Probabilistic Programming

Daniel M. Roy

Department of Statistical Sciences
Department of Computer Science
University of Toronto

Workshop on Uncertainty in Computation
2016 Program on *Logical Structures in Computation*
Simons Institute for the Theory of Computing

1. Simple story:
   Probabilistic programming automates Bayesian inference
2. Real story:
   It's complicated

Probabilistic programming

1. Represent probability distributions by ~~*formulas*~~ **programs** ***that generate samples***.
2. Build **generic algorithms for probabilistic conditioning** using probabilistic programs as representations.

Bayesian statistics

1. Express statistical assumptions via **probability distributions**.

$$\underbrace{\Pr(\text{parameters}, \text{data})}_{\text{joint}} = \underbrace{\Pr(\text{parameters})}_{\text{prior}} \underbrace{\Pr(\text{data} \mid \text{parameters})}_{\text{model/likelihood}}$$

2. Statistical inference from data $\rightarrow$ parameters via **conditioning**.

$$\Pr(\text{parameters}, \text{data}),\ x \xmapsto{\text{conditioning}} \underbrace{\Pr(\text{parameters} \mid \text{data} = x)}_{\text{posterior}}$$

Example: simple probabilistic Python program

```
1 def binomial(n, p):
2     return sum( [bernoulli(p) for i in range(n)] )
```

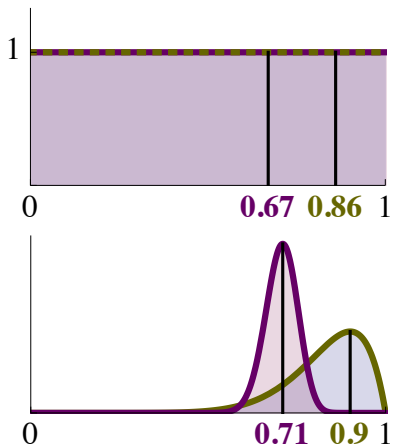
- *returns* a **random integer** in $\{0, \ldots, n\}$.
- *defines* a **family of distributions** on $\{0, \ldots, n\}$, in particular, the *Binomial family*.
- *represents* a **statistical model** of

  the # of successes among
  n independent and identical experiments

Example: simple probabilistic Python program

```
1  def binomial(n, p):
2      return sum( [bernoulli(p) for i in range(n)] )
3  def randomized_trial()
4      p_control   = uniform(0,1)  # prior
5      p_treatment = uniform(0,1)  # prior
6      return ( binomial(100, p_control),
7               binomial(10,  p_treatment) )
```

represents a **Bayesian model** of a randomized trial.



simulation $\longrightarrow$ $(\mathbf{71}, \mathbf{9})$

$\longleftarrow$ inference $(\mathbf{71}, \mathbf{9})$

The stochastic inference problem

INPUT: `guesser` and `checker` probabilistic programs.

OUTPUT: a sample from the same distribution as the program

```
accept = False
while (not accept):
    guess = guesser()
    accept = checker(guess)
return guess
```

This computation captures **Bayesian statistical inference**.

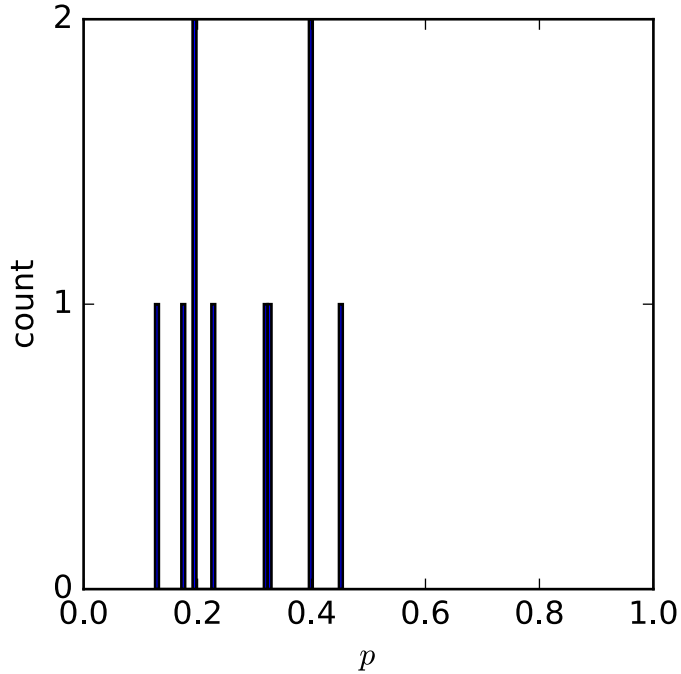
"prior" distribution $\longleftrightarrow$ distribution of `guesser()`

"likelihood(g)" $\longleftrightarrow$ $\Pr($ `checker(`g`)` is `True` $)$

"posterior" distribution $\longleftrightarrow$ distribution of return value

Example: inferring bias of a coin

```
accept = False
while (not accept):
    guess = guesser()
    accept = checker(guess)
return guess
```

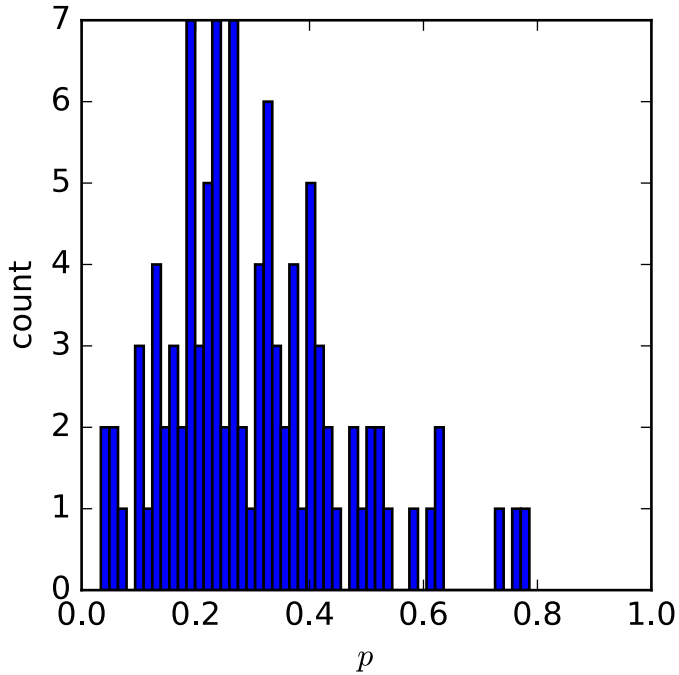


Given $x_1, \ldots, x_n \in \{0, 1\}$,
report probability of $x_{n+1} = 1$? E.g., $0, 0, 1, 0, 0$

```
def guesser():
  p = uniform()
  return p

def checker(p):
  return [0,0,1,0,0] == bernoulli(p,5)
```

Example: inferring bias of a coin



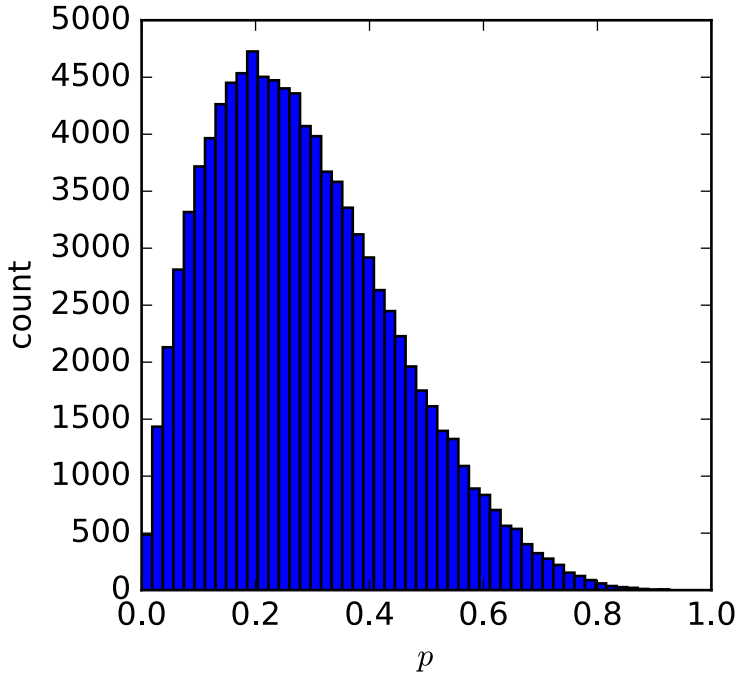

```
accept = False
while (not accept):
    guess = guesser()
    accept = checker(guess)
return guess
```

Given $x_1, \ldots, x_n \in \{0, 1\}$,
report probability of $x_{n+1} = 1$? E.g., $0, 0, 1, 0, 0$

```
def guesser():
  p = uniform()
  return p

def checker(p):
  return [0,0,1,0,0] == bernoulli(p,5)
```

Example: inferring bias of a coin




```
accept = False
while (not accept):
    guess = guesser()
    accept = checker(guess)
return guess
```

Given $x_1, \ldots, x_n \in \{0, 1\}$,
report probability of $x_{n+1} = 1$? E.g., $0, 0, 1, 0, 0$

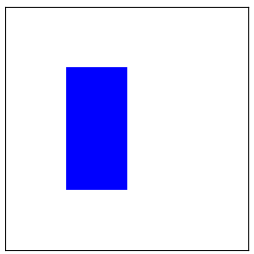
```
def guesser():
  p = uniform()
  return p

def checker(p):
  return [0,0,1,0,0] == bernoulli(p,5)
```

Example: inferring objects from an image

```
accept = False
while (not accept):
    guess = guesser()
    accept = checker(guess)
return guess
```

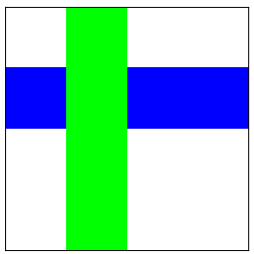
How many objects in this image?



Example: inferring objects from an image

```
accept = False
while (not accept):
    guess = guesser()
    accept = checker(guess)
return guess
```

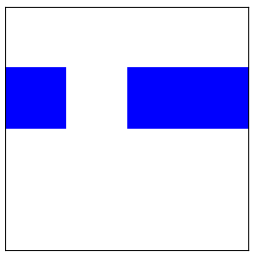
How many objects in this image?



Example: inferring objects from an image

```
accept = False
while (not accept):
    guess = guesser()
    accept = checker(guess)
return guess
```

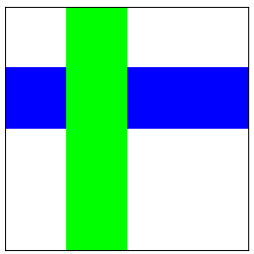
How many objects in this image?



Example: inferring objects from an image

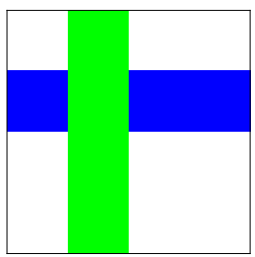
```
accept = False
while (not accept):
    guess = guesser()
    accept = checker(guess)
return guess
```

How many objects in this image?



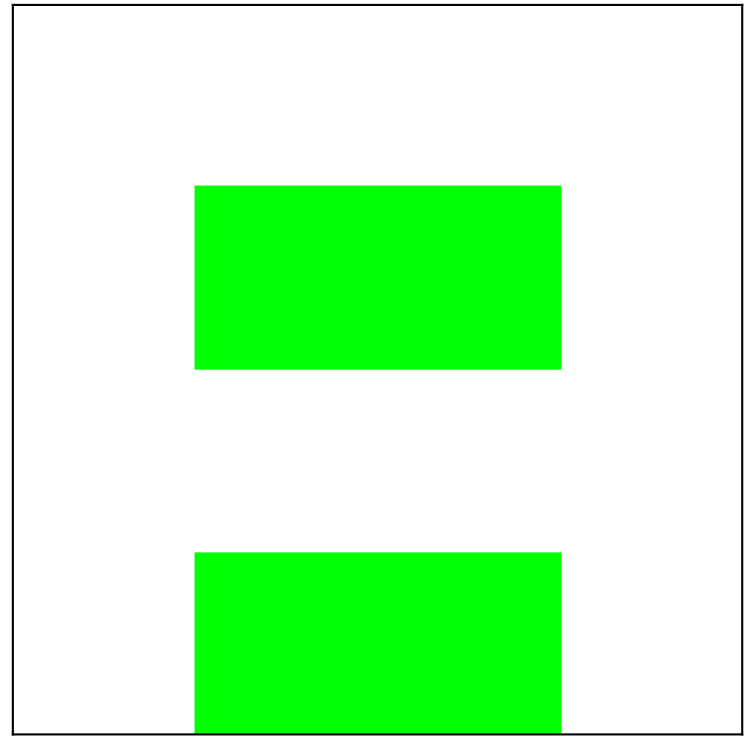
```
def guesser():
  k = geometric()
  blocks = [ randomblock() for _ in range(k) ]
  colors = [ randomcolor() for _ in range(k) ]
  return (k,blocks,colors)

def checker(k,blocks,colors):
  return rasterize(blocks,colors) ==
```



Example: inferring objects from an image

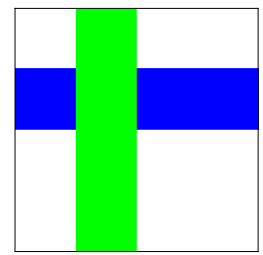
```
accept = False
while (not accept):
    guess = guesser()
    accept = checker(guess)
return guess
```



How many objects in this image?

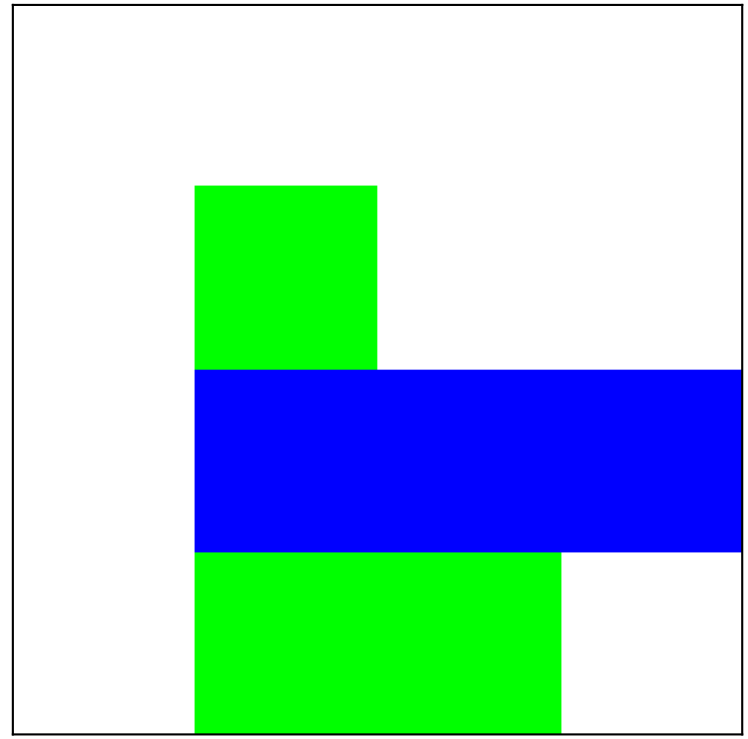
```
def guesser():
  k = geometric()
  blocks = [ randomblock() for _ in range(k) ]
  colors = [ randomcolor() for _ in range(k) ]
  return (k,blocks,colors)

def checker(k,blocks,colors):
  return rasterize(blocks,colors) ==
```



Example: inferring objects from an image

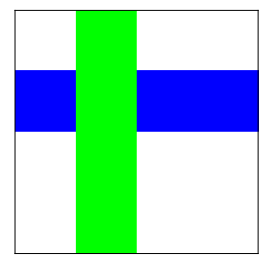
```
accept = False
while (not accept):
    guess = guesser()
    accept = checker(guess)
return guess
```



How many objects in this image?

```
def guesser():
  k = geometric()
  blocks = [ randomblock() for _ in range(k) ]
  colors = [ randomcolor() for _ in range(k) ]
  return (k,blocks,colors)

def checker(k,blocks,colors):
  return rasterize(blocks,colors) ==
```



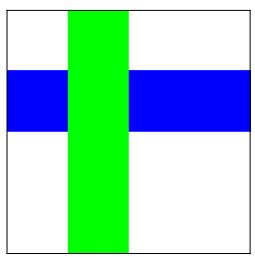
Example: inferring objects from an image

```
accept = False
while (not accept):
    guess = guesser()
    accept = checker(guess)
return guess
```

How many objects in this image?

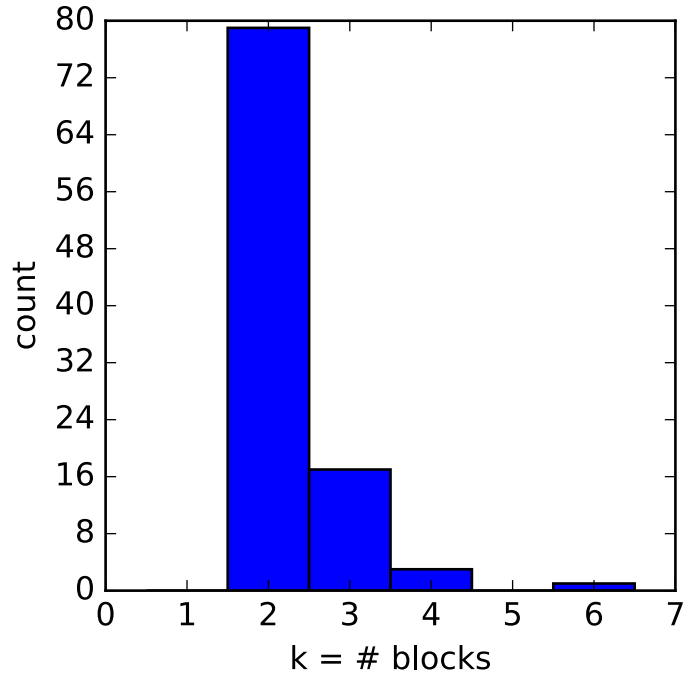
```
def guesser():
  k = geometric()
  blocks = [ randomblock() for _ in range(k) ]
  colors = [ randomcolor() for _ in range(k) ]
  return (k,blocks,colors)

def checker(k,blocks,colors):
  return rasterize(blocks,colors) ==
```



Example: inferring objects from an image

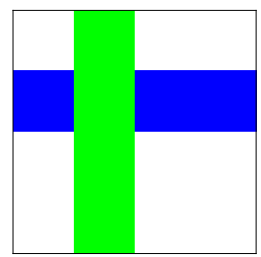
```
accept = False
while (not accept):
    guess = guesser()
    accept = checker(guess)
return guess
```



How many objects in this image?

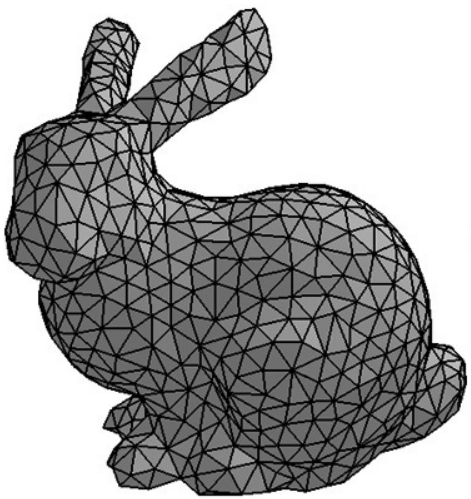
```
def guesser():
  k = geometric()
  blocks = [ randomblock() for _ in range(k) ]
  colors = [ randomcolor() for _ in range(k) ]
  return (k,blocks,colors)

def checker(k,blocks,colors):
  return rasterize(blocks,colors) ==
```



Fantasy example: extracting 3D structure from images

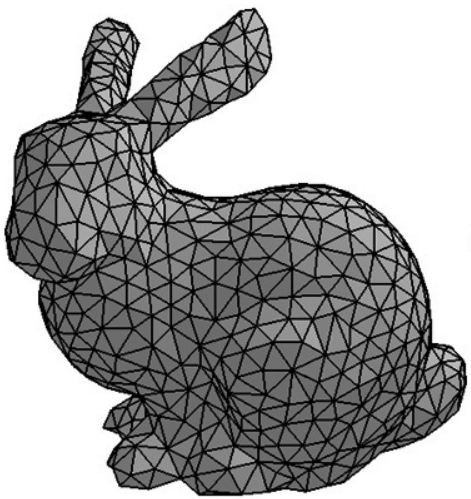
```
accept = False
while (not accept):
    guess = guesser()
    accept = checker(guess)
return guess
```



checker

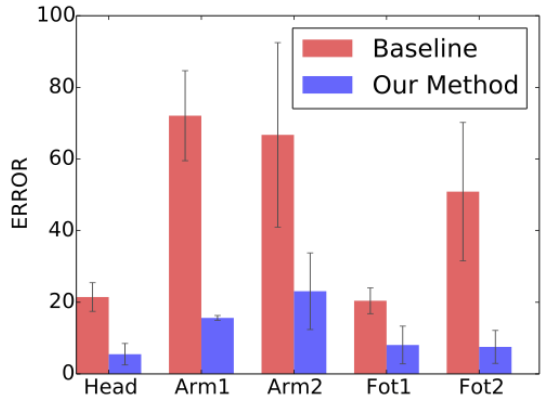
Fantasy example: extracting 3D structure from images

```
accept = False
while (not accept):
    guess = guesser()
    accept = checker(guess)
return guess
```

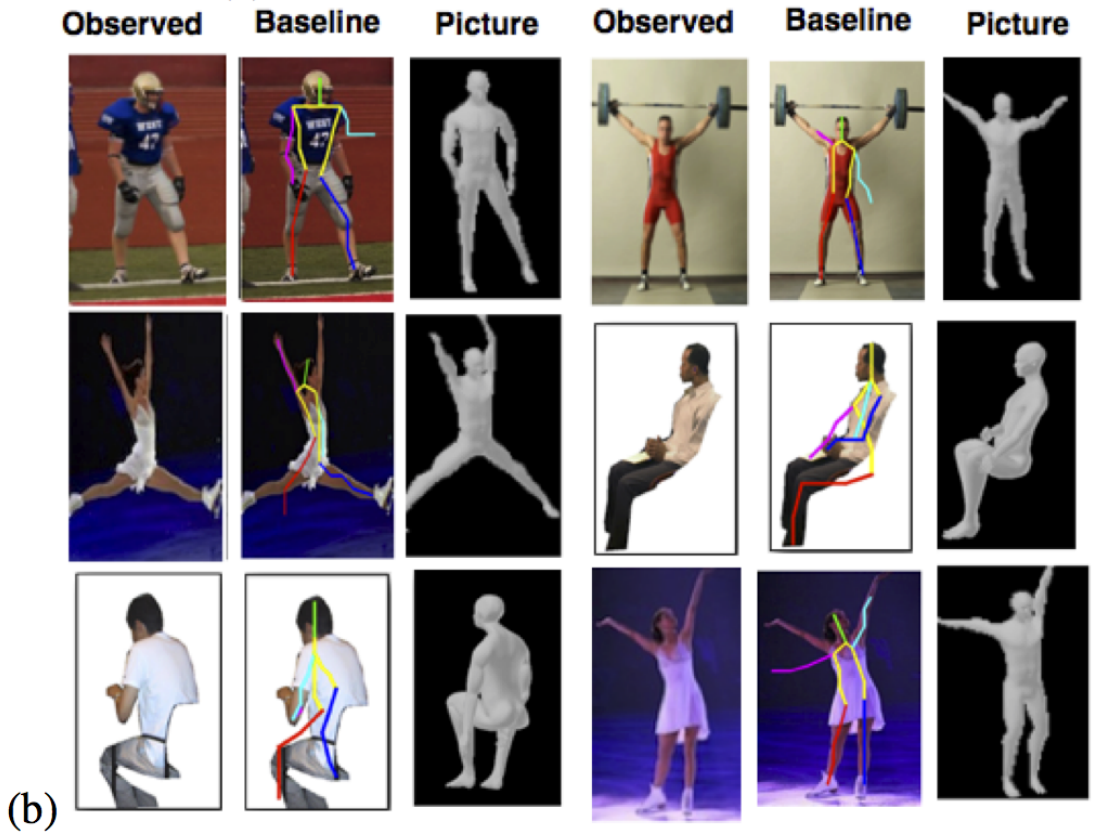


inference ←

Example from Mansinghka's group at MIT



(a)



(b)

Probabilistic programs defining unbounded distributions

```
1  def geometric(p):
2      if bernoulli(p) == 1: return 0
3      else: return 1 + geometric(p)
```

represents the Geometric distribution with mean $1/p - 1$.

Note: no bound on running time! Only halts with probability 1!

A sampler that halts with probability one is called **a.e. computable**.

A sampler that always halts is called **computable**.

Theorem. *The set of distributions with a.e. computable samplers is a strict superset of those with computable samplers.*

Conditioning as a higher-order procedure

```
1  def condition(guesser,checker):
2      # guesser: Unit -> S
3      # predicate : S -> Boolean
4      accept = False
5      while (not accept)
6          guess = guesser()
7          accept = checker(guess)
8      return guess
```

represents the higher order operation of **conditioning**. When `checker` is deterministic, then

$$(P, \mathbf{1}_A) \mapsto P(\,\cdot\,|A) \equiv \frac{P(\,\cdot \cap A)}{P(A)}.$$

Halts with probability 1 provided $P(A) > 0$.

`condition` as an algorithm

Key point: `condition` is not a serious proposal for an *algorithm*, but it denotes the operation we care about in Bayesian analysis.

How efficient is condition? Let `model()` represent a distribution P and `pred` represent an indicator function $\mathbf{1}_A$.

Proposition. In expectation, `condition(model,pred)` takes $\frac{1}{P(A)}$ times as long to run as `pred(model())`.

Corollary. *If* `pred(model())` *is efficient and* $P(A)$ *not too small, then* `condition(model,pred)` *is efficient.*

An efficient "version" of `condition`

State-of-the-art CHURCH engines work by MCMC, performing a random walk over the possible executions of `model()`. These engines are complex, but we can ignore polynomial factors to get a much simpler algorithm:

```
# assume: r_model(random()) =d= model()
#       : perturb(random()) =d= random()
#         (and ergodic w.r.t. random() on every meas. set)
def mcmc_condition(r_model,predicate,n):
    entropy = random()
    for _ in range(n):
        new_entropy = perturb(entropy)
        if predicate(r_model(new_entropy)) == True:
            entropy = new_entropy
    return r_model(new_entropy)
```

approximates `condition` to arbitrary accuracy as $n \to \infty$.

Perturb and traces

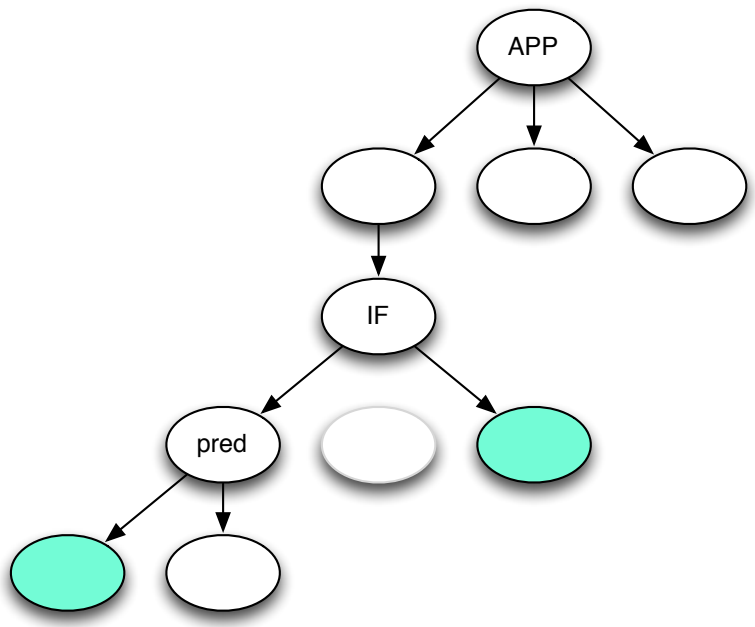
The key to this working reasonably well is `perturb` and to understand perturb we've got to understand "traces".

A traces is a tree data structure the captures the random choices encountered and the path taken by the interpreter while evaluating the probabilistic program.

Traces have two key parts.

1. Random primitives.
2. Applications/control-flow.

The trace is determined by the values of the random primitives. Changes to these primitives can modify the control flow.



Goal of perturb is to efficiently take small steps in the space of traces.

Perturb and traces

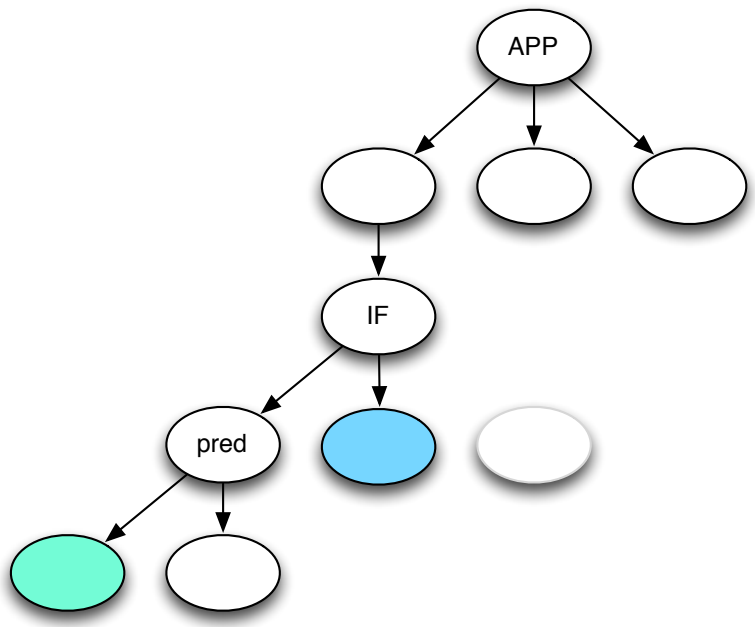
The key to this working reasonably well is `perturb` and to understand perturb we've got to understand "traces".

A traces is a tree data structure the captures the random choices encountered and the path taken by the interpreter while evaluating the probabilistic program.

Traces have two key parts.

1. Random primitives.
2. Applications/control-flow.

The trace is determined by the values of the random primitives. Changes to these primitives can modify the control flow.



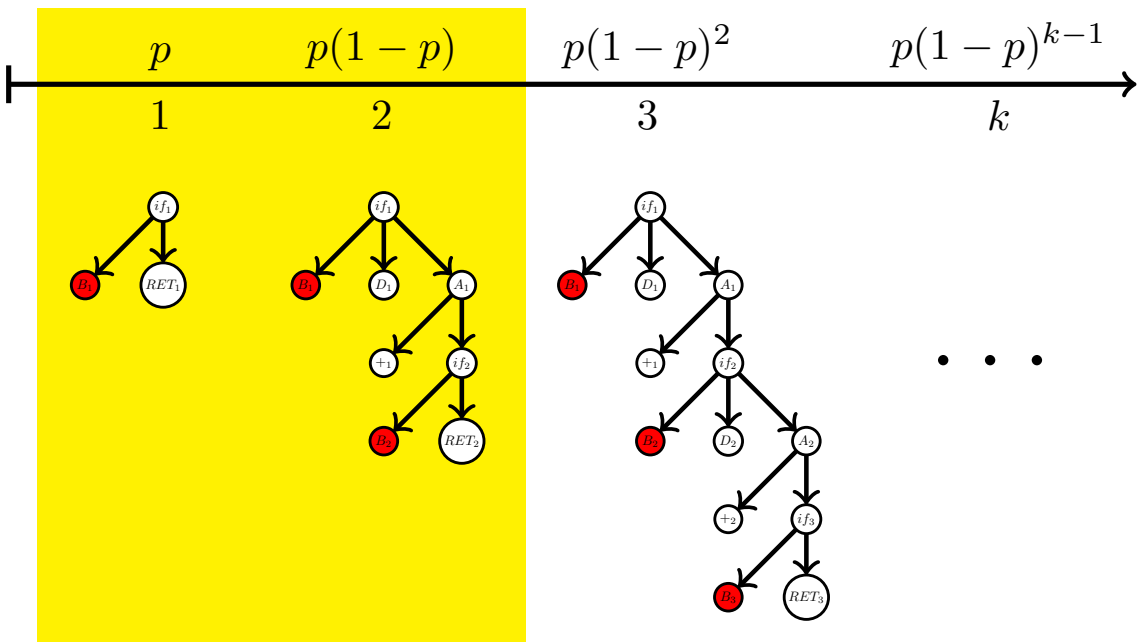
Goal of perturb is to efficiently take small steps in the space of traces.

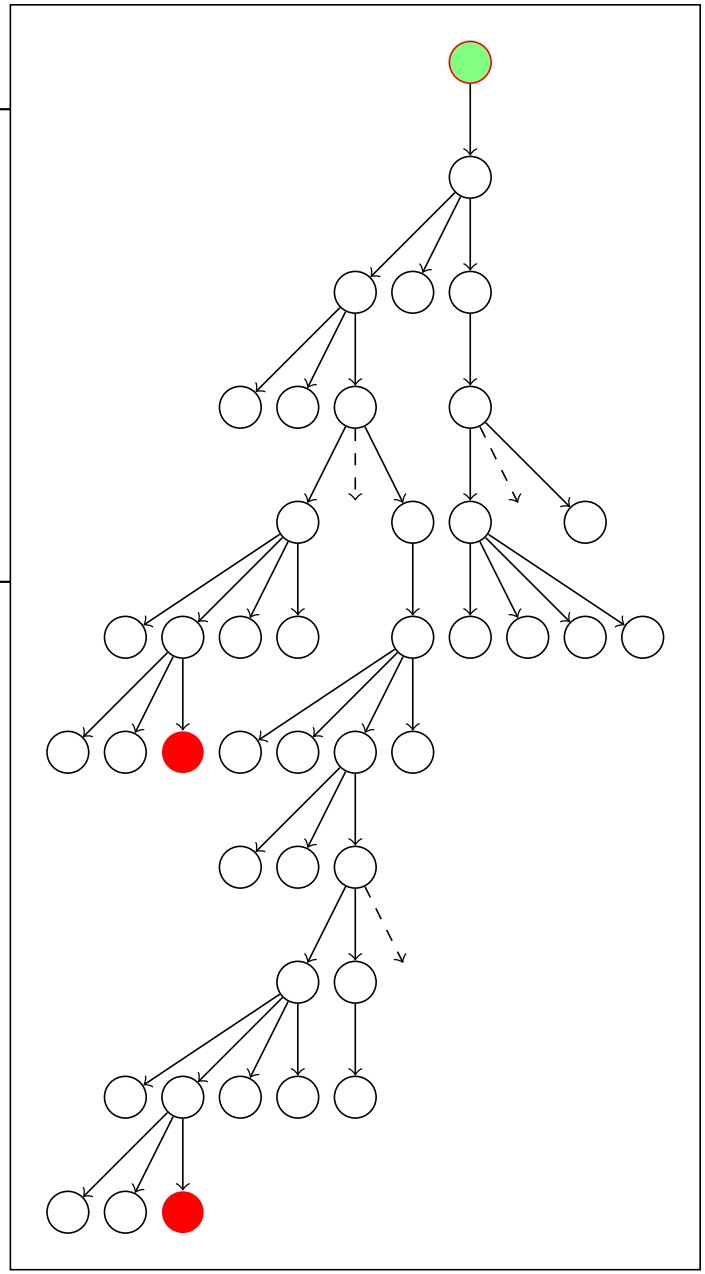
“Universal” MCMC inference for probabilistic programs

MIT-Church [GM**R**+08]

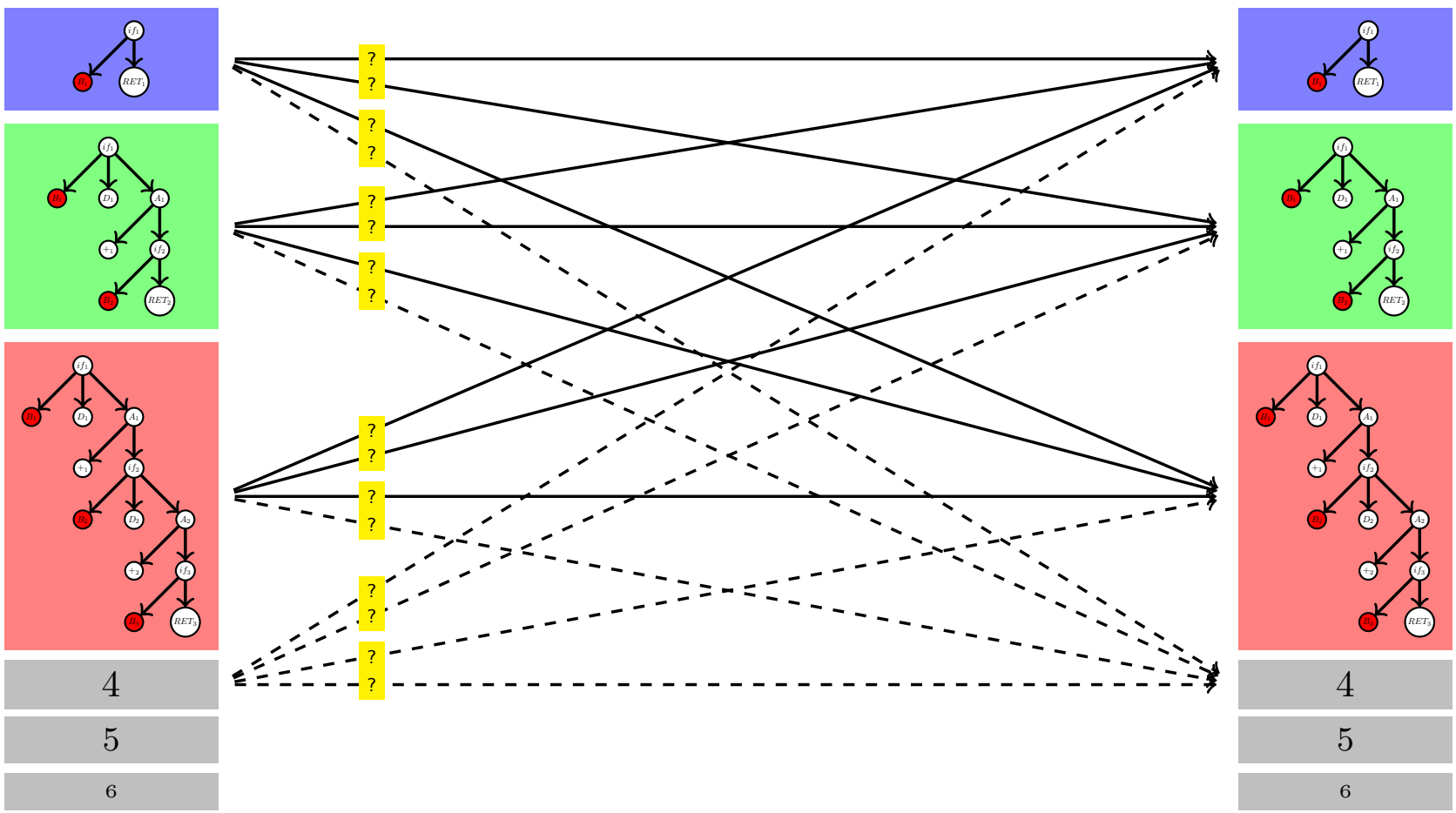
```
1  def geometric(p):
2      if bernoulli(p) == 1: return 1
3      else: return 1 + geometric(p)
```

```
1  def aliased_geometric(p):
2      g = geometric(p)
3      return 1 if g < 3 else 0
```

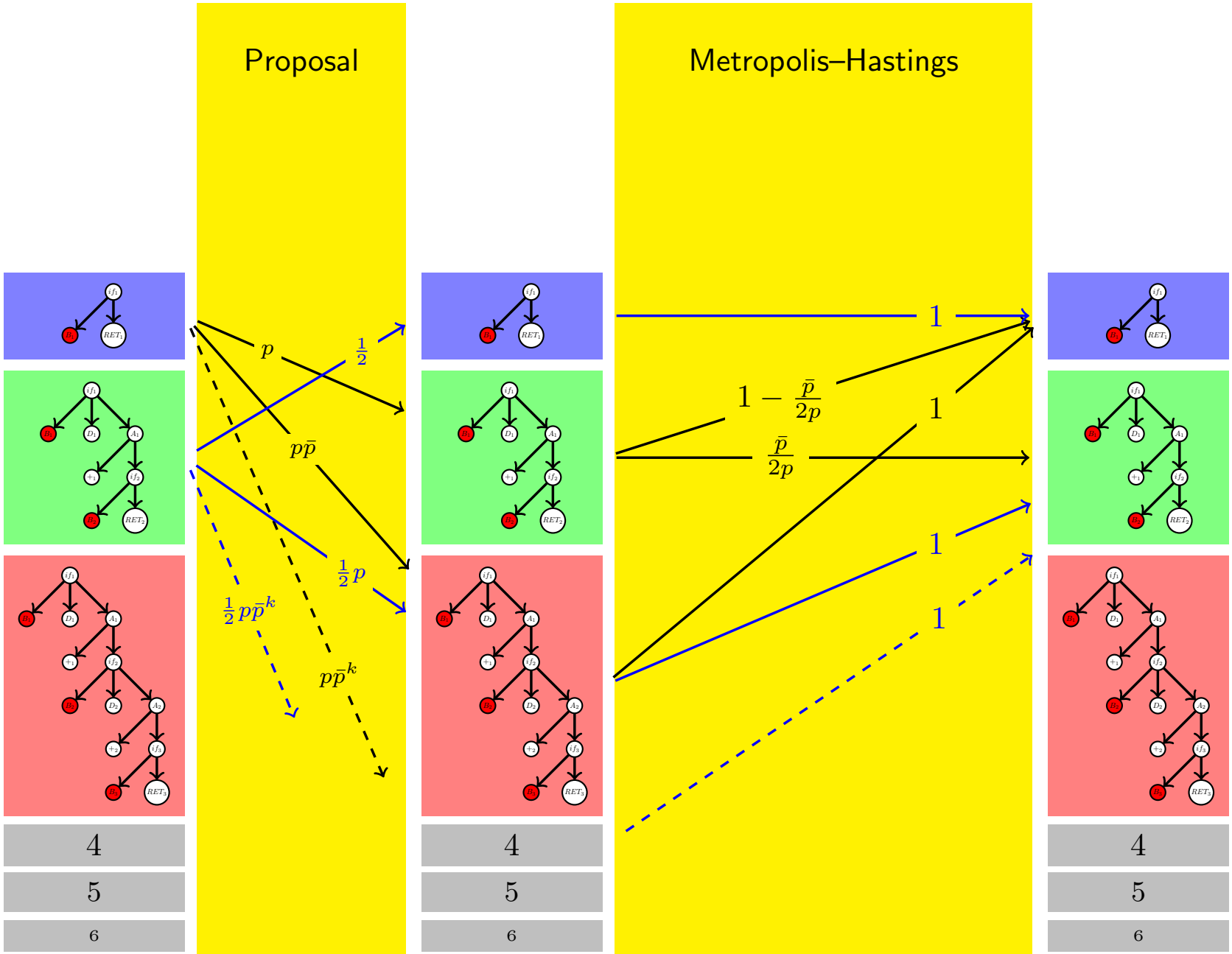




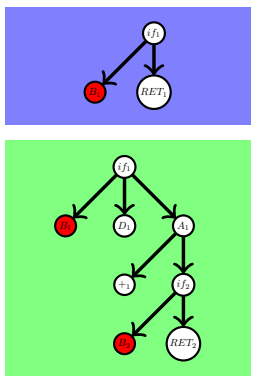
MIT-Church [GM**R**+08]

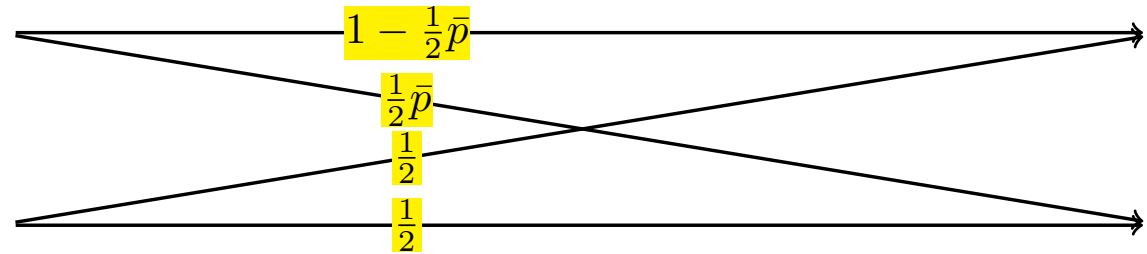


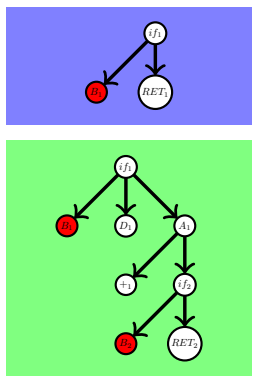
MIT-Church [GM**R**+08]

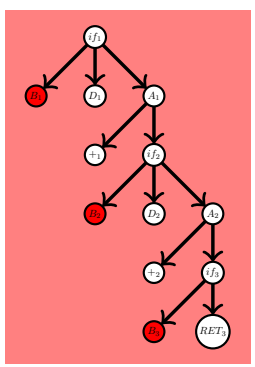


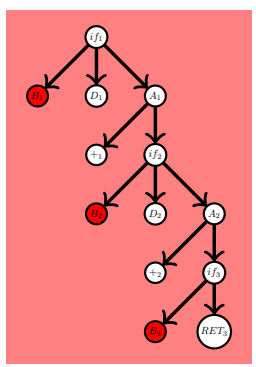
MIT-Church [GM**R**+08]











$$\begin{pmatrix} 1-\frac{1}{2}\bar{p} & \frac{1}{2}\bar{p} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}^k \to \begin{pmatrix} \frac{p}{p+p\bar{p}} & \frac{\bar{p}}{p+p\bar{p}} \\ \frac{p}{p+p\bar{p}} & \frac{\bar{p}}{p+p\bar{p}} \end{pmatrix} \quad \text{as } k \to \infty$$

4

5

6

4

5

6

The real story:

Probabilistic programming ~~automates~~ is game changing but messy

What is probabilistic programming?

Simple story from Gordon et al. (2014). Probabilistic Programming. ICSE.

The goal of probabilistic programming is to enable probabilistic modeling and machine learning to be accessible to the working programmer, who has sufficient domain expertise, but perhaps not enough expertise in probability theory or machine learning. We wish to hide the details of inference inside the compiler and run-time, and enable the programmer to express models using her domain expertise and dramatically increase the number of programmers who can benefit from probabilistic modeling.

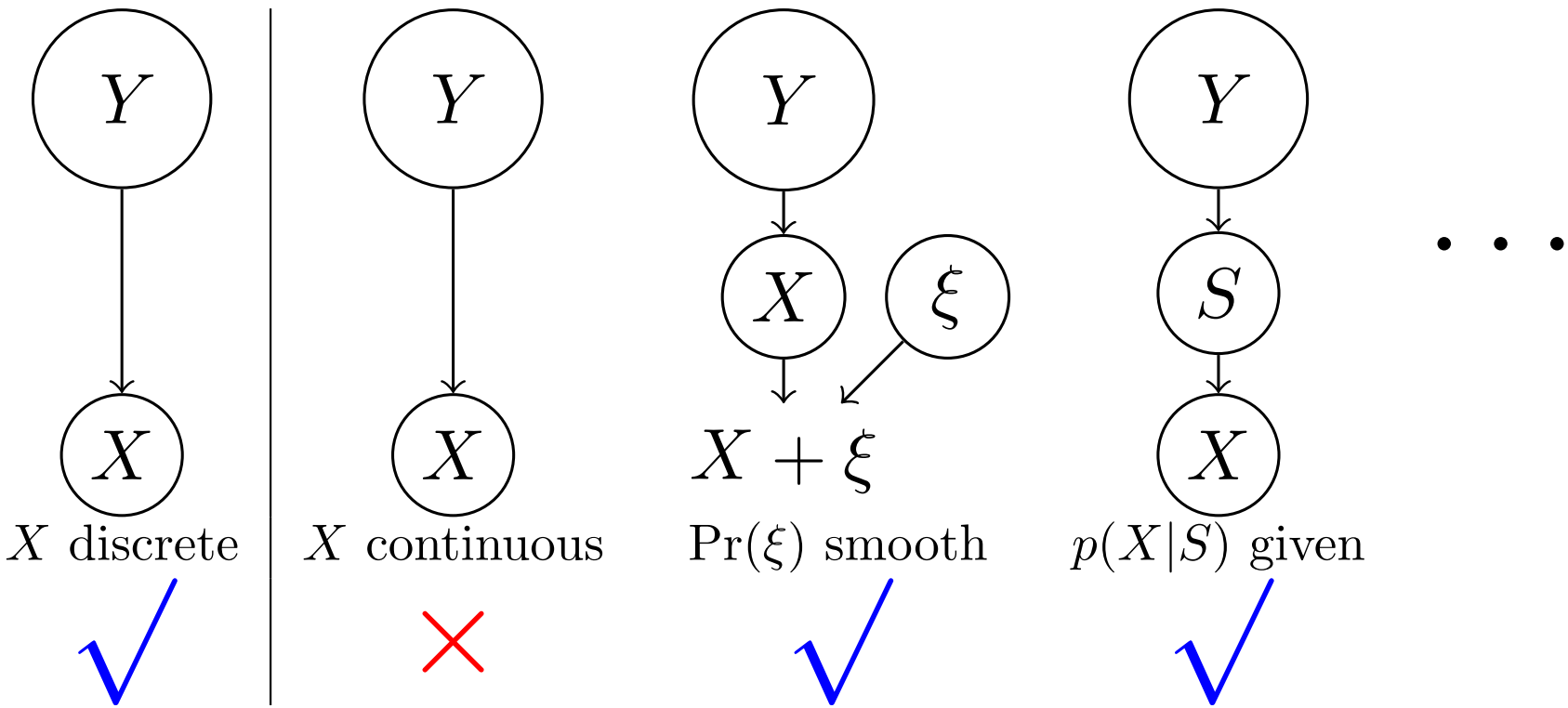
This is essentially the perspective the subfield took in 2008 at the 1st NIPS Workshop on Probabilistic Programming. It has turned out to be possible for simple models with fixed structure, but constant work is required on the part of system designers to keep cracks from appearing in this facade as users then push beyond these simple models.

Many systems now diverge from presenting this facade. E.g., in Venture, probabilistic programs are interactions with an approximate inference engine. Users control aspects of inference, which is emphasized to be approximate, but convergent.

Q: Can we automate Bayesian reasoning?

$$\Pr(X, Y),\ x \longmapsto \Pr(Y|X = x)$$

A: No, but almost.



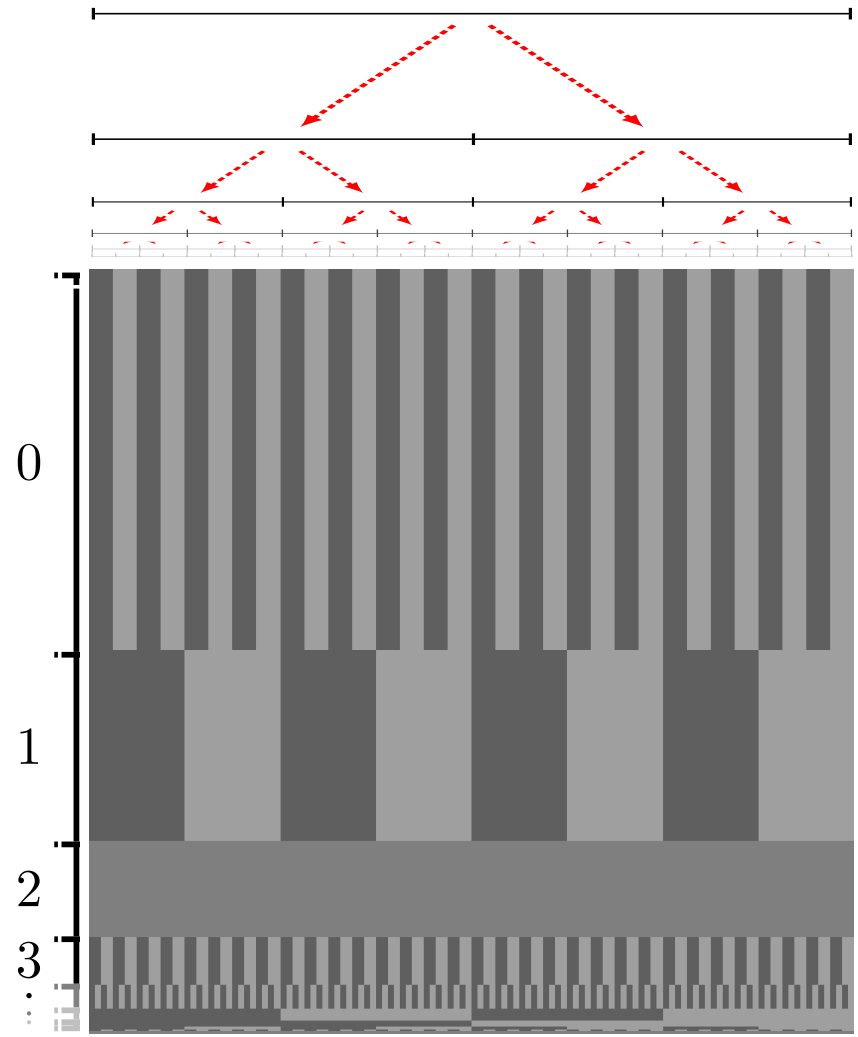
[Freer and **R.**, 2010] [Ackerman, Freer, and **R.**, 2011] ...

The halting distribution

(Ackerman, Freer, and **R.**, 2011)

$$Y \sim \text{Uniform}[0,1]$$
$$N := \lfloor -\log_2 Y \rfloor$$
$$X|Y \sim \mathsf{HaltingTime}(M_N)$$

$$\frac{\Pr(N = \mathbf{n} \mid X = x)}{\Pr(N =^* \mid X = x)} \cdot 2^{\mathbf{n}-^*}$$
$$\in \begin{cases} \{2/3,\ 4/3\} & M_{\mathbf{n}} \text{ halts;} \\ \{1\} & \text{otherwise.} \end{cases}$$

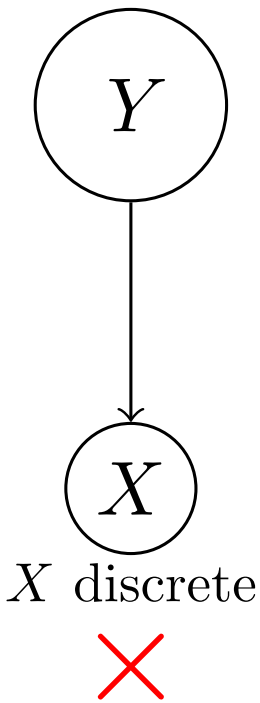


Theorem (Ackerman, Freer, and R., 2011).
The halting problem is computable from $Pr(Y \mid X = x)$.

Q: What about efficient inference?

$$\Pr(X, Y),\ x \longmapsto \Pr(Y|X = x)$$

A: No, of course, but...




```
def hash_of_random_string(n):
    str = random_binary_string(n)
    return cryptographic_hash(str)
```

Bayes nets, Tree Width, ...

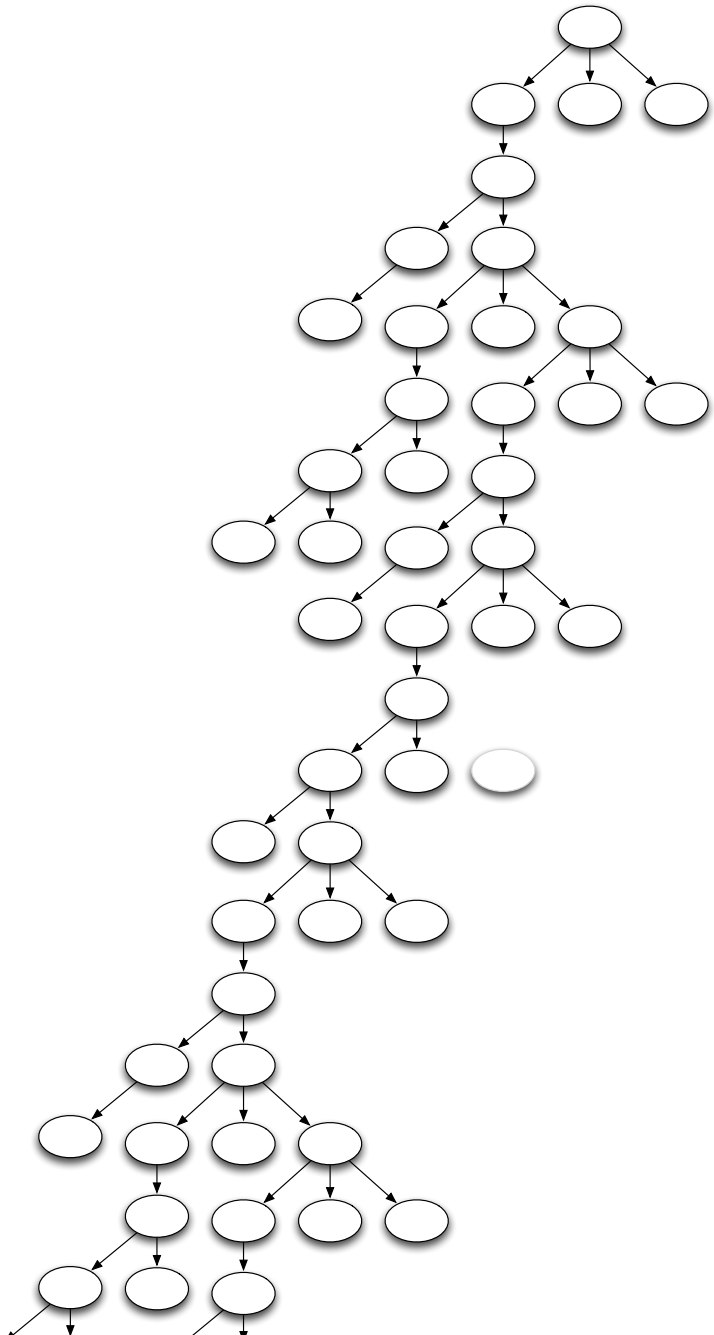
Conditional independence is importance!

Conditional independence and traces

Conditional independence looks flat in a trace.

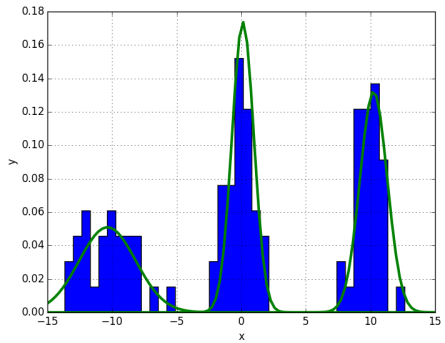
1. bad shape: deep
   no conditional independence
2. good shape: shallow
   lots of conditional
   independence

Key idea: There are multiple ways to implement the same distribution. Choose one that keeps the trace shallow.

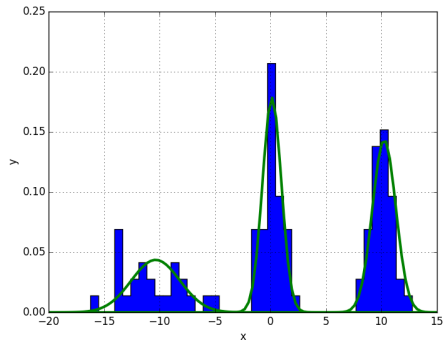


Measuring the reliability of MCMC is hard

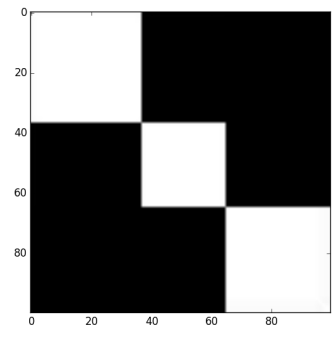
Standard diagnostics are heuristic and can only hint that something is wrong, rather than guarantee that answers are reliable.



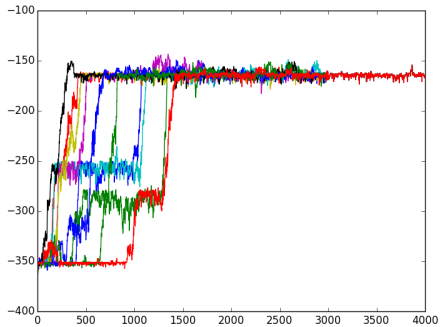
(a) Posterior distribution



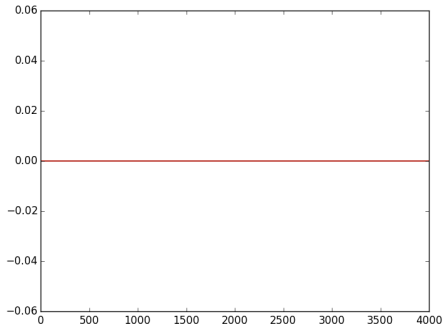
(b) Predictive distribution



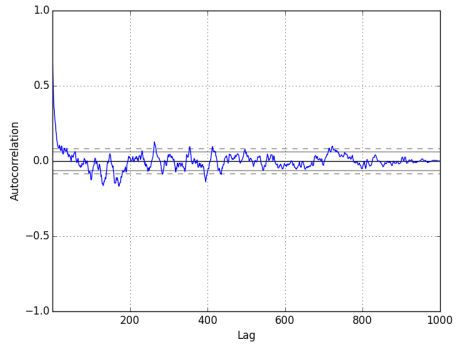
(c) Clusters



(d) Log-likelihood



(e) Complete Log-likelihood



(f) Autocorrelation (LogLik)

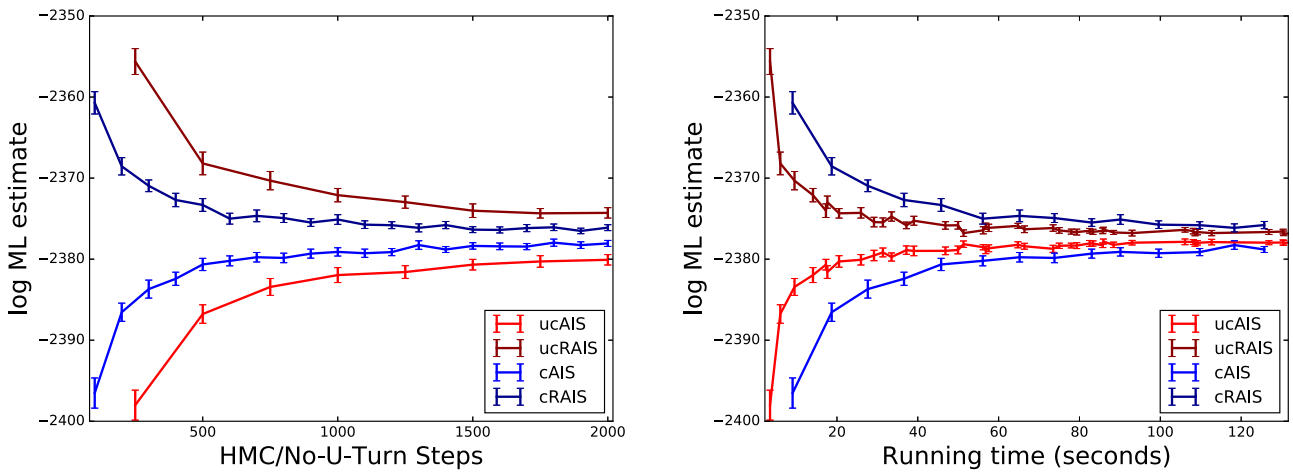
Bidirectional Monte Carlo [GAG16]

New tools based on BDMC (joint work with Grosse and Ancha, NIPS 2016, see also Cusumano–Mansinghka arXiv:1606.00068) provide first rigorous approach to diagnosing chains on synthetic data.

REVERSE AIS: $p_t \to p_1$ yields *upper* bound Z^+ on marginal likelihood

FORWARD AIS: $p_1 \to p_t$ yields *lower* bound Z^- on marginal likelihood

Theorem [GAR16,CM16]. Difference $Z^+ - Z^-$ bounds MCMC approximation error in expectation.



Aspects of probabilistic programming

- Statistical models are *models*
  - Box: "All models are wrong, some are useful."
    - Claim: We have a license to approximate
      (Bigf difference with logic programming)
    - Stronger claim: exact inference not possible in general
    - Stronger still: accuracy of approximate inference usually unknown, but good empirical performance on task suffices
- "Right" semantics for probabilistic program depends on inference
  - Inference is an *extremely* complex transformation:
    probabilistic progra $\longleftrightarrow$ inference program
  - Distributional semantics: too coarse to capture this interaction
  - Most general languages demand a lot of users.
    - In contrast, Stan.
- Do we know how to write good probabilistic programs?
- Efficient approximations rule
  - Conditioning is hard
  - Whether an approximation is reasonable is often problem specific

Conclusions

- Probabilistic programming for machine learning and statistics is a grand challenge for programming languages
- Underlying computational problem is much harder than execution
- Extreme approximation is licensed
- Current users must understand how to interact with underlying inference