,1\}^n \to \{0,1\}$ is $(n\text{-}k)$-symmetric if there is a set $J \subseteq [n]$ of k variables such that $f(x) = f(y)$ whenever $x_J = y_J$ and $|x|=|y|$.

An algebraic definition

- **Def'n.** $f^{\pi}(x) = f(\pi x) = f(\, x_{\pi(1)},\ldots,x_{\pi(n)}\,)$.
- **Def'n.** f is poly-symmetric if

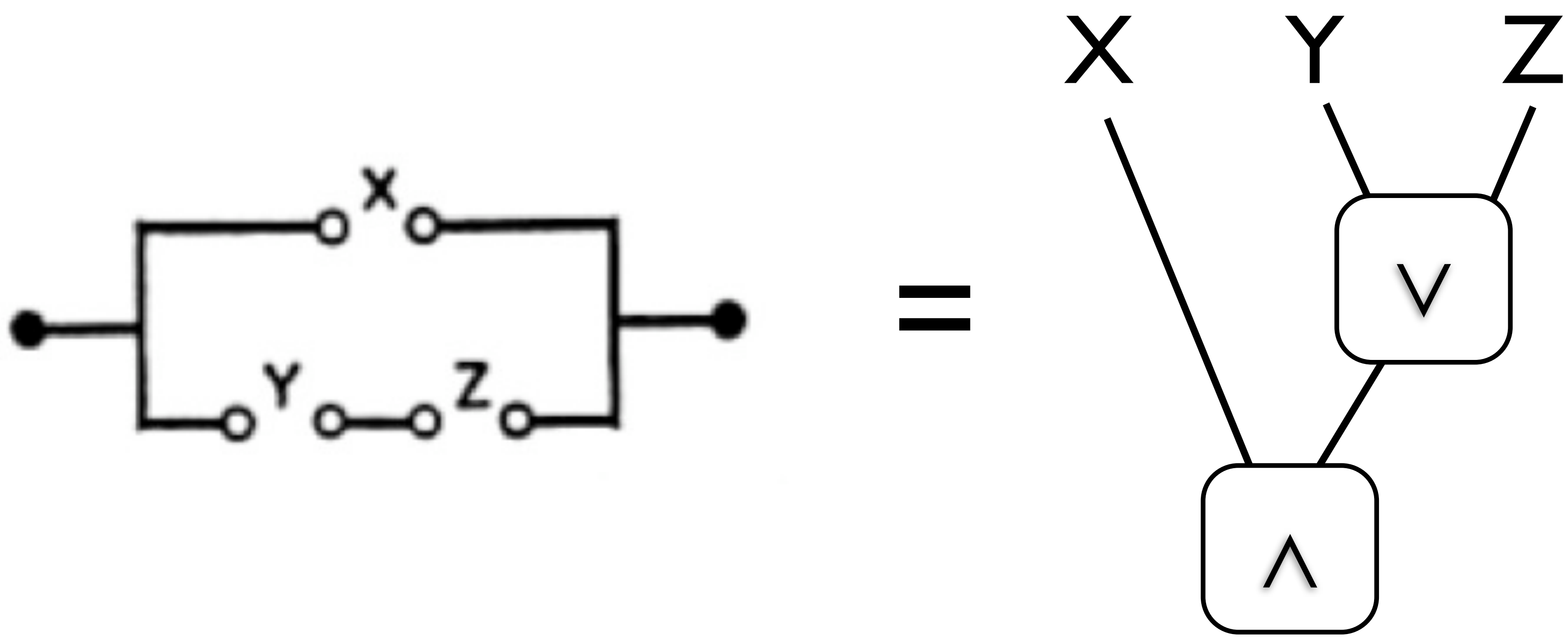
$$|\mathbf{ISO}_f| = |\{f^{\pi} : \pi \in S_n\}| \leq \text{poly}(n).$$

- **Theorem.** f is poly-symmetric if and only if it is $(n\text{-}k)$-symmetric for some k=O(1).

[Clote, Kranakis '91]
[Chakraborty, Fischer, Garcia-Soriano, Matslieh '12]

Partial Symmetry in Theoretical Computer Science

Circuit complexity



Theorem (Shannon ’49)**.** Almost every function *f* has circuit complexity $\Omega(2^n/n)$.

Circuit complexity

- **Theorem.** Every symmetric function has circuit complexity at most n^2. [Shannon '38]

- **Theorem.** Every k-junta has circuit complexity at most $2^{k+3}/k$. [Shannon '49]

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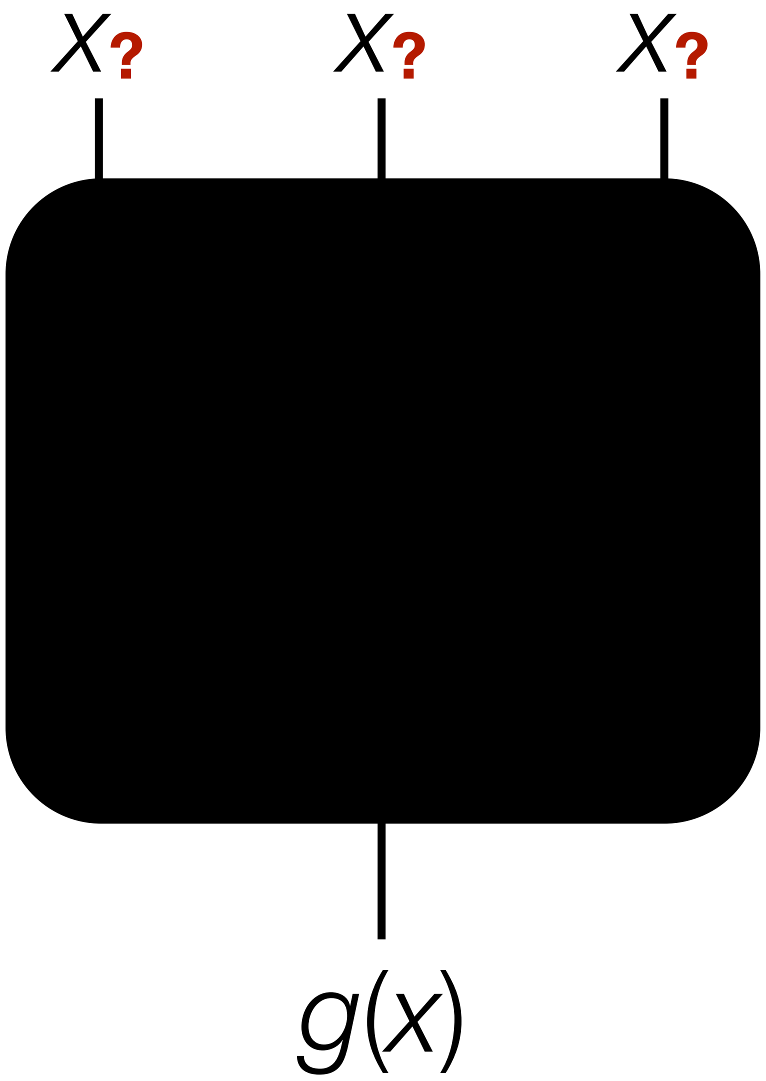
- **Theorem.** Every k-junta has circuit complexity at most $2^{k+3}/k$. [Shannon '49]
- **Theorem.** Every $(n\text{-}k)$-symmetric function has circuit complexity $\leq (n-k)2^k + (n-k)^2$. [Shannon '49]

Parallel complexity and Proof complexity

- **Theorem.** If *f* is (*n*-*k*)-symmetric for some *k*=O(1), then *f* is in $TC^0 \subseteq NC^1$.
  [Clote, Kranakis '91]

- **Corollary.** "Frege probably does not effectively-*p* simulate Extended Frege."
  [Pitassi, Santhanam '10]

Testing function isomorphism



?
=

x	f(x)
0 0 0	1
0 0 1	0
0 1 0	0
0 1 1	1
1 0 0	1
1 0 1	1
1 1 0	0
1 1 1	0

Testing function isomorphism

Def'n. A *q*-query tester for the property $\mathbf{ISO}_f = \{ f^\pi : \pi \in S_n \}$ queries $g : \{0,1\}^n \to \{0,1\}$ on at most q inputs and

(i) Accepts w.p. $^2/_3$ when $g \in \mathbf{ISO}_f$,

(ii) Rejects w.p. $^2/_3$ when for every $\pi \in S_n$, $\Pr[\, g(x) \neq f^\pi(x) \,] \geq {}^1/_{100}$.

Main Question. For which functions f can we test $\mathbf{ISO}_f$ with O(1) queries?

Testing function isomorphism

- **Fact.** For every symmetric function f, we can test $\mathbf{ISO}_f$ with O(1) queries.
- **Theorem.** For every k-junta f, we can test $\mathbf{ISO}_f$ with $O(k \log k)$ queries.

[Fischer, Kindler, Ron, Safra, Samorodnitsky ’04]
[B. ’09]
[Chakraborty, Garcia-Soriano, Matslieh ’10]

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- **Theorem.** For every $(n\text{-}k)$-symmetric *f*, we can test $\mathbf{ISO}_f$ with $O(k \log k)$ queries.

[B., Weinstein, Yoshida ’12]
[Chakraborty, Fischer, Garcia-Soriano, Matslieh ’12]

Testing function isomorphism

Conjecture. Fix any $k \geq 1$. If f is ε-far from $(n-k)$-symmetric, then testing $\mathbf{ISO}_f$ requires $\Omega(\log \log k)$ queries.

[B., Weinstein, Yoshida '12]
[Chakraborty, Fischer, Garcia-Soriano, Matslieh '12]

Influence and partial symmetry

Three Notions of Influence

Influence of coordinate i:

- $\mathrm{Inf}_i(f) = \Pr_x[\ f(x) \neq f(x^{\oplus i})\ ]$.

Total influence / average sensitivity:

- $\mathrm{Inf}(f) = \sum_i \mathrm{Inf}_i(f)$.

Influence of a set $S \subseteq [n]$ of coordinates:

- $\mathrm{Inf}_S(f) = \Pr_{x,y}[\ f(x) \neq f(x_{[n]\setminus S}\, y_S)\ ]$.

Three Notions of Influence

Influence of coordinates *i,j*:

- $\mathrm{Inf}^*_{i,j}(f) = \Pr_x[\ f(x) \neq f(\ (x^{(i\leftrightarrow j)})\ )\ ].$

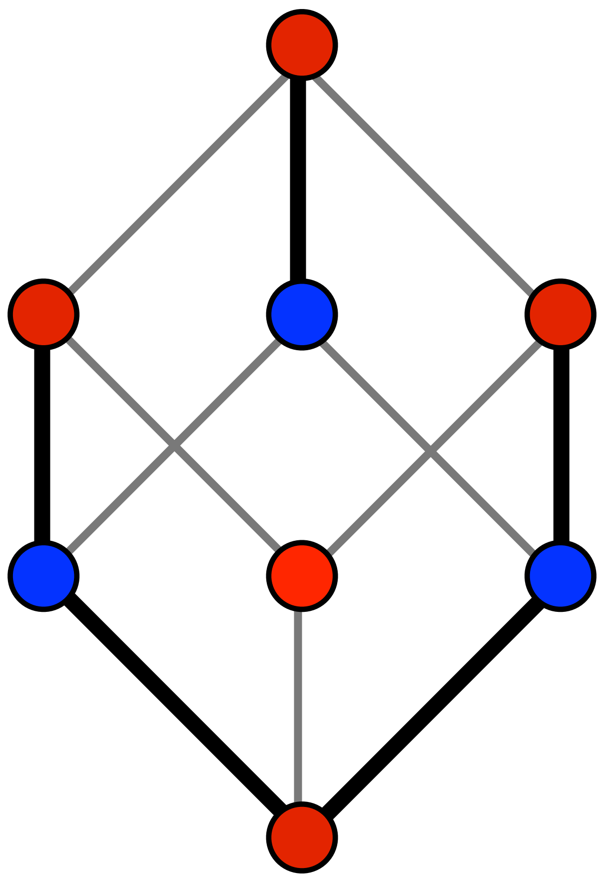
Total influence:

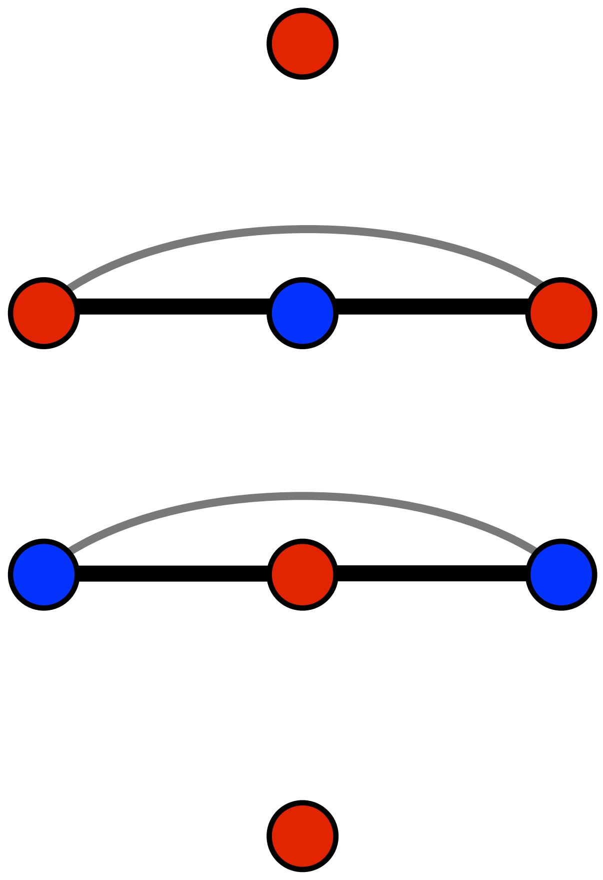
- $\mathrm{Inf}^*(f) = \sum_{i\neq j} \mathrm{Inf}^*_{i,j}(f).$

Influence of a set *S* of coordinates:

- $\mathrm{Inf}^*_S(f) = \Pr_{x,\pi\in\mathcal{S}_S}[\ f(x) \neq f(\pi x)\ ].$

Inf vs. Inf*





Properties of $\text{Inf}^*_{i,j}$ and Inf^*

- **Fact.** When f is symmetric, $\text{Inf}^*(f) = 0$.
- **Fact.** $\text{Inf}^*_{i,j}(f) = \sum_{T: i,j \notin T} \left(\hat{f}(T \cup \{i\}) - \hat{f}(T \cup \{j\}) \right)^2$.
- **Theorem** (KKL for Inf*)**.** When f is far from symmetric, there exist $i \neq j$ such that $\text{Inf}^*_{i,j}(f) = \Omega(\log(n)/n)$. [O'Donnell, Wimmer '08]

Properties of Inf^*_S

- **Fact.** When f is $(n\text{-}k)$-symmetric, there is a set J of size $|J|=k$ s.t. $\mathrm{Inf}^*_{[n]\setminus J}(f) = 0$.
- **Fact.** $\mathrm{Inf}^*_S(f) = \Sigma_T\, \mathrm{Var}_{\pi\in\mathcal{S}_S}(\, \hat{f}(\pi T)\,)$.
- **Lemma** (Monotonicity)**.** $\mathrm{Inf}^*_S(f) \leq \mathrm{Inf}^*_{S\cup T}(f)$.
- **Lemma** (Subadditivity)**.** If $|S|,|T|\geq(1-\gamma)n$ then $\mathrm{Inf}^*_{S\cup T}(f) \leq \mathrm{Inf}^*_S(f) + \mathrm{Inf}^*_T(f) + o(\gamma^{1/2})$.

Properties of Inf*

Theorem. Let f be ε-far from (n-k)-symmetric and let P be a random $O(k^2)$-partition of $[n]$. Then whp every union J of k parts in P satisfies $\mathrm{Inf}^*_{[n]\setminus J}(f) \geq \varepsilon/9$.

Proof sketch.

1. $F_{1/3}=\{S\subseteq[n]: \mathrm{Inf}^*_{[n]\setminus S}(f) < \varepsilon/3\}$ is $(k+1)$-intersecting.

2. If $F_{1/3}$ contains a set S s.t. $|S|\leq 2k$, the bound holds.

3. Else, $F_{1/9}=\{S\subseteq[n]:\mathrm{Inf}^*_{[n]\setminus S}(f)<\varepsilon/9\}$ is $(2k+1)$-intersecting and the bound holds by the Intersection Theorem. □

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$$\mathrm{Inf}^*_{[n]\setminus(S\cap T)}(f) = \mathrm{Inf}^*_{([n]\setminus S)\cup([n]\setminus T)}(f) \leq \mathrm{Inf}^*_{[n]\setminus S}(f)+\mathrm{Inf}^*_{[n]\setminus T}(f)+\varepsilon/3 < \varepsilon$$

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W.h.p., S is shattered by P ⇒ $J\cap S\leq k$ ⇒ $J\notin F_{1/3}$.

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Each J is O(1/k)-biased random set ⇒ $\Pr[J\in F_{1/9}]\leq k^{-2k}$.

Discussion

Open Problems

- Which other results in the analysis of boolean functions can we extend to partial symmetry?
  - Friedgut's junta theorem?
  - Structure of the Fourier spectrum?
- Can we use such extensions to prove the function isomorphism testing conjecture?
- In which other areas of TCS do partially symmetric functions appear?
  - Local reconstruction. [Alon, Weinstein '12]
  - Active property testing. [Alon, Hod, Weinstein '13]

Thanks!