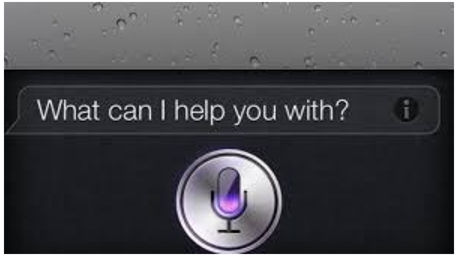
Random Projections for Probabilistic Inference

Stefano Ermon

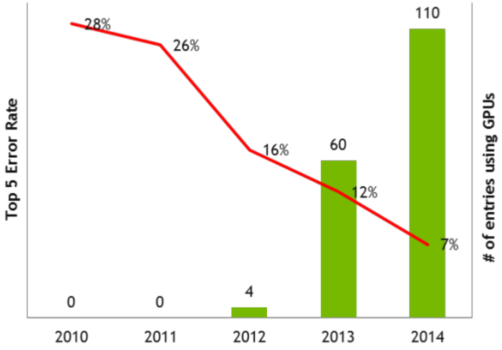
STANFORD UNIVERSITY

Recent progress in Artificial Intelligence

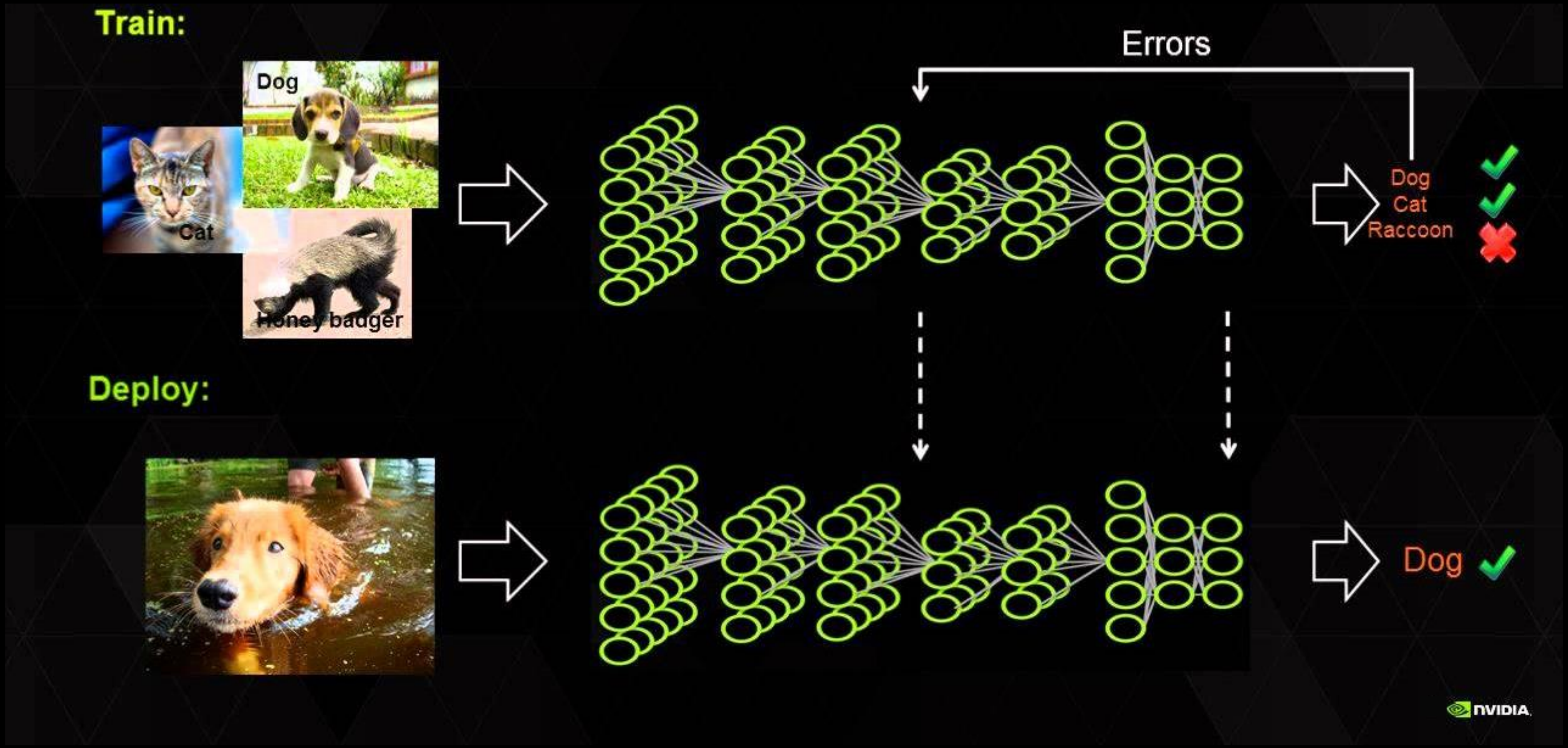


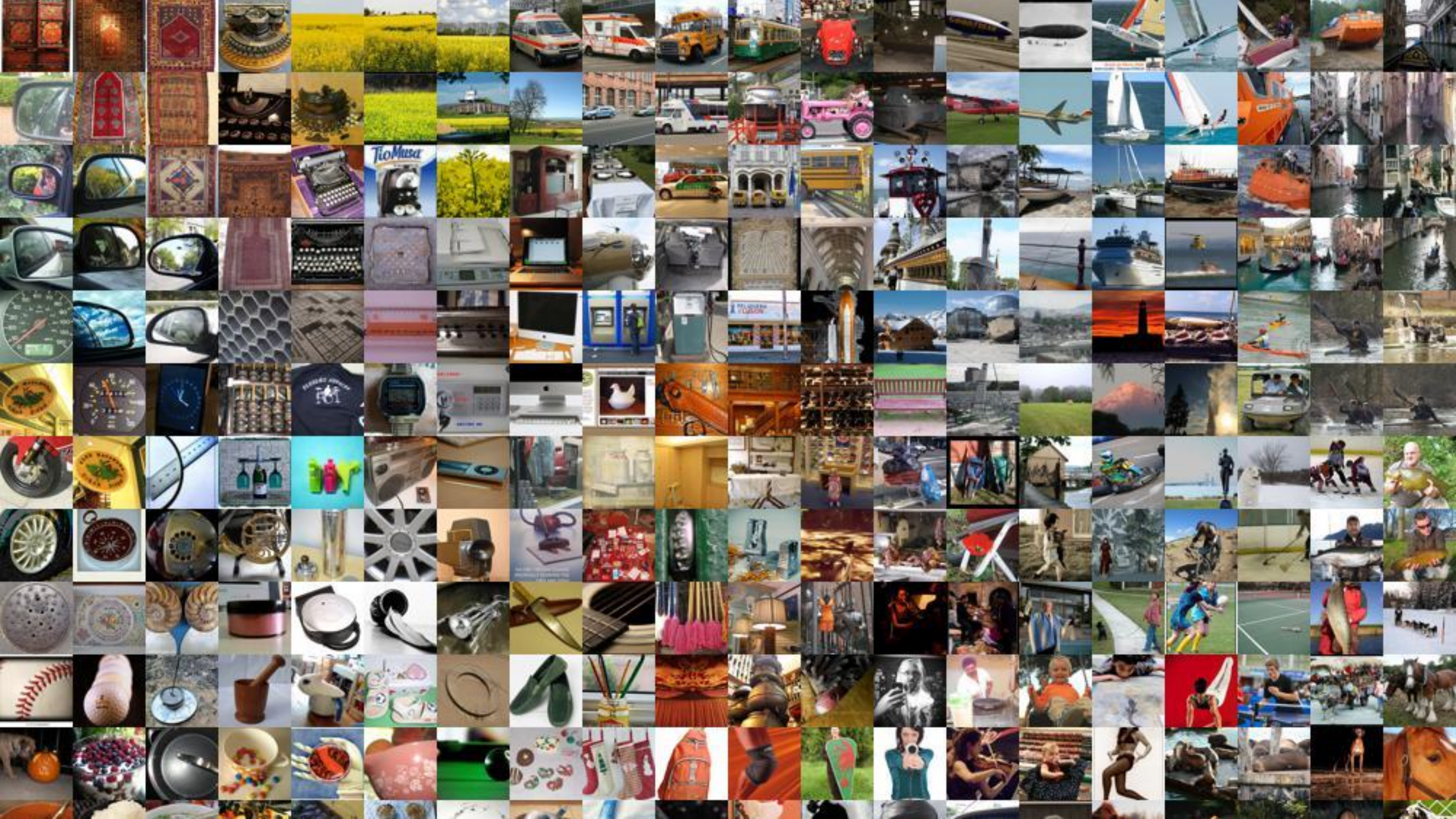
Skype Translator

IMAGENET



Supervised Learning

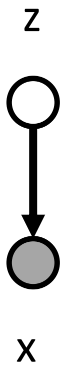



TioMusa

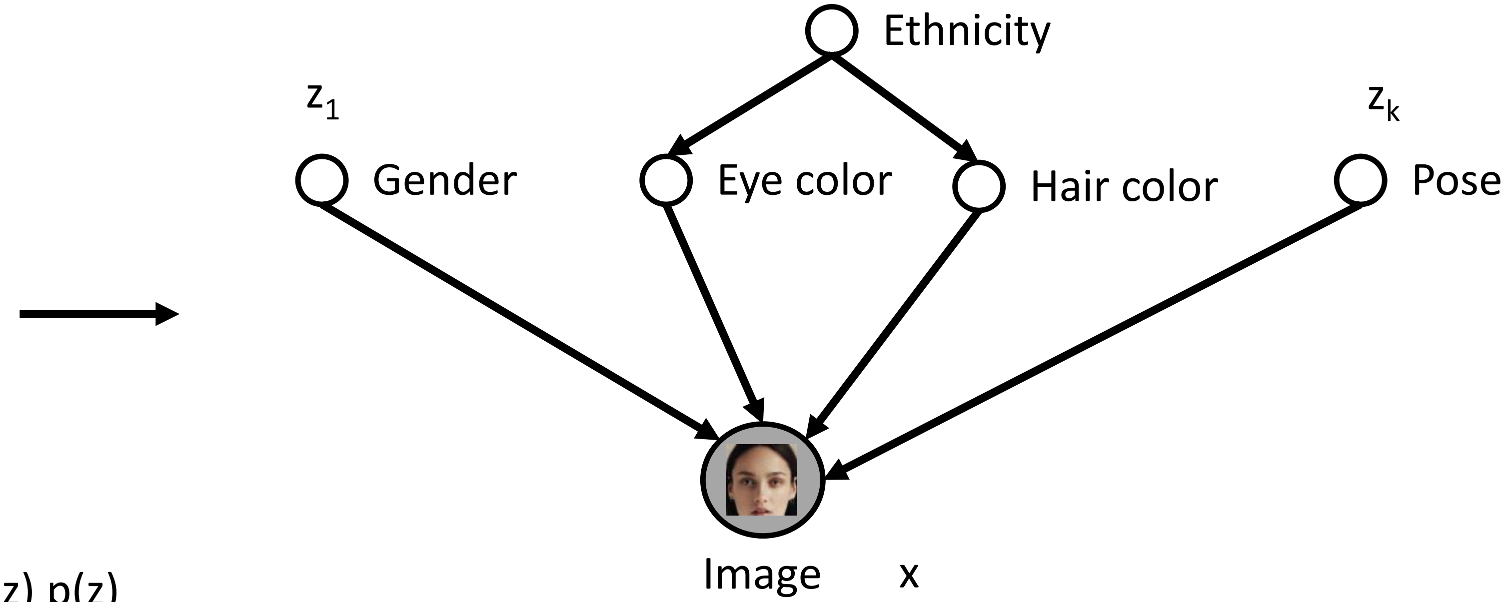
200k photos of celebrities

$\sim p(x)$

Latent Variable Model



$p(x,z) = p(x|z)\, p(z)$



A

B

C

D

E

F

G

H

I

J

K

L

True/sample? 

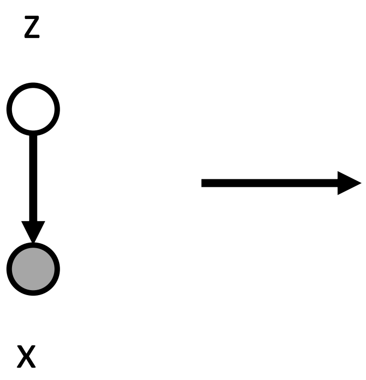
True/sample? 

“Seeds”

A

B

Learning Latent Variable Models



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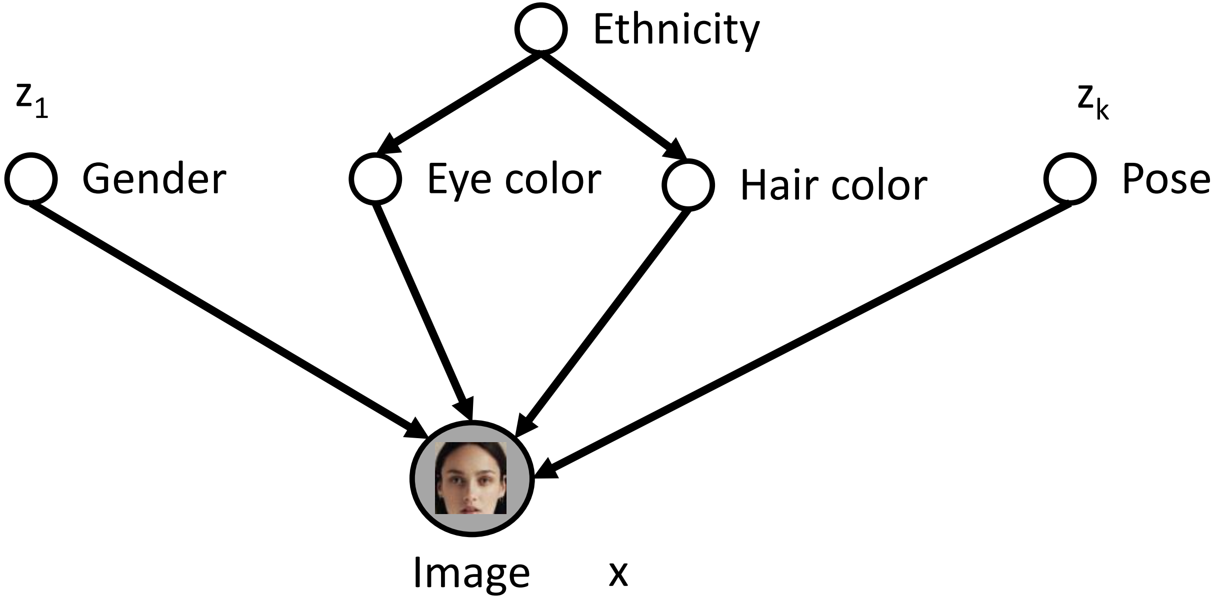


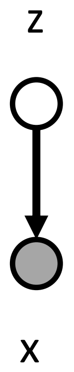
Choose parameters to maximize the (log) likelihood of the data:

$$max_{\theta} \log p(x;\theta) = max_{\theta} \log \sum_{z} p(x,z;\theta)$$

Learning Latent Variable Models

Approach: maximize the (log) likelihood of the data:

$$\log \sum_z p(x,z) = \log \sum_z q(z) \frac{p(x,z)}{q(z)}$$

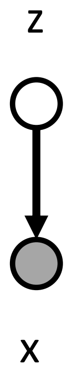


Learning Latent Variable Models

Approach: maximize the (log) likelihood of the data:

$$\log \sum_z p(x,z) = \log \sum_z q(z) \frac{p(x,z)}{q(z)}$$

$$\log \sum_z p(x,z) \geq \sum_z q(z) \log \frac{p(x,z)}{q(z)}$$



Learning Latent Variable Models

Approach: maximize the (log) likelihood of the data:

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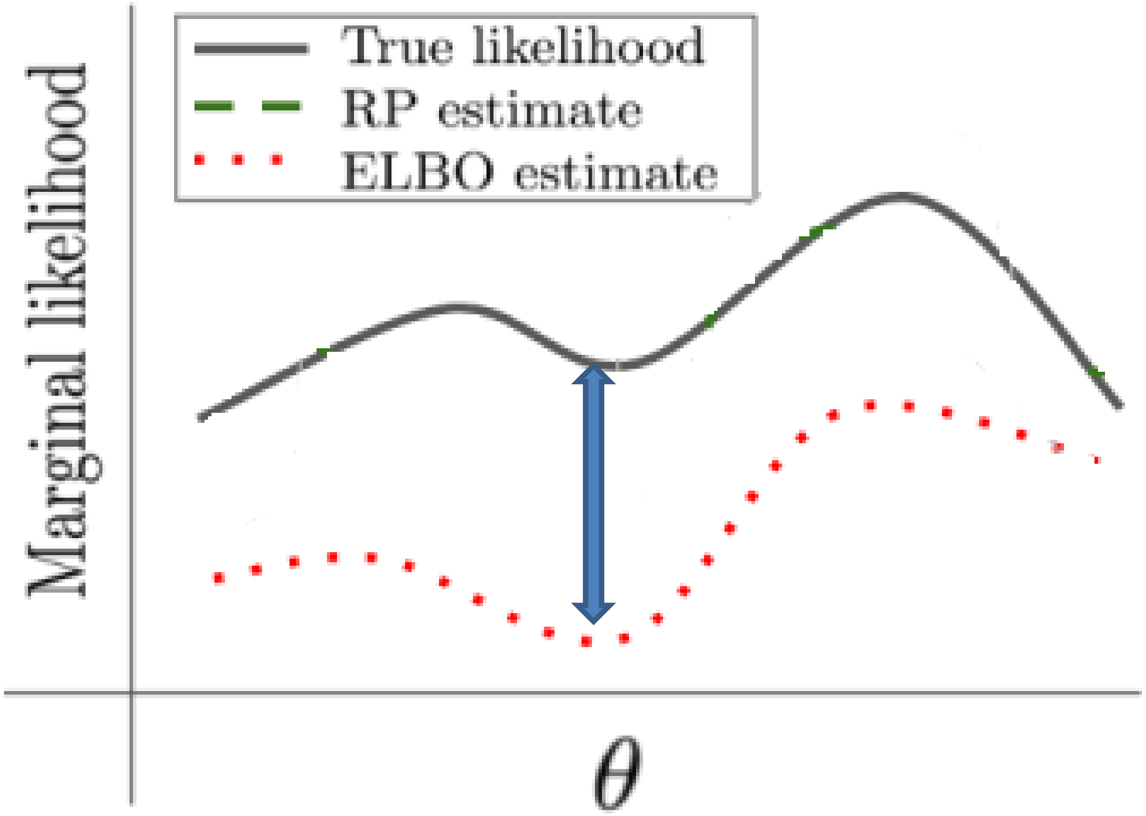
Idea: choose q to be simple


True likelihood
RP estimate
ELBO estimate
Marginal likelihood
θ

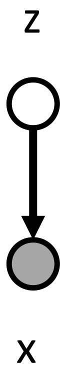
Learning Latent Variable Models

Approach: maximize the (log) likelihood of the data:

$$\log \sum_z p(x,z) = \log \sum_z q(z) \frac{p(x,z)}{q(z)}$$

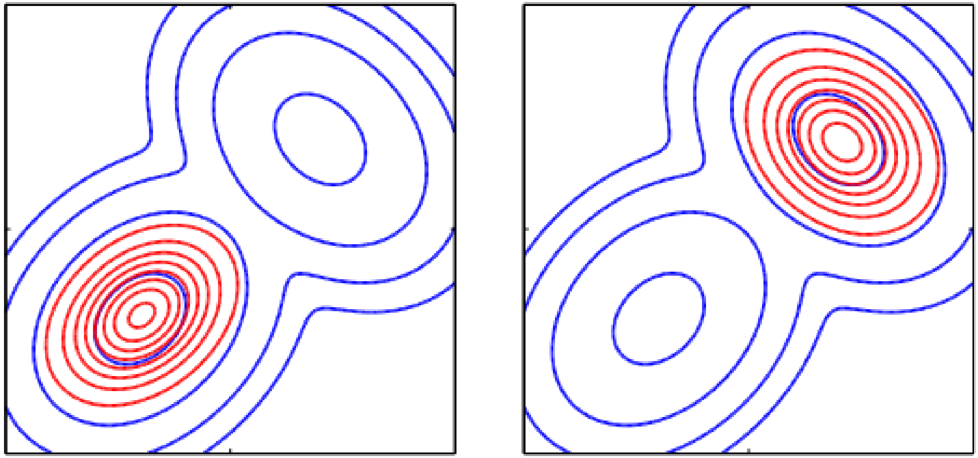
$$\log \sum_z p(x,z) \geq \sum_z q(z) \log \frac{p(x,z)}{q(z)}$$

$$\log \sum_z p(x,z) = \sum_z q(z) \log \frac{p(x,z)}{q(z)} \boxed{+ KL(q, p(z|x))}$$



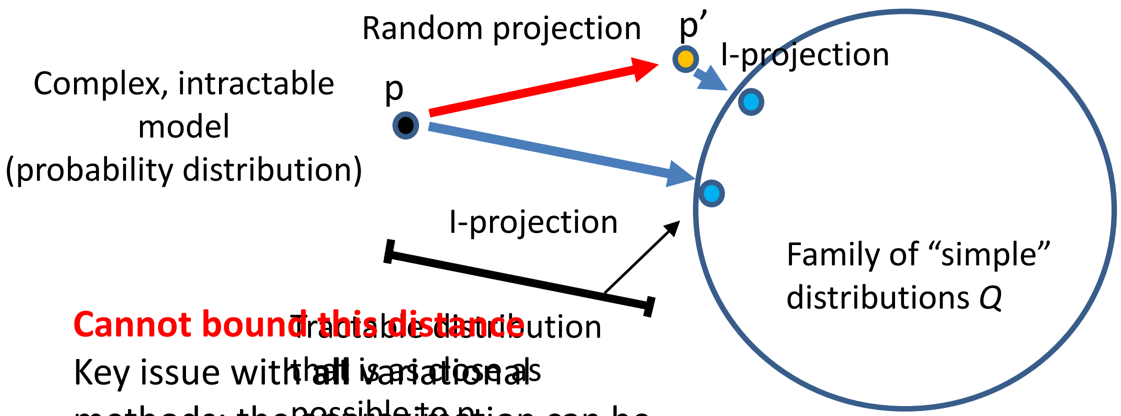
Variational Inference

Approximate a complex distribution ***p*** using a simpler (tractable) distribution ***q****
(e.g., a Gaussian distribution)



p=Blue, q^*=Red (two equivalently good solutions!)

Variational Methods and Random Projections



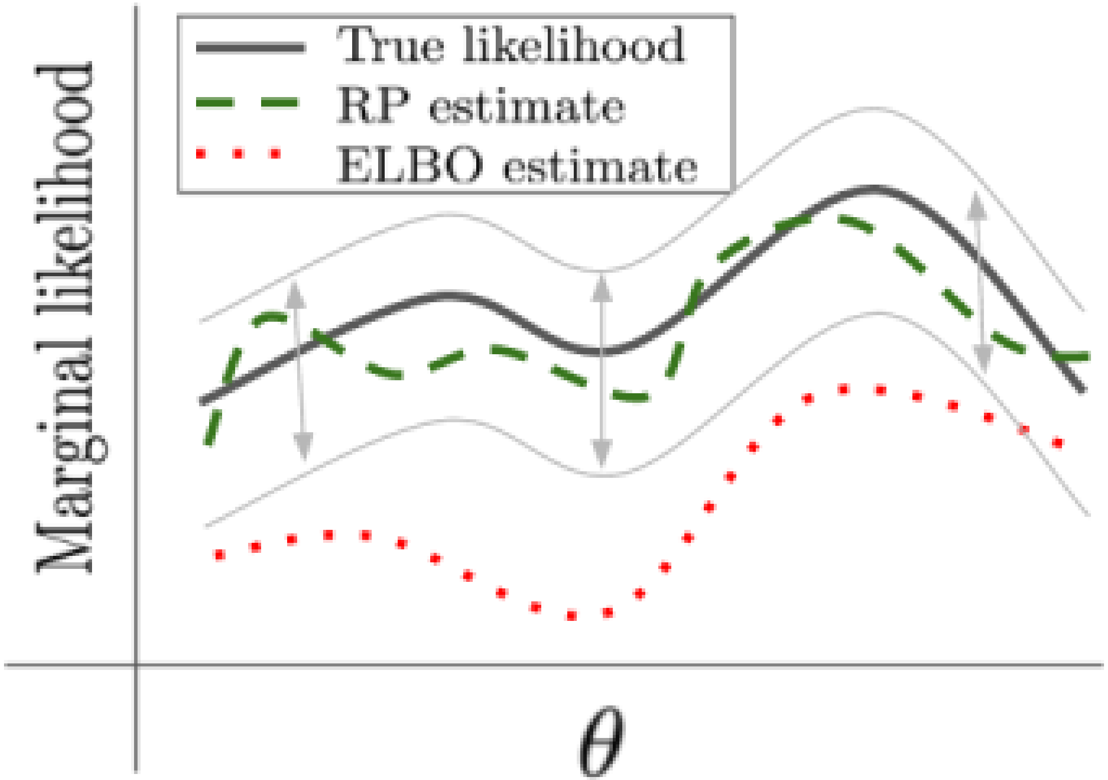
Key issue with **all** variational methods: the approximation can be arbitrarily bad

Taking a random projection of *p*:

1. Properties of p are preserved
2. p' is "simpler" and therefore "closer" to *Q*

Idea: take a random projection first, then an I-projection [AISTATS-16, NIPS-16]

Main result (informal): ***good approximation with high probability***



True likelihood
RP estimate
ELBO estimate
Marginal likelihood
θ

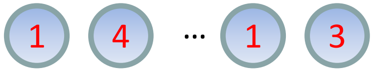
Outline

1. Introduction ✓
2. Probabilistic inference by hashing and optimization
   I. Formalism
   II. Random Projections
3. Combining random projections and I-projections
4. Other classes of random projections
5. Conclusions

Probabilistic Inference in High Dimensions

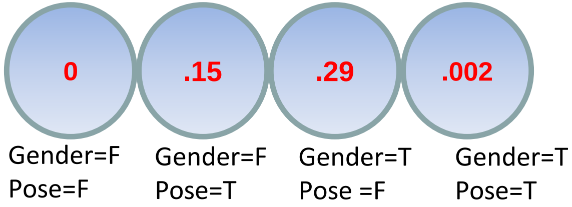
- We are given
  - a set of 2^n configurations (= assignments of z vars)
  - non-negative weights *w*
    - *from Bayes Net, factor graph, weighted CNF..*
- Goal: compute **total weight**

2^n configurations



$$\sum_z p(x,z) = \sum_z w(z)$$

- *Example 1*: n=2 binary variables, sum over 4 configurations



P[Face]

$= \sum_{x=\{T,F\}} \sum_{y=\{T,F\}}$ P[Gender=x, Pose=y, Image=Face]

$= 0 + 0.15 + 0.29 + 0.002 = 0.442$

- *Example 2*: n=100 variables, sum over $2^{100} \approx 10^{30}$ terms (**curse of dimensionality**)

$$w(z) \in \{0,1\}$$

How Might One Count?

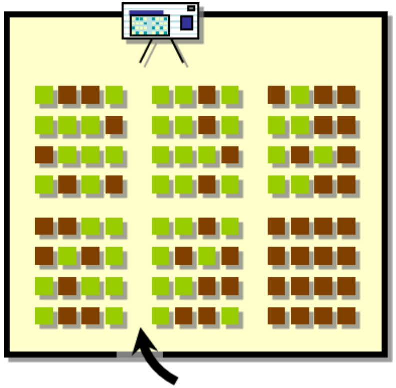
Problem characteristics:

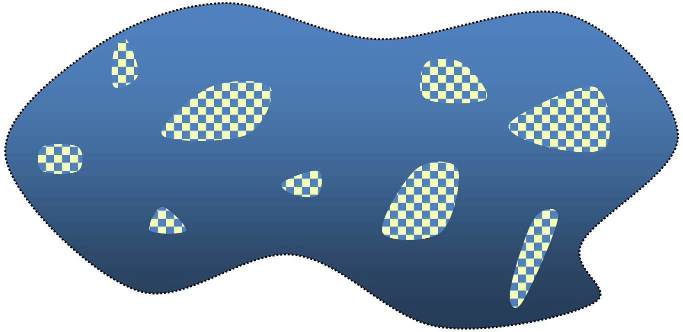
- Space naturally divided into rows, columns, sections, …
- Many seats empty
- Uneven distribution of people (e.g. more near door, aisles, front, etc.)

Analogy: How many people are present in the hall?

From Counting People to Marginal Inference

– Auditorium : search space
– Seats : 2^n truth assignments of the z variables
– Occupied seats : assignments with non-zero weight

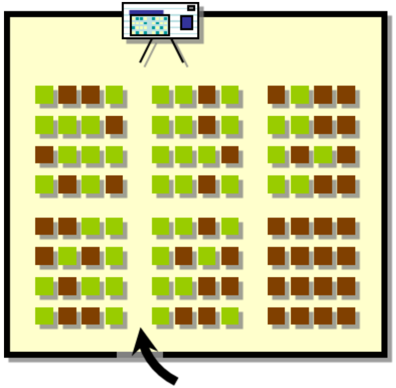


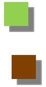


: occupied seats (47) = assignments with non-zero weight

: empty seats (49)

#1: Brute-Force Counting





Idea:

- Go through every seat
- If occupied, increment counter

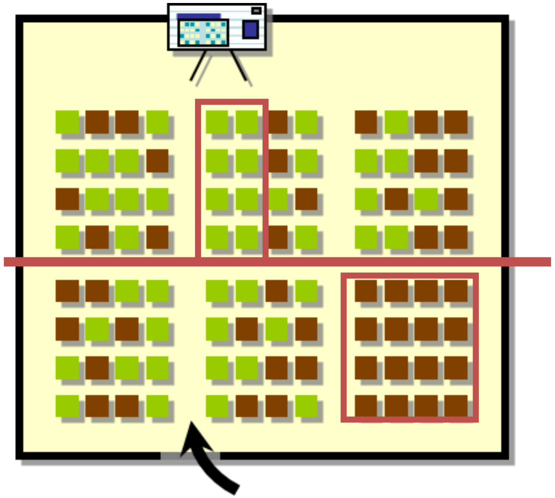
Advantage:

- Simplicity, accuracy

Drawback:

- Scalability

#2: Branch-and-Bound (DPLL-style)



Idea:

- Split space into sections
  e.g. front/back, left/right/ctr, …
- Use smart detection of full/empty sections
- Add up all partial counts

Advantage:

- Relatively faster, exact

Drawback:

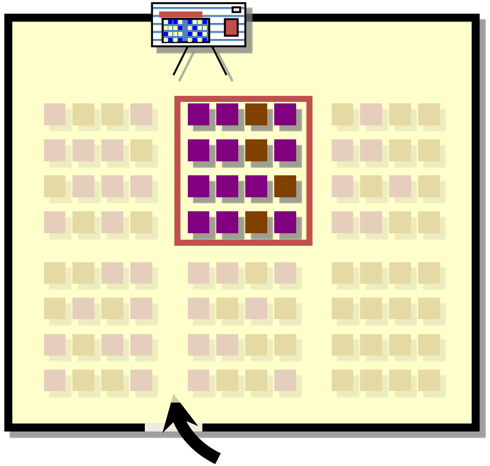
- Still "accounts for" every single person present: need extremely fine granularity
- Scalability

Framework used in DPLL-based systematic exact counters
e.g. Cachet [Sang-et]

Related to compilation approaches [Darwiche et. al]

Approximate model counting?

#3: Estimation By Sampling -- Naïve



Idea:

- Randomly select a region
- Count within this region
- Scale up appropriately

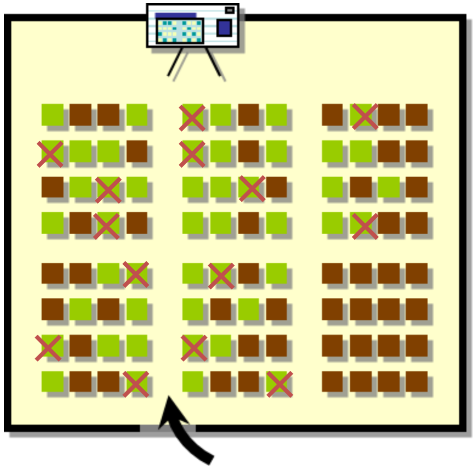
Advantage:

- Quite fast

Drawback:

- Robustness: can easily under- or over-estimate
- Scalability in sparse spaces: e.g. 10^{60} solutions out of 10^{300} means need region much larger than 10^{240} to “hit” any solutions

Let's Try Something Different …



A Distributed Coin-Flipping Strategy
(Intuition)

Idea:

Everyone starts with a hand up

– Everyone tosses a coin
– If heads, keep hand up,
  if tails, bring hand down
– Repeat till no one hand is up

Return $2^{\#(\text{rounds})}$

Does this work?

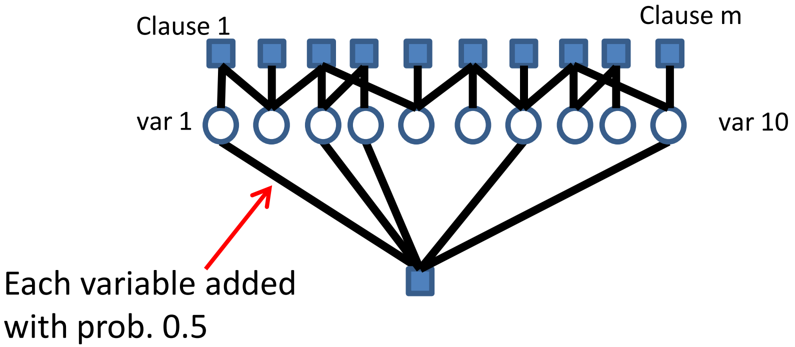
- On average, Yes!
- With M people present, need roughly $\log_2 M$ rounds for a unique hand to survive

Making the Intuitive Idea Concrete

- How can we make each configuration “flip” a coin?
- How do we transform the average behavior into a robust method with **provable correctness guarantees**?
- Many approaches based on this idea (originated from theoretical work due to [Stockmeyer-83, Valiant-Vazirani-86, etc.]):
  - Mbound, XorSample [Gomes et al-2007]
  - WISH, PAWS [Ermon et al-2013]
  - ApproxMC, UniWit,UniGen [Chakraborty et al-2014,2016]
  - Achilioptas et al UAI-15 (error correcting codes)
  - Belle et al. at UAI-15 (SMT solvers)

Random parity constraints

- XOR/parity constraints:
  - *Example:* $a \oplus b \oplus c \oplus d = 1$ satisfied if an odd number of a,b,c,d are set to **1**



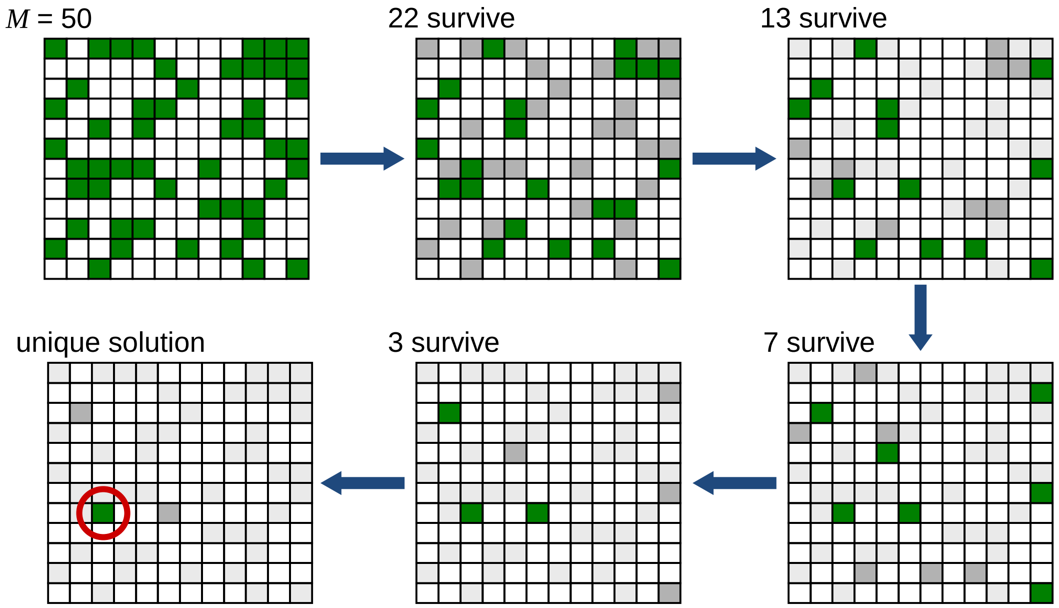
Randomly generated parity constraint X

$$x_1 \oplus x_3 \oplus x_4 \oplus x_7 \oplus x_{10} = \mathbf{1}$$

- Each solution satisfies this random constraint with probability ½
- Pairwise independence: For every two configurations ***A*** and ***B***, "***A*** satisfies ***X***" and "***B*** satisfies ***X***" are independent events

The Desired Effect

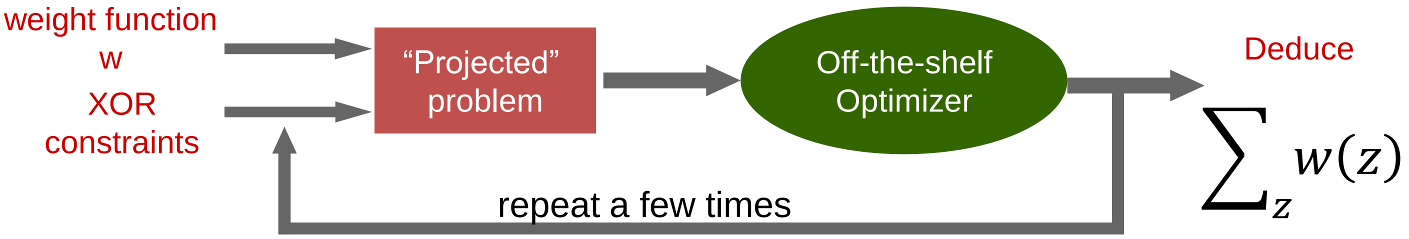
If each XOR cut the solution space roughly in half, would get down to a unique solution in roughly $\log_2 M$ steps!

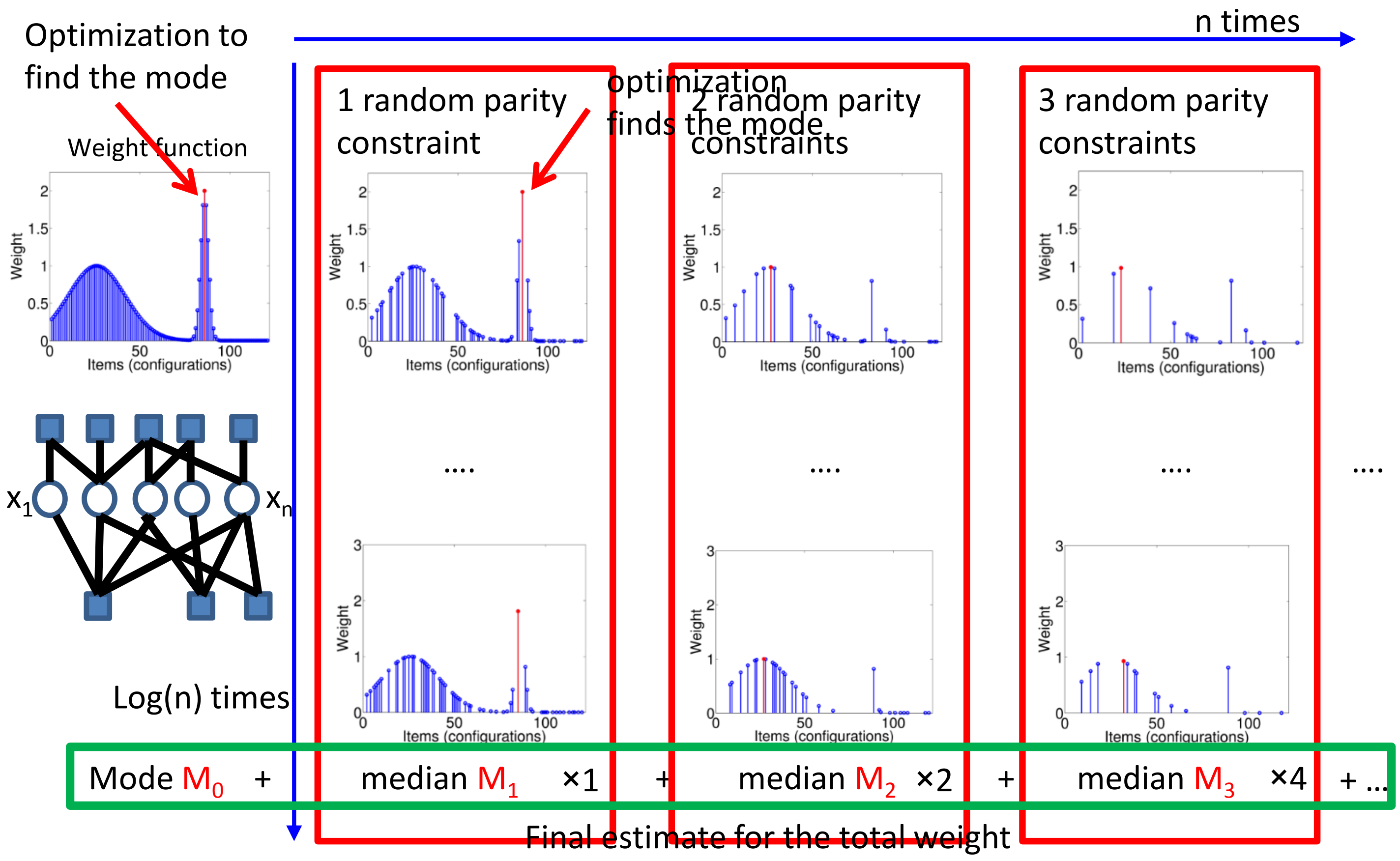


Hashing for Weighted Problems

Given a **weight function** $w(z) = p(x, z)$

1. Add some XOR constraints to w to get w'
   (this reduces the degrees of freedom)
2. Find MAX-weight assignment of w'
3. Conclude "something" about the **total weight** $\sum_z w(z)$





Optimization to find the mode
n times
1 random parity constraint
optimization finds the mode
2 random parity constraints
3 random parity constraints
Weight function
Weight
Items (configurations)
x_1
x_n
....
Log(n) times
Mode M_0 + median M_1 ×1 + median M_2 ×2 + median M_3 ×4 + ...
Final estimate for the total weight

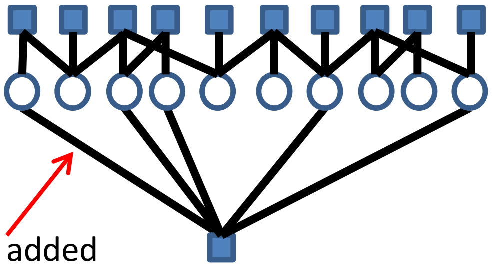
Accuracy Guarantees

Result (stated informally):

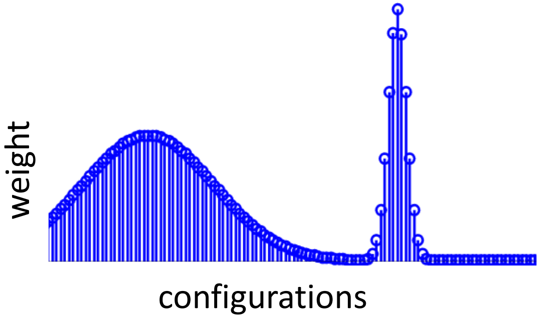
With high probability WISH (**W**eighted-**S**ums-**H**ashing) computes a constant factor approximation to the total weight and it requires solving $\theta(n \log n)$ **optimization instances.**

Hashing as a random projection

- XOR/parity constraints:
  - *Example:* $a \oplus b \oplus c \oplus d$ = 1 satisfied if an odd number of a,b,c,d are set to **1**



Each variable added with prob. 0.5

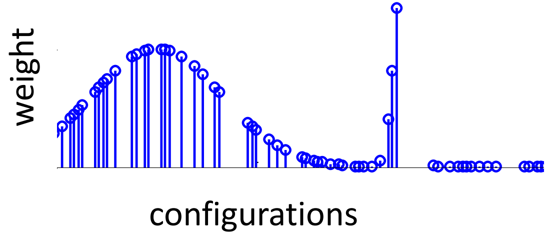


Randomly generated parity constraint X

$$x_1 \oplus x_3 \oplus x_4 \oplus x_7 \oplus x_{10} = \mathbf{1}$$

Set weight to zero if constraint is not satisfied

Random projection



This random projection:

1. "**simplifies**" the model
2. preserves its "key properties"

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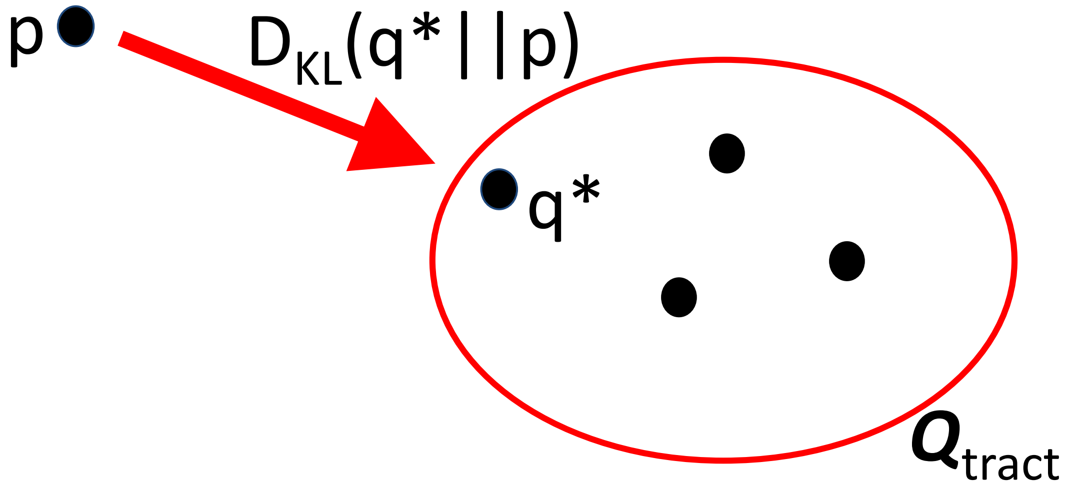
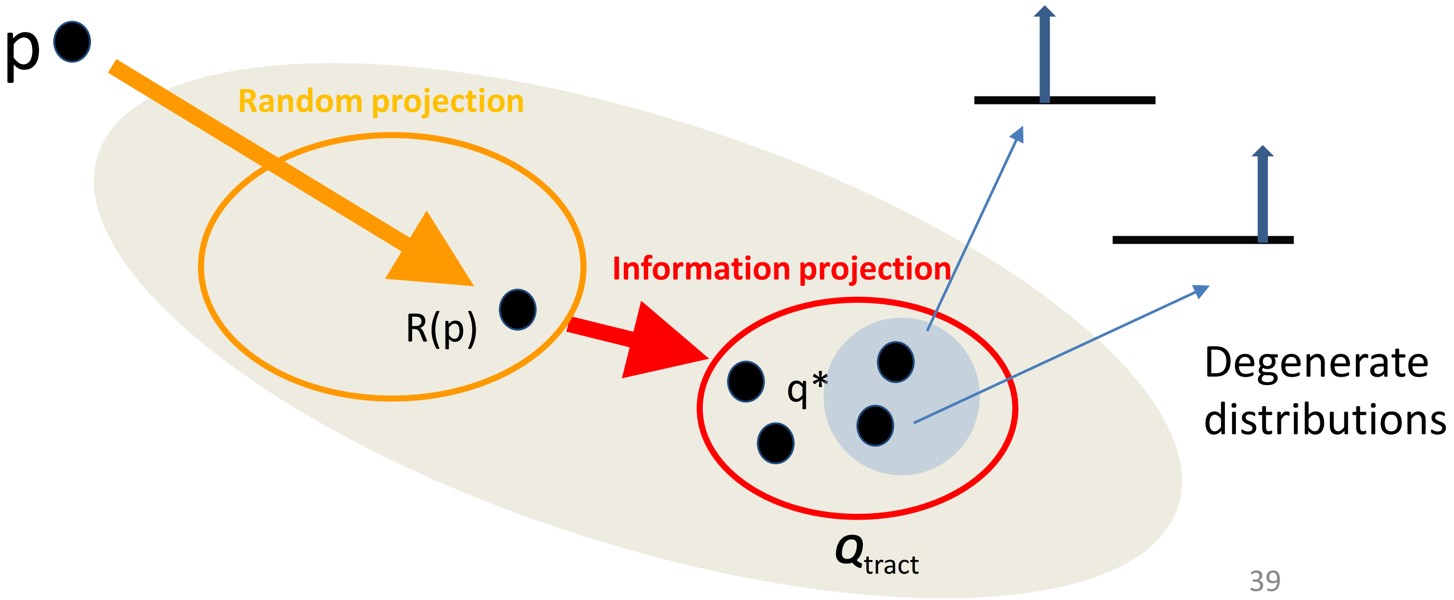
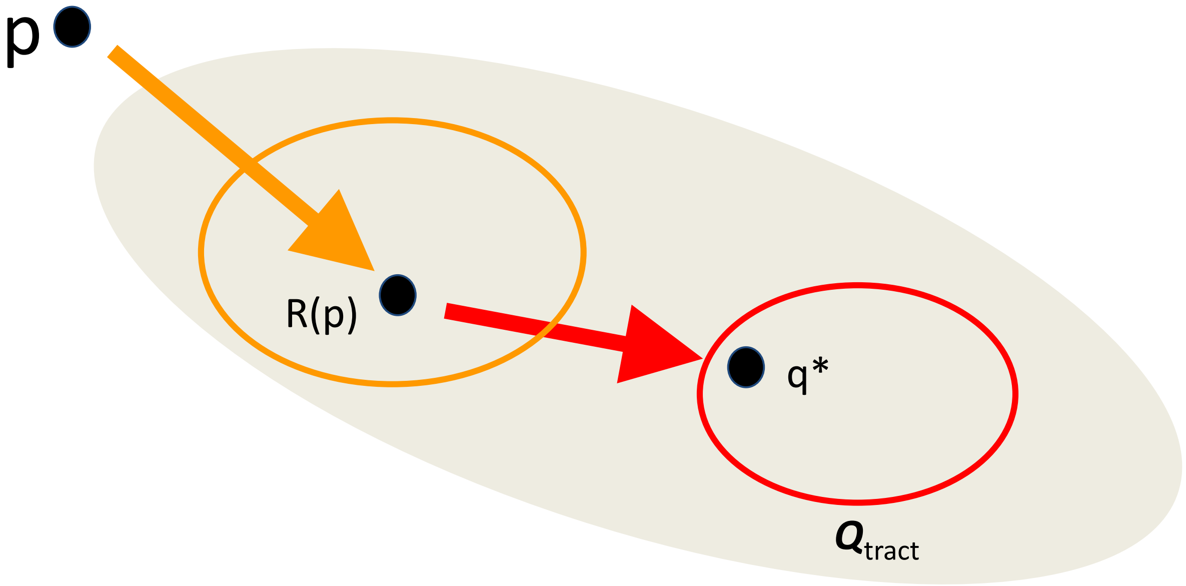
Variational Inference



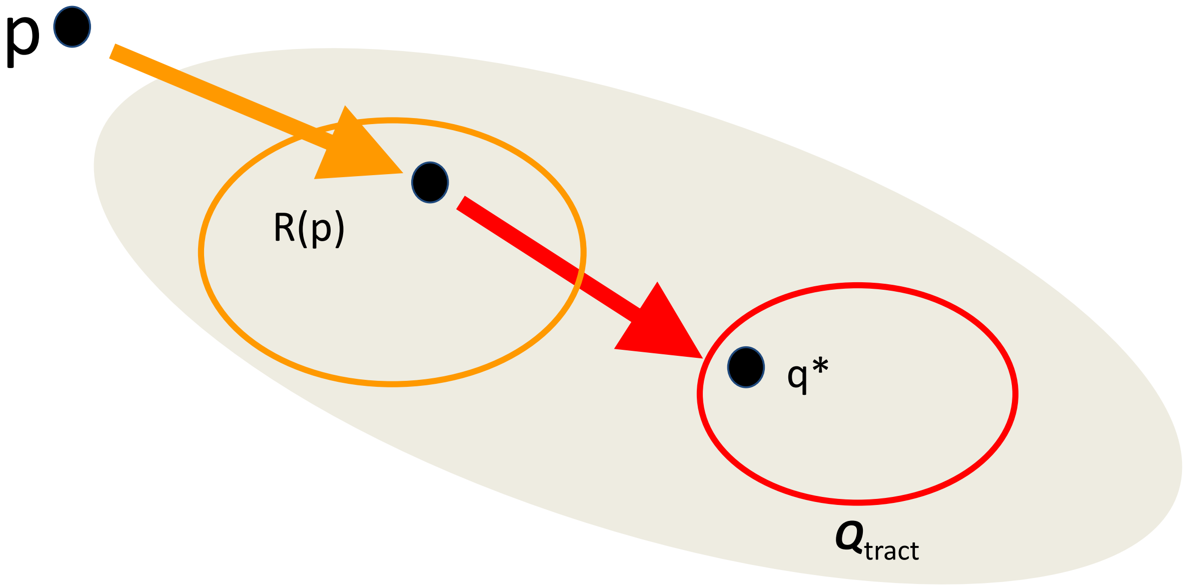
Variational Inference with Random Projections



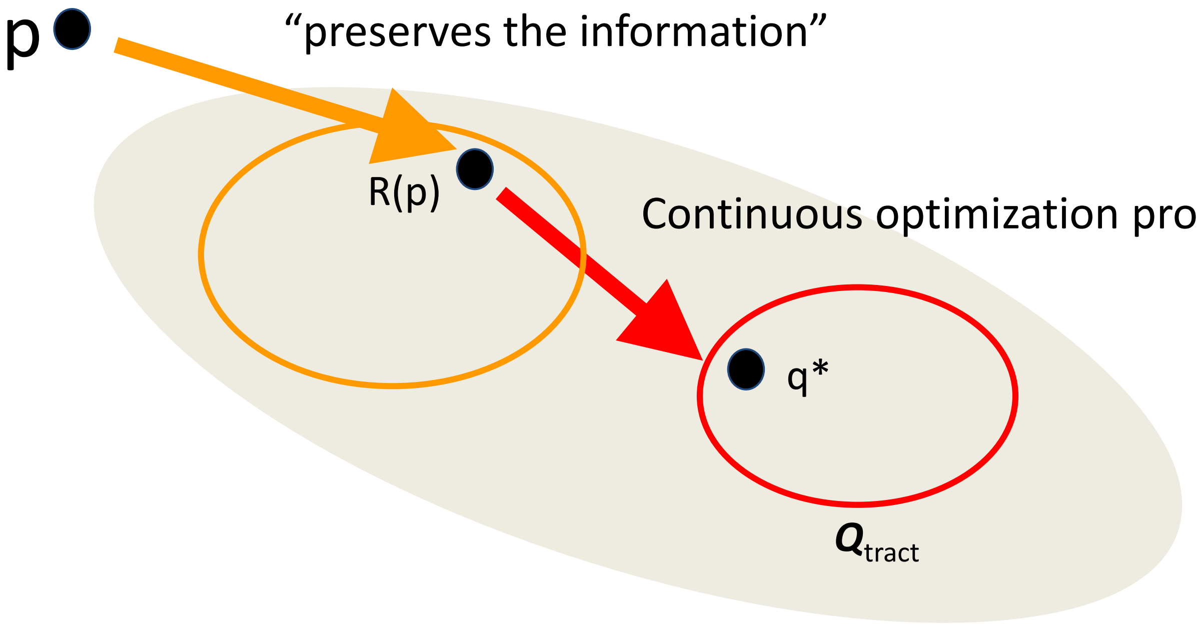
Variational Inference with Random Projections



Variational Inference with Random Projections

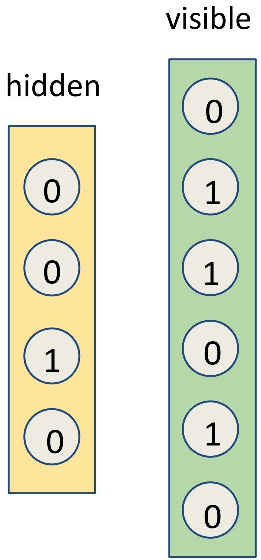


Variational Inference with Random Projections



Result (informal statement): after the random projection (e.g., using the right number of XORs), the resulting distribution can be **well approximated** using standard variational inference (with accuracy guarantees)

Algorithm: Mean Field with Random Projections



Fully factored distribution:

$$q(z) = q(z_1)\, q(z_2) \ldots q(z_n)$$

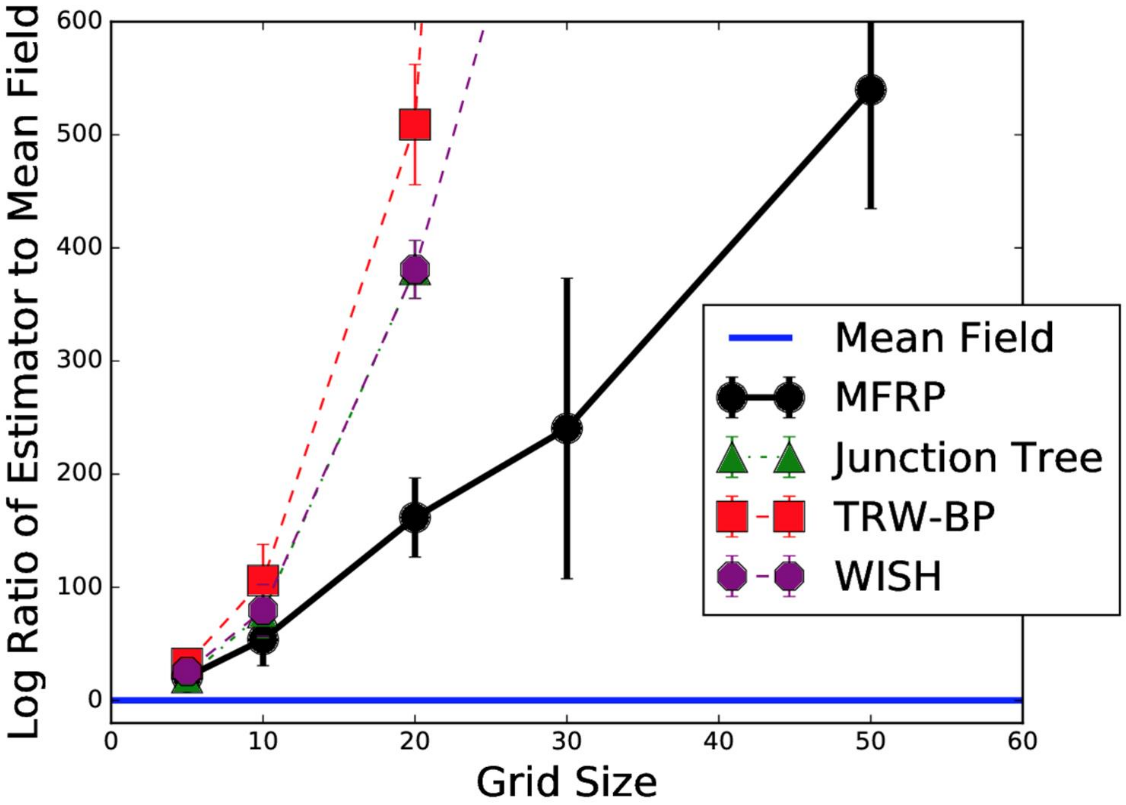
Randomly-projected mean field:

Variational objective is

- coordinate-wise concave
- has fewer variables

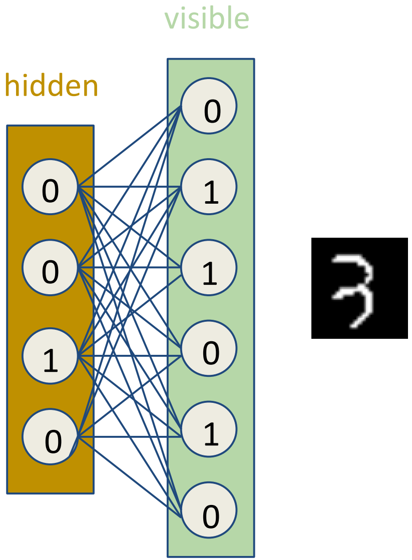
<u>If</u> we find **global optimum**, then accuracy guarantees

Ising Model



Log Ratio of Estimator to Mean Field
Grid Size
0
100
200
300
400
500
600
0
10
20
30
40
50
60
Mean Field
MFRP
Junction Tree
TRW-BP
WISH

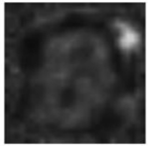
Boltzmann Machines



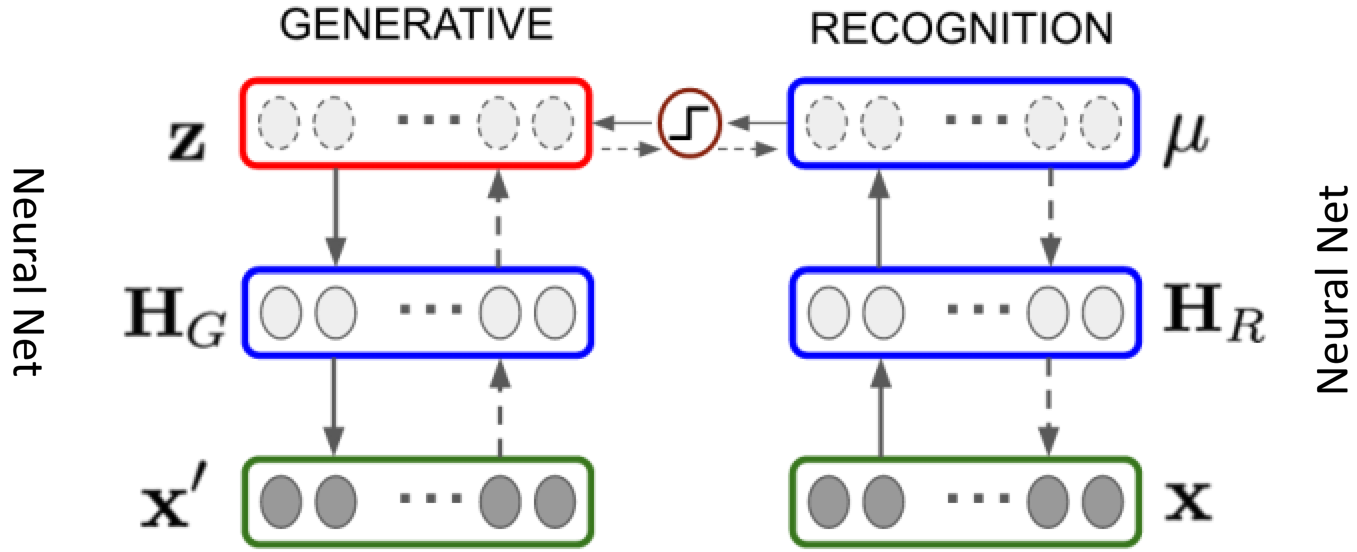
Discrete graphical model:

p(h, v) = f(h, v) / Z

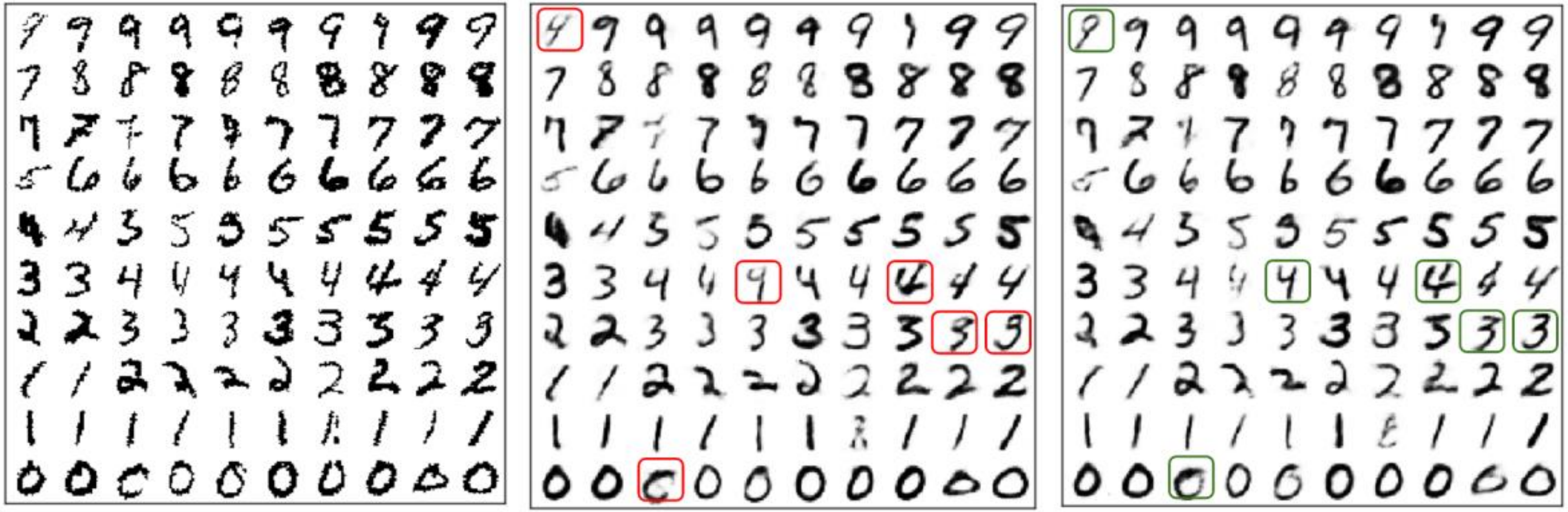
RBMs

No. Hidden Nodes	100	100	100	200	200	200
CD-k	1	5	15	5	15	25
MF $\log Z$	501	348	297	203	293	279
MFRP $\log Z$	501	**433**	**342**	**380**	**313**	**295**
MF μ						
MFRP μ						

Variational Autoencoders



Denoising results



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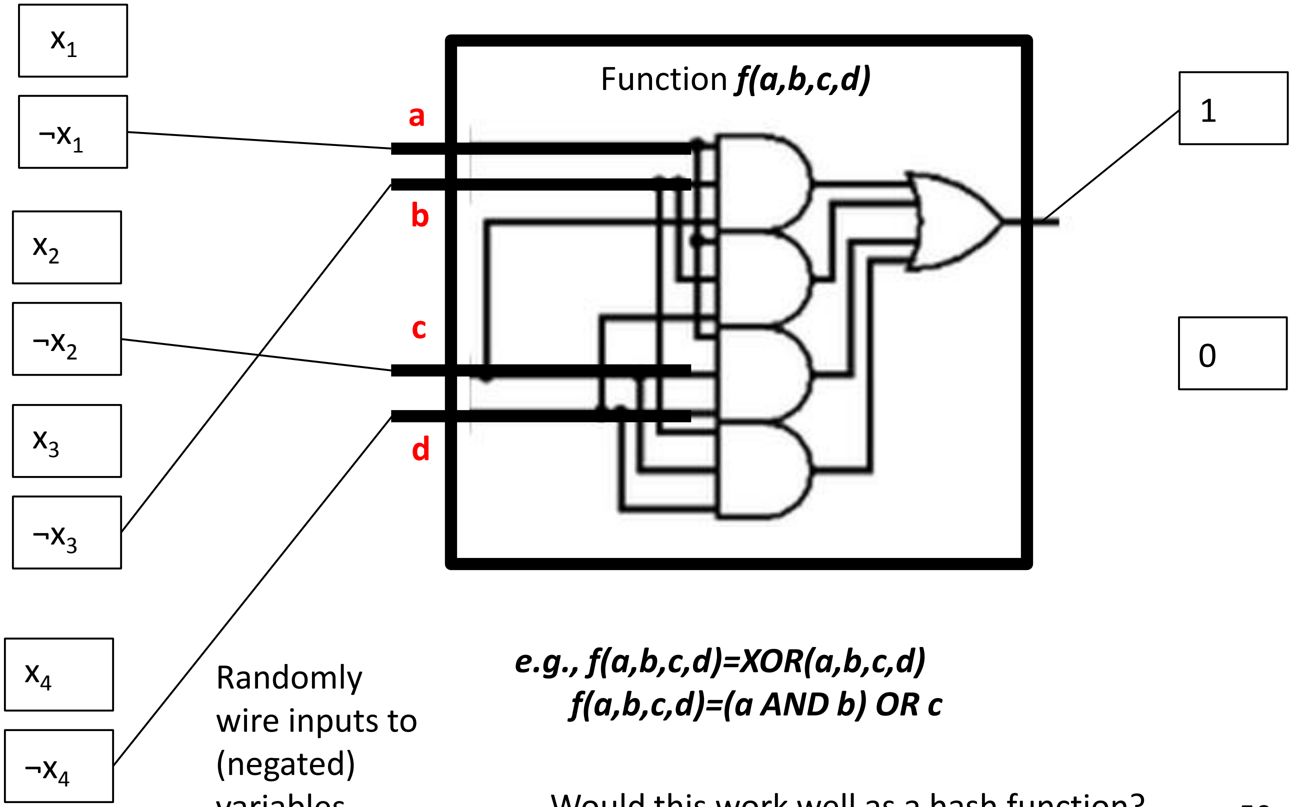
Random Constraints for Probabilistic Inference



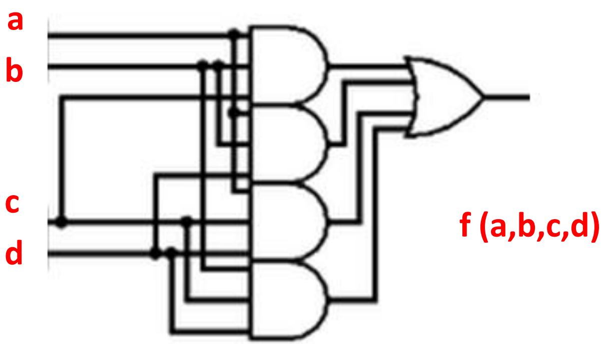
Randomly wire inputs to (negated) variables

e.g., f(a,b,c,d)=XOR(a,b,c,d)

f(a,b,c,d)=(a AND b) OR c

Would this work well as a hash function?

Noise sensitivity

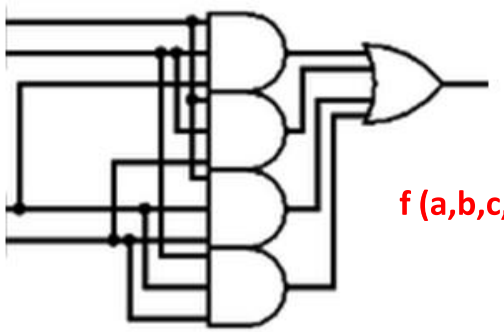


Performance of this family of hash functions depends on the **noise sensitivity** of **f (a,b,c,d)**

- Given a random input x, e.g. **x**=(0, 1,1,0)
- Randomly flip each bit of x with probability p → **x'**=(1, 1,1,0)
- How likely is that f(x) = f(x')?

Result (informal statement): The more noise sensitive a function f is, the better the corresponding family of hash functions (with random wiring) behaves. [ICML-16]

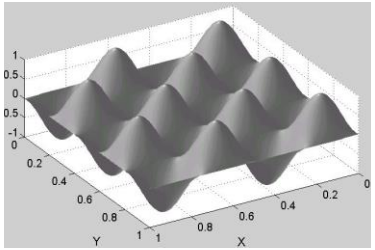
Random Constraints for Probabilistic Inference and Model Counting

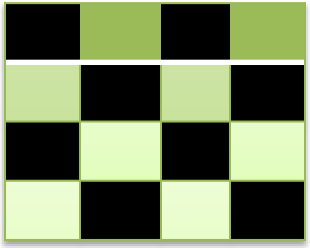


f (a,b,c,d)

- Noise sensitivity can be computed in *closed form* from the Fourier spectrum of **f.** Known for many common functions!
- Intuitively, the more "oscillatory" **f** is, the more noise sensitive it is

2-D sine wave





2-D parity function

Experimental results

INSTANCE	GT	LB_{trib}	LB_{xor}	t_{trib}	t_{xor}
LS10	24	18	19	**1**	209
LS11	33	25	24	**28**	623
LS12	—	32	29	**34**	658
LS13	—	33	34	**3**	74
LS14	—	36	34	**12**	761
LS15	—	**39**	34	**51**	829
20RDR45	—	29	29	**1**	523
23RDR45	—	**27**	10	**7**	800

Tribes: One or two orders of magnitude faster!

$$(x_1 \wedge x_2 \wedge \neg x_3) \vee (\neg x_4 \wedge \neg x_5) \vee (\neg x_6 \wedge x_7)$$

Conclusions

- Random projections and hashing have been extremely useful in Information Retrieval and Machine Learning
- Can also be applied to more complex objects, like probability distributions
- New algorithms with improved guarantees and practical performance
- Exciting ML applications: learning generative models of data