Generalizations of KKL Theorem & Friedgut's Junta Theorem

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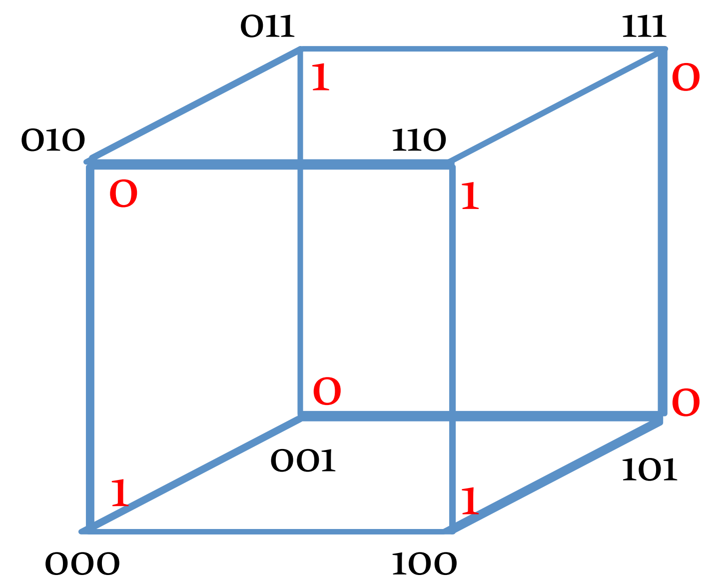
Boolean Functions and Influence

$$f : \{0,1\}^n \to \{0,1\}$$

Influence (of coordinate i)

$$\mathrm{Inf}_i(f) = \Pr_x[f(x) \neq f(x \oplus e_i)]$$

Flip i^{th} bit



KKL Theorem

Assume f is balanced $\Pr_x[f(x) = 1] = 1/2$

Simplest such function $g(x) = x_j$ $\sum_i \text{Inf}_i(g) = 1$

Easy: For any balanced f $\sum_i \text{Inf}_i(f) \geq 1$

$\Rightarrow$ $\exists i \;\; \text{Inf}_i(f) \geq \frac{1}{n}$

[Kahn-Kalai-Linial]

For every boolean f, $\exists i \;\; \text{Inf}_i(f) \geq \frac{\log n}{n}$

Friedgut's Junta Theorem

Easy: For any balanced f s.t. $\sum_i \mathrm{Inf}_i(f) = 1$ ⟹ $\exists i \; f(x) = x_i$ or $1 - x_i$

f depends only on 1 coordinate

What if $\mathbb{I} = \sum_i \mathrm{Inf}_i(f)$ is larger but still constant?

[Friedgut]

Every boolean f is ^{2}-close to a boolean g that depends only on $\exp(\mathbb{I}/\epsilon)$ coordinates

g is a Junta

^{2}-close ⟺ $\Pr_x[f(x) \neq g(x)] \leq \epsilon$

Influence

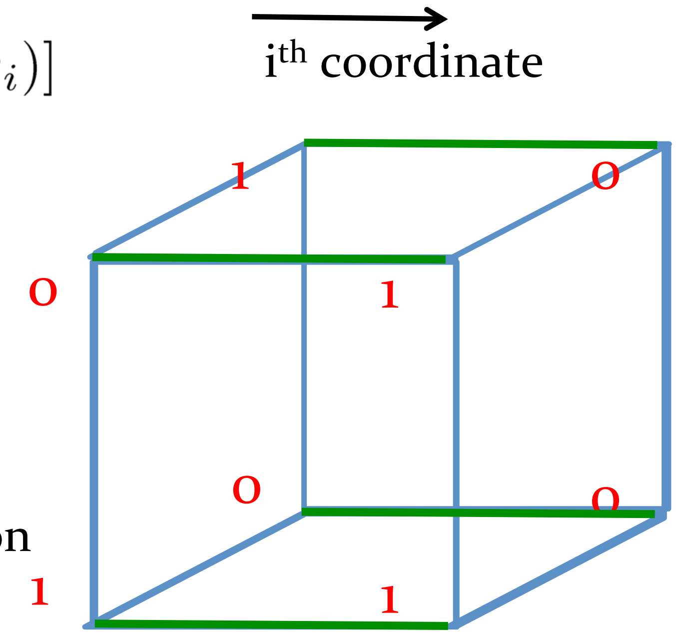
$$\mathrm{Inf}_i(f) = \Pr_x[f(x) \neq f(x \oplus e_i)]$$

$$= \mathbf{E}_x\, \mathbb{1}[\text{Is f non-constant on } (x, x \oplus x_i)]$$

$$= \mathbf{E}_x\, \mathbf{Var}_{x_i}\, f(x)$$

$=$ fraction of edges cut in i$^{\text{th}}$ direction

$=$ fractional size of boundary in i$^{\text{th}}$ direction



Identify f with the cut

$$\{\ x \mid f(x) = 1\ \}$$

Generalizations of KKL

For every balanced f :Domain^n → {0,1}, ∃ coordinate i s.t.

$$\mathrm{Inf}_i(f) \geq \frac{\log n}{n}$$

	Domain	$\mathrm{Inf}_i(f)$
	p-biased {0,1}	$\Pr_x[f(x) \neq f(x \oplus e_i)]$
[Mossel]	Finite probability space Ω	$\mathbf{E}_x\, \mathbf{Var}_{x_i}\, f(x)$
[Bourgain-Kahn-Kalai-Katznelson-Linial]	[0,1]	$\mathbf{E}_x\, \mathbb{1}[\text{Is } f(x\|_{i \leftarrow y})$ non-constant for $y \in [0,1]]$
[Keller-Mossel-Sen]	Gaussian	measure of boundary in ith direction
[S-Tulsiani] [Cordero Erausquin-Ledoux]	Vertices of a Graph	fraction of edges cut in ith direction

$\frac{\sqrt{\log n}}{n}$ instead of $\frac{\log n}{n}$

G^n = n-fold Cartesian product of G

depend on the particular domain chosen

Other Generalizations

[O'Donnell-Wimmer]	Certain classes of generalizations of Cayley graphs (non-product setting)
[Cordero Erausquin-Ledoux]	A framework that encompasses [KKL], [S-Tulsiani],[O'Donnell-Wimmer]
[Keller]	Other generalizations of Influence in the product setting

Generalizations of Junta Thm

Every f :Domain^n → {0,1} is ^{2}-close to a boolean g depending only on $\exp(\mathbb{I}/\epsilon)$ coordinates

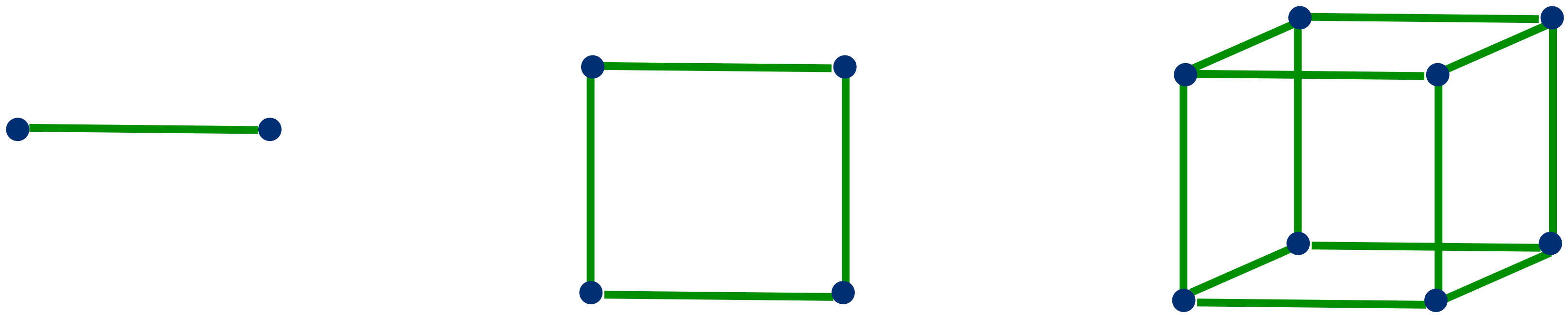
$$\mathbb{I} = \sum_i \mathrm{Inf}_i(f)$$

	Domain	$\mathrm{Inf}_i(f)$
[Friedgut]	p-biased {0,1}	$\Pr_x[f(x) \neq f(x \oplus e_i)]$
[S-Tulsiani]	Vertices of a Graph	fraction of edges cut in ith direction

Applications to hardness of approximation results [S-Saket][Khot-Saket]

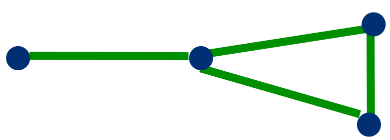
* The constants depend on the particular domain chosen

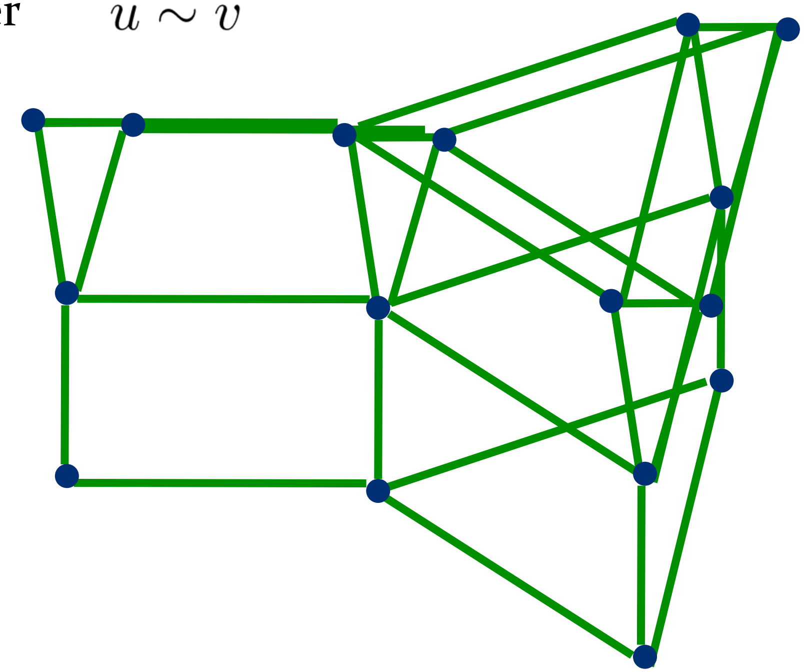
Cartesian Product of Graphs



$$(x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_n) \sim (x_1, \ldots, x_{i-1}, v, x_{i+1}, \ldots, x_n)$$

whenever $u \sim v$





Log Sobolev Inequality

Given a graph G(V,E), for any f: V(G) → R

$$\mathop{\mathbf{E}}_{(x,y)\in E(G)} (f(x) - f(y))^2 \geq \lambda(G)\,\mathbf{Var}(f)$$

Spectral Gap

Spectral Gap of G

far f is

nt

Log Sobolev Inequality

Given a graph G(V,E), for any f: V(G) → R

$$\mathop{\mathbf{E}}_{(x,y)\in E(G)} (f(x) - f(y))^2 \geq \alpha(G)\,\mathbf{Ent}(f^2)$$

Log-Sobolev

Log Sobolev constant of G

$$\mathbf{Ent}(f^2) = \mathbf{E}\, f^2 \log f^2 - (\mathbf{E}\, f^2) \log(\mathbf{E}\, f^2)$$

Behavior under Cartesian Products

$$\alpha(G^n) = \frac{1}{n}\alpha(G)$$

Results for Cartesian Products

Given graph G(V,E), define

$\mathrm{Inf}_i(f) =$ fraction of edges cut along ith direction

$$\mathbb{I} = \sum_i \mathrm{Inf}_i(f)$$

[S-Tulsiani]

For every f: V(G^n) → {0,1}, $\exists i \;\; \mathrm{Inf}_i(f) \geq \alpha(G) \frac{\log n}{n}$

Also follows from [Cordero Erausquin-Ledoux]

[S-Tulsiani]

Every f: V(G^n) → {0,1} is ^{2}-close to a boolean g that depends only on $\exp(\mathbb{I}/\alpha(G)\epsilon)$ coordinates

Proof Idea for Hypercube

Given f: $\{0,1\}^n \rightarrow \{0,1\}$

Define

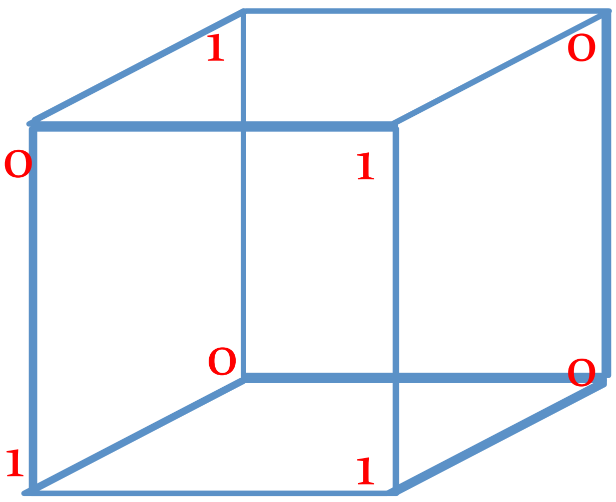
$$g_1(x) = \mathbf{E}_{x_1}\, f(x)$$

$$g_2(x) = \mathbf{E}_{x_2}\, g_1(x)$$

$$g_3(x) = \mathbf{E}_{x_2}\, g_2(x)$$

$$\ldots$$

$$g_n(x) = \mathbf{E}_{x_n}\, g_{n-1}(x)$$



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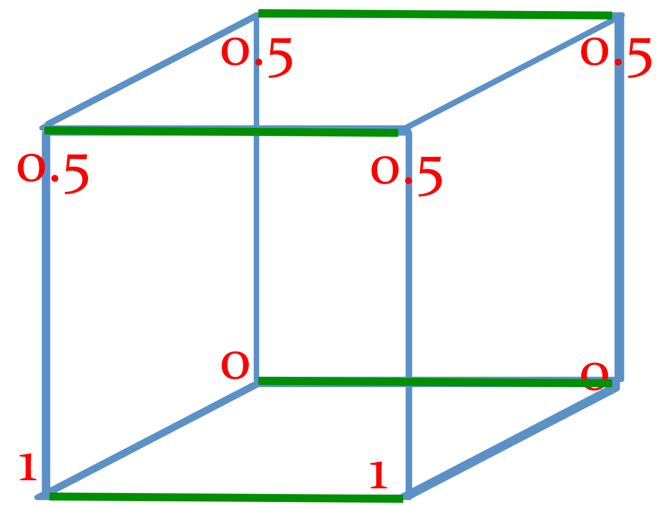

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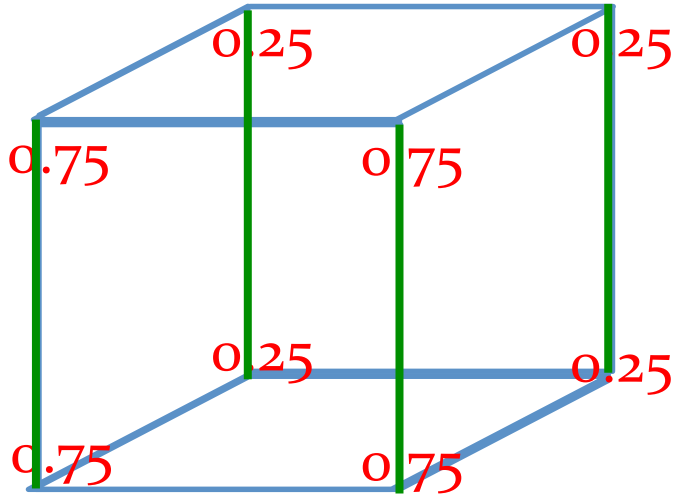


Proof Idea for Hypercube

Given f: {0,1}^{n} → {0,1}

Define

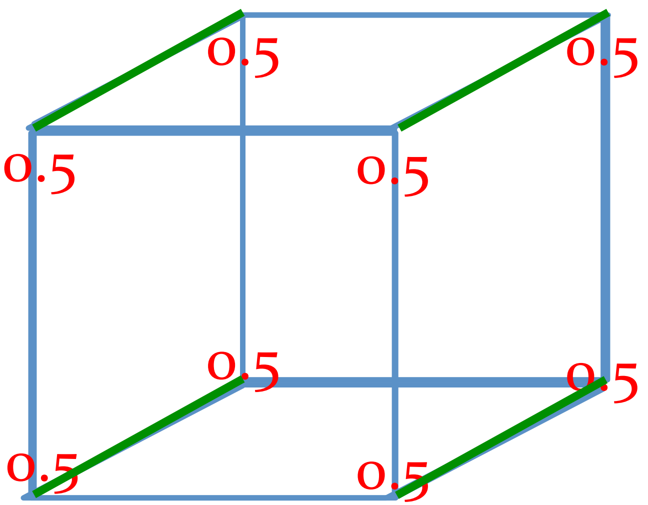
$$g_1(x) = \mathbf{E}_{x_1}\, f(x) \qquad f_1(x) = f(x) - g_1(x)$$

$$g_2(x) = \mathbf{E}_{x_2}\, g_1(x) \qquad f_2(x) = g_1(x) - g_2(x)$$

$$g_3(x) = \mathbf{E}_{x_2}\, g_2(x)$$

$$\cdots \qquad \cdots$$

$$g_n(x) = \mathbf{E}_{x_n}\, g_{n-1}(x) \quad f_n(x) = g_n(x) - g_{n-1}(x)$$



Proof Idea for Hypercube

Given f: $\{0,1\}^n \rightarrow \{0,1\}$

Define

$$f_j(x) = \mathbf{E}_{x_1,\ldots,x_j}\, f(x) - \mathbf{E}_{x_1,\ldots,x_{j-1}}\, f(x)$$

“Martingale increments”

Apply log-Sobolev inequality to each of f_j

For a general graph G, define f_j with the measure $\mu(v) = \frac{\deg(v)}{\sum_u \deg(u)}$

Proof Idea for KKL

Given f: $G^n \rightarrow \{0,1\}$

Define $f_j(x) = \mathbf{E}_{x_1,\ldots,x_j} f(x) - \mathbf{E}_{x_1,\ldots,x_{j-1}} f(x)$

Some nice properties

Zero mean $\mathbf{E}_x f_i(x) = 0$

Orthogonality $\mathbf{E}_x f_i(x) f_j(x) = 0 \text{ for } i \neq j$

$$\sum_i \mathbf{E}_x f_i(x)^2 = \mathbf{E}_x (\sum_i f_i(x))^2 = \mathbf{Var}(f) = 1$$

E-Orthogonality

$$\sum_i \mathop{\mathbf{E}}_{(x,y)\sim E(G^n)} (f_i(x) - f_i(y))^2 = \mathop{\mathbf{E}}_{(x,y)\sim E(G^n)} (f(x) - f(y))^2$$

Norm bounds

$\mathbf{E}_x f_j(x)^2 \leq \mathbf{Var}_j(f)$ $\mathbf{E}_x |f_j(x)| \leq \mathbf{Var}_j(f)$

Results for Cartesian Products

Given graph G(V,E), define

$$\mathrm{Inf}_i(f) = \text{ fraction of edges cut along } i^{\text{th}} \text{ direction}$$

$$\mathbb{I} = \sum_i \mathrm{Inf}_i(f)$$

[S-Tulsiani]

For every f: V(G^n) → {0,1}, $\exists i \;\; \mathrm{Inf}_i(f) \geq \alpha(G)\frac{\log n}{n}$

Also follows from [Cordero Erausquin-Ledoux]

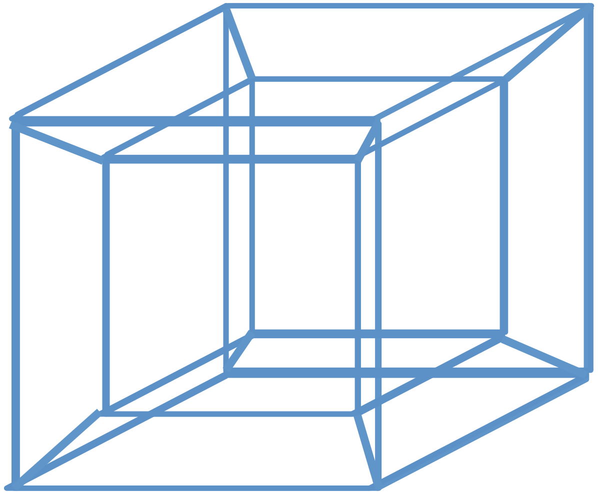
[S-Tulsiani]

Every f: V(G^n) → {0,1} is ^{2}-close to a boolean g that depends only on $\exp(\mathbb{I}/\alpha(G)\epsilon)$ coordinates

Future Directions

- Generalize the theorems to more general product settings
- How about other non-product settings?
- If ®(G) the best constant?

Thanks!



Proof idea for KKL

Let $V = \max_j \mathbf{Var}_j(f)$

Case 1: $V \geq \frac{\log n}{n}$

$$\exists i \;\; \mathrm{Inf}_i(f) \geq \Phi(G)V \geq \alpha(G)\frac{\log n}{n}$$

Case 2: $V \leq \frac{\log n}{n}$

Lemma: $\mathbf{E}\, f_j^2 \log f_j^2 \geq -\frac{\alpha(G)}{n}\left(\frac{1}{n} + \mathbf{E}\, f_j^2\right)\log nV$

$$\mathop{\mathbf{E}}_{(x,y)\in E(G^n)}(f_j(x) - f_j(y))^2 \geq -\frac{\alpha(G)}{n}\left(\frac{1}{n}\log nV + \mathbf{E}\, f_j^2 \log nV^2\right)$$

$$\mathop{\mathbf{E}}_{(x,y)\in E(G^n)}(f(x) - f(y))^2 \geq -\frac{\alpha(G)}{n}\left(\log nV + \log nV^2\right) = \Omega\left(\frac{\alpha(G)}{n}\log n\right)$$

$$\max_j \mathop{\mathrm{Inf}}_j(f) \geq \mathop{\mathbf{E}}_{(x,y)\in E(G^n)}(f(x) - f(y))^2$$