Controlling probabilistic systems under partial observation an automata and verification perspective

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Partially observable probabilistic systems

- Why probabilities? randomized algorithms, unpredictable behaviours, abstraction of non-determinism
- Why partial observation? abstraction of large systems, security concerns

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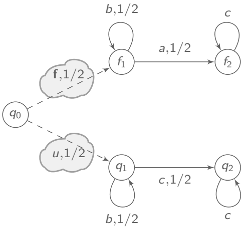

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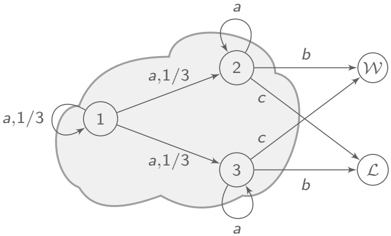

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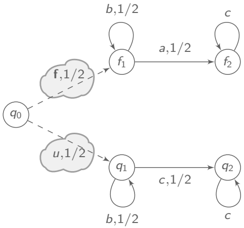
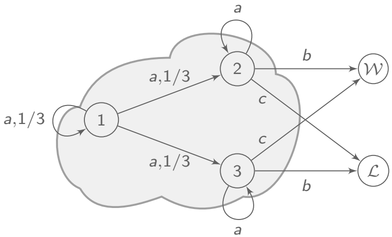




- language-theoretic questions: languages defined by prob. automata
- monitoring issues: fault diagnosis, supervision, etc.
- **control problems**: optimization for a given objective

Outline

Probabilistic automata

Partially observable MDP

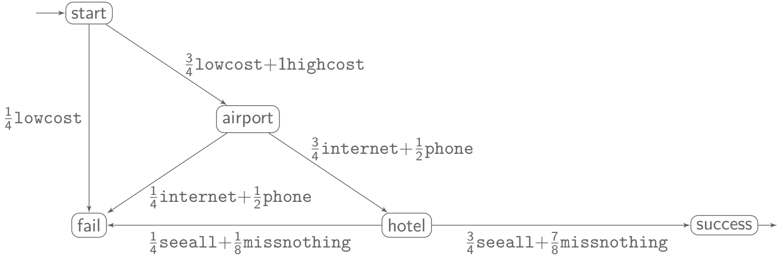
Discussion

Motivating example for probabilistic automata (PA)

Planning holidays *in advance*:

1. choose an airline type (`lowcost`/`highcost`);
2. book accommodation (`internet`/`phone`);
3. choose tour (`seeall`/`missnothing`).

each action fails with some probability



success probability of plan `lowcost` · `internet` · `seeall` is $\frac{27}{64}$.

Control strategies in PA

Strategies are words
what is the probability to reach a final state after word w?

The acceptance probability of $w = a_1 \ldots a_n$ by $\mathcal{A}$ is:

$$\mathbf{Pr}_{\mathcal{A}}(w) = \sum_{q \in Q} \pi_{\mathbf{0}}[q] \sum_{q' \in F} \left(\prod_{i=1}^{n} \mathbf{P}_{a_i} \right) [q, q'] = \pi_0 \mathbf{P}_w \mathbf{1}_F^T$$

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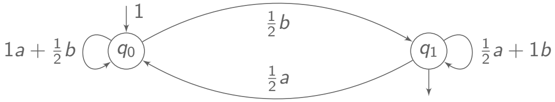
Optimal strategies may not exist



$$\mathbf{Pr}_{\mathcal{A}}(a_1 \dots a_n) = \sum_{i=1}^{n} 2^{i-n-1} \cdot \mathbf{1}_{a_i = b}$$

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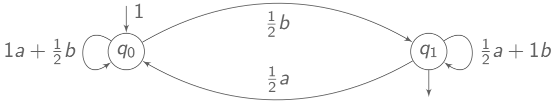

$$\mathbf{Pr}_{\mathcal{A}}(a_1 \dots a_n) = \sum_{i=1}^{n} 2^{i-n-1} \cdot \mathbf{1}_{a_i = b}$$

$\longrightarrow$ Find good enough strategies, *i.e.* that guarantee a given probability

Existence of good-enough strategies

$$L_{\bowtie\theta}(\mathcal{A}) = \{w \in A^* \mid \mathbf{Pr}_{\mathcal{A}}(w) \bowtie \theta\}$$

The problem, given a PA $\mathcal{A}$ of telling whether $L_{\geq\frac{1}{2}}(\mathcal{A}) \neq \emptyset$ is undecidable. Paz'71

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Undecidability is robust

refined emptiness assuming that for $\epsilon > 0$ either $\exists w\ \mathbf{Pr}_{\mathcal{A}}(w) \geq 1 - \varepsilon$ or $\forall w\ \mathbf{Pr}_{\mathcal{A}}(w) < \varepsilon$, decide which is the case Condon *et al.*'03

value one problem does there exist $(w_n)_{n\in\mathbb{N}}$ such that $\limsup_n \mathbf{Pr}_{\mathcal{A}}(w_n) = 1$? Gimbert and Oualhadj'10

parametric probability values does there exist a valuation of probabilities such that $\mathcal{A}$ has value one? Fijalkow *et al.*'14

Anything decidable?

Almost-sure language: $L_{=1}(\mathcal{A})$

Emptiness of almost-sure language is PSPACE-complete.

equivalent to universality problem for NFA

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Quantitative language equivalence

Input: $\mathcal{A}$ and $\mathcal{A}'$ PA

Output: yes iff $\forall w \in A^* \ \mathbf{Pr}_{\mathcal{A}}(w) = \mathbf{Pr}_{\mathcal{A}'}(w)$

Quantitative language equivalence is decidable in PTIME.

Schützenberger'61, Tzeng'92

linear algebra argument

polynomial bound on length of counterexample to equivalence

Recap on Probabilistic Automata

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What if we execute a plan, but have feedback, and can modify the plan?
partially observable Markov decision processes

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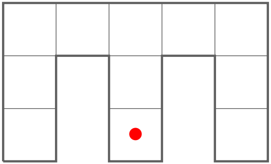
Example of partially observable MDP (POMDP)

McCallum maze: robot with limited sensor abilities, and imperfect moves



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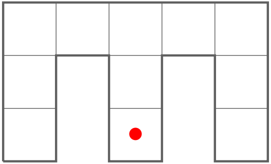

- robot only sees walls surrounding it, not the precise cell
  observations $\Omega = \{\{L, U\}, \{U, D\}, \{U, R\}, \{L, D, R\} \cdots\}$
- actions $A = \{N, W, S, E\}$ are not implemented accurately
  action N leads to north with probability $\frac{2}{3}$ and others with $\frac{1}{3}$

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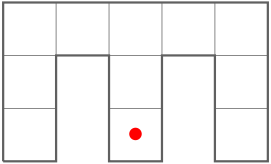

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Reachability objective: move to target cell ●
Optimization: minimum expected time

Strategies

Strategy: maps *history* $\rho \in (A\Omega)^*$ with distribution over actions;

$$\nu : (A\Omega)^* \to \mathsf{Dist}(A)$$

$\nu(\rho, a)$: probability that a is chosen given history ρ

- **pure** strategy: all distributions are Dirac
- **belief-based** strategy: based on set of current possible states

word in PA $\Longleftrightarrow$ pure strategy in POMDP with $|\Omega| = 1$

Consequence: all hardness results lift from PA to POMDP

Infinite horizon objectives

Objectives Reachability F visited at least once:

$$\Diamond F = \{q_0 q_1 q_2 \cdots \in S^\omega \mid \exists n,\ q_n \in F\}$$

Safety always stay in F:

$$\Box F = \{q_0 q_1 q_2 \cdots \in S^\omega \mid \forall n,\ q_n \in F\}$$

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Goal: For φ an objective, evaluate $\sup_\nu \mathbb{P}^\nu(\mathcal{M} \models \varphi)$.

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Pure strategies suffice!

For every strategy ν, there exists a pure strategy ν' such that $\mathbb{P}^\nu(\mathcal{M} \models \varphi) \leq \mathbb{P}^{\nu'}(\mathcal{M} \models \varphi)$. Chatterjee *et al.*'15

Undecidability results

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positive repeated reachability does there exist ν such that
$\mathbb{P}^{\nu}(\mathcal{M} \models \Box\Diamond F) > 0$? Baier *et al.*'08

combined objectives does there exist ν such that $\mathbb{P}^{\nu}(\mathcal{M} \models \Box\Diamond F_1) = 1$ and $\mathbb{P}^{\nu}(\mathcal{M} \models \Box F_2) > 0$? Bertrand *et al.*'14

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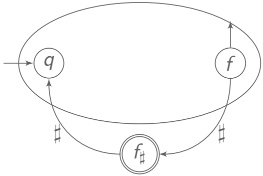
Proof of first statement: reduction from the value one problem for PA



pure strategies in $\mathcal{M}$: $\nu_{\mathbf{w}} = w_1\sharp\sharp w_2\sharp\sharp w_3\cdots$

$$\text{val}(\mathcal{A}) = 1 \iff \exists (w_i)_{i\in\mathbb{N}} \prod_i \mathbb{P}_{\mathcal{A}}(w_i) > 0$$

$$\iff \exists \nu_{\mathbf{w}}\ \mathbb{P}^{\nu_{\mathbf{w}}}(\mathcal{M} \models \Box\Diamond f_\sharp) > 0$$

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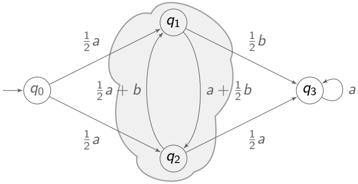

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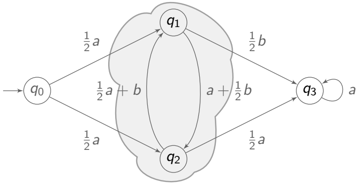

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Open: decidability of non-null proportion with positive probability

$$\exists\nu,\ \mathbb{P}^\nu(\mathcal{M} \models \limsup_n \frac{\#\text{visits to } F \text{ in } n \text{ first steps}}{n} > 0) > 0?$$

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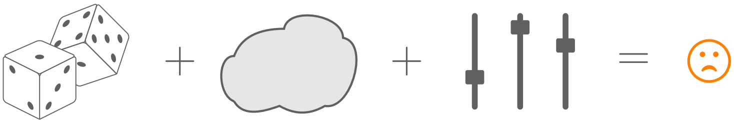
Life is hard...



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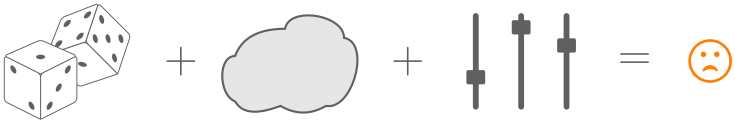

most optimization problems are undecidable

- notably quantitative questions
- but also some qualitative questions
- and undecidability is robust

... but there is still hope

- usual way arounds
  - decidable subclasses — Fijalkow *et al.*'12
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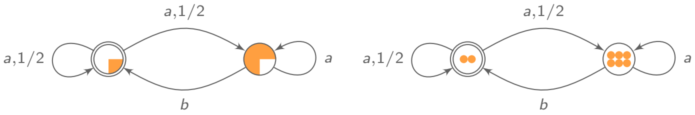

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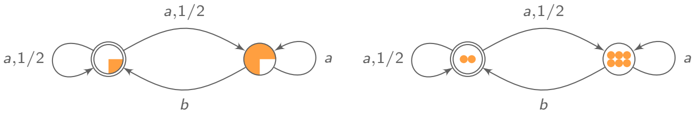

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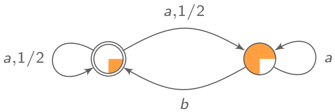

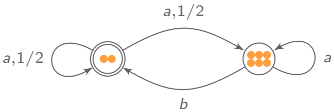

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