Logic and Quantum Information
Lecture V: Mere Possibilities

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Generalizing Distributions

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Recall firstly that a probability distribution of finite support on a set X can be specified as a function

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where $\mathbb{R}_{\geq 0}$ is the set of non-negative reals, satisfying the normalization condition

$$\sum_{x \in X} d(x) \;=\; 1.$$

This guarantees that the range of the function lies within the unit interval $[0, 1]$.

The finite support condition means that d is zero on all but a finite subset of X. The probability assigned to an event $E \subseteq X$ is then given by

$$d(E) \;=\; \sum_{x \in E} d(x).$$

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This is easily generalized by replacing $\mathbb{R}_{\geq 0}$ by an arbitrary **commutative semiring**, which is an algebraic structure $(R, +, 0, \cdot, 1)$, where $(R, +, 0)$ and $(R, \cdot, 1)$ are commutative monoids satisfying the distributive law:

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We can now define a functor $\mathcal{D}_R$ of R-distributions, parameterized by a commutative semiring R. Given a set X, $\mathcal{D}_R(X)$ is the set of R-distributions of finite support. The functorial action is defined exactly as for the probabilistic case, as the push-forward of a measure. If $f : X \to Y$, $\mathcal{D}_R(f) : \mathcal{D}_R(X) \to \mathcal{D}_R(Y)$:

$$\mathcal{D}_R(f)(d)(U) \; = \; d(f^{-1}(U))$$

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Note that there is a homomorphism of semirings from $\mathbb{R}_{\geq 0}$ to B, which sends positive probabilities to 1 (possible), and 0 to 0 (impossible). This lifts to a map on distributions, which sends a probability distribution to its **support**. This in turn sends probabilistic empirical models $\{d_C\}_{C \in \mathcal{M}}$ to possibilstic empirical models.

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We refer to this induced map as the **possibilistic collapse**.

Generalizing Empirical Models

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Given a measurement scenario $(X, \mathcal{M}, O)$, and a semiring R, we have the notion of a compatible family of R-distributions $\{e_C\}_{C \in \mathcal{M}}$, where $e_C \in \mathcal{D}_R(O^C)$.

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All the notions relating to contextuality, global sections etc. work in the same way as before, across this broader variety of situations.

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Given a global section d_g for an empirical model $e \in EM(\Sigma, R)$, it is easy to see that $\bar{h}(d_g)$ is a global section for $\bar{h}(e)$. Thus we have the following result.

Proposition

If $\bar{h}(e)$ is contextual, then so is e. In particular, if the possibilistic collapse of a probabilistic empirical model e is contextual, then e is contextual. The converse is not true.

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Cambridge	10991-06284	Newton	£2,567.53
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Consider again the Hardy model:

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_1, b_1)	1	1	1	1
(a_1, b_2)	0	1	1	1
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Change of perspective:

$a_1,\ a_2,\ b_1,\ b_2$	attributes
0, 1	data values
joint outcomes of measurements	tuples

The Hardy model as a relational database

The four rows of the model turn into four **relation tables**:

a_1	b_1
0	0
0	1
1	0
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a_1	b_2
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There is no **universal relation**: no table

a_1	a_2	b_1	b_2
⋮	⋮	⋮	⋮

whose projections onto $\{a_i, b_i\}$, $i = 1, 2$, yield the above four tables.

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attribute	measurement
set of attributes defining a relation table	compatible set of measurements
database schema	measurement cover
tuple	local section (joint outcome)
relation/set of tuples	boolean distribution on joint outcomes
universal relation instance	global section/hidden variable model
acyclicity	Vorob'ev condition

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We can also consider probabilistic databases and other generalisations; cf. provenance semirings.

The no-signalling polytope

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- explicitly represent models as points in $\mathbb{R}^N$, with $N = \sum_{C \in \mathcal{M}} |C|$.
- $\mathcal{N}$ is a polytope: defined by a finite number of linear constraints.

The structure of the no-signalling polytope

- **NS**: set of probabilistic empirical models
- $\mathcal{F}$: the face lattice of this polytope (vertices, edges, ...)

- $\mathcal{S}$: possibilistic models of the form poss(e) for some $e \in \mathbf{NS}$
- ordered contextwise by support

Then

$$\mathcal{F} \cong \mathcal{S}_{\perp}$$

In fact, the result applies to a much wider class of polytopes.

NS is defined by constraints:

- Non-negativity;
- Linear equations: viz. normalisation and no-signalling.

In geometric terms: $\mathbf{NS} = \mathcal{H}_{\geq \mathbf{0}} \cap \mathsf{Aff}\,(\mathbf{NS})$
where $\mathsf{Aff}\,(\mathbf{NS})$ is the affine subspace generated by **NS**,
and $\mathcal{H}_{\geq \mathbf{0}} = \{\mathbf{v} : \mathbf{v} \geq \mathbf{0}\}$.

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For any P is **standard form**, there is an order-isomorphism between:

- $\mathcal{F}(P)$, the face lattice of P.
- $\mathcal{S}(P)$, set of "supports" of points in P, ordered by inclusion.

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$$\{\mathbf{x} \,:\, \mathbf{a} \cdot \mathbf{x} \geq b\} \quad \text{for some } \mathbf{a} \in \mathbb{R}^n,\ b \in \mathbb{R}.$$

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Fundamental Theorem of Polytopes: the two notions coincide.

Face lattice

- $\mathbf{a} \cdot \mathbf{x} \geq b$ is **valid** for P if it is satisfied by every $\mathbf{x} \in P$.
- A valid inequality defines a **face** F of P:

$$F \;:=\; \{\mathbf{x} \in P \;:\; \mathbf{a} \cdot \mathbf{x} = b\}.$$

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- Meets in $\mathcal{F}(P)$ are given by intersection of faces, joins defined indirectly.

Called the **face lattice** of P, aka the **combinatorial type** of P.

Relative interior

Relative interior of a set S:

$$\mathrm{relint}\,(S) = \{\mathbf{x} \in S \; : \; \exists \epsilon > 0.\; \mathrm{Aff}\,(S) \cap B_\epsilon(x) \subseteq S\}$$

For a convex set:

$$\mathrm{relint}\,(S) = \{\mathbf{x} \in S \; : \; \forall \mathbf{y} \in S.\; \exists \epsilon > 0.\; (1+\epsilon)\mathbf{x} - \epsilon\mathbf{y} \in S\}$$

Intuitively: a point $\mathbf{x}$ is in the relative interior if the line segment $[\mathbf{y}, \mathbf{x}]$ from any point $\mathbf{y}$ of S in to $\mathbf{x}$ can be extended beyond $\mathbf{x}$ while remaining in S.

Carrier face

Every polytope P can be written as the disjoint union of the relative interiors of its non-empty faces:

$$P = \bigsqcup_{F \in \mathcal{F}^+(P)} \operatorname{relint} F.$$

This means that for any polytope P we can define a map

$$\operatorname{carr} : P \longrightarrow \mathcal{F}^+(P)$$

which assigns to each point $\mathbf{x}$ of P its **carrier face** — the unique face F such that $\mathbf{x} \in \operatorname{relint} F$.

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- $\mathcal{S}(P) := \{\mathsf{supp}\mathbf{x} \; : \; \mathbf{x} \in P\}$, ordered componentwise.

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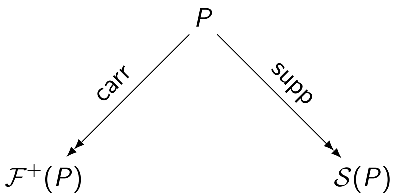
- So $\mathcal{S}(P)_\bot$ is a finite lattice.



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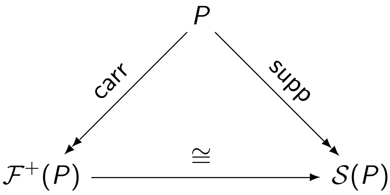

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Clearly, $\mathbf{x}^\sigma \cdot \mathbf{z} \geq 0$ is valid for all $\mathbf{z} \in P$, and defines a face

$$\begin{aligned} F_{\mathbf{x}} &= \{\mathbf{z} \in P \,:\, \mathbf{x}^\sigma \cdot \mathbf{z} = 0\} \\ &= \{\mathbf{z} \in P \,:\, \mathsf{supp}\mathbf{z} \leq \mathsf{supp}\mathbf{x}\}. \end{aligned}$$

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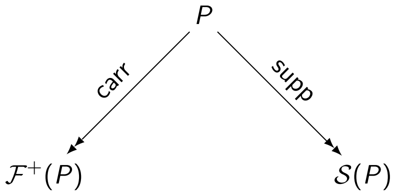
- $\mathbf{v} := (1+\epsilon)\mathbf{x} - \epsilon\mathbf{z} \geq \mathbf{0}$.
- Hence, $\mathbf{v} \in F_{\mathbf{x}}$.



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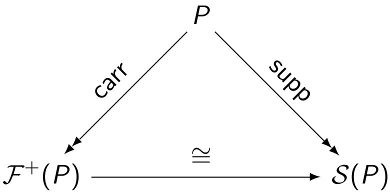

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Lattice iso: $\mathcal{F}(P) \cong \mathcal{S}(P)_{\perp}$

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- Therefore, the extremal empirical models are exactly those models which are completely and uniquely determined by their supports.
- These vertices of the polytope can be written as the disjoint union of the non-contextual, deterministic models – the vertices of the polytope of classical models – and the strongly contextual models with minimal support.

But . . .

- Note the mention of support!
- We still start from probabilistic models and take their supports.

Can we characterise the combinatorial type of **NS** using **only** possibilistic notions?

- Recall that empirical models are families of **consistent distributions**.
- These can be defined over any commutative semiring R.
- $\mathbb{R}_{\geq 0}$ gives probabilistic models.
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The support lattice $\mathcal{S}(\mathbf{NS}_{\mathbb{R}_{\geq 0}})$ is the image of this map.

Can we give an **intrinsic characterisation** of the image of the possibilistic collapse map, using only possibilistic notions?

$$\mathcal{S}(\mathbf{NS}_{\mathbb{R}_{\geq 0}}) \neq \mathbf{NS}_{\mathsf{B}}$$

i.e. there exist possibilistic empirical models that are not the support of any (probabilistic) empirical model (Abramsky, 2012).

A	B	0 0	0 1	1 0	1 1
a_1	b_1	1	0	0	1
a_1	b_2	1	1	0	1
a_2	b_1	1	0	0	1
a_2	b_2	1	0	0	1

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A	B	00	01	10	11
a_1	b_1	c	0	0	c'
a_1	b_2	d	g	0	d'
a_2	b_1	e	0	0	e'
a_2	b_2	f	0	0	f'

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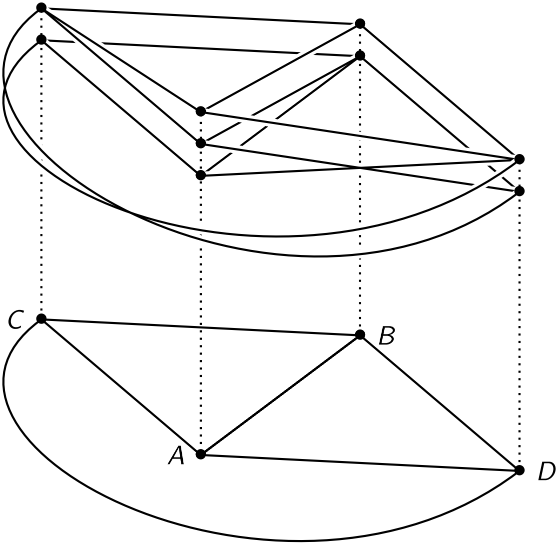
- A sensible question would be: given a possibilistic empirical model, is there always a (probabilistic) empirical model whose support is at most the original one?
- That is, are minimal possibilistic models always realisable as supports?
- Also, NO!

$$
\begin{aligned}
X &= \{A, B, C, D\} \\
\mathcal{M} &= \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\} \\
O &= \{0, 1, 2\}
\end{aligned}
$$

Possible assignments:

$$
\begin{array}{cccccc}
AB & \mapsto & 00, & 10, & 21 \\
 & & a & b & c \\
AC & \mapsto & 00, & 11, & 21 \\
 & & d & e & f \\
AD & \mapsto & 01, & 10, & 21 \\
 & & k & l & m \\
BC & \mapsto & 00, & 11 & \\
 & & g & h & \\
BD & \mapsto & 00, & 11 & \\
 & & i & j & \\
CD & \mapsto & 01, & 10 &
\end{array}
$$


C
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A
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$$
\begin{array}{llllllll}
a = k, & b = l, & g = i, & h = j, & c = n, & d = k, & e = l, & f = m \\
c = h, & h = o, & g = n, & i = o, & j = n, & c = j, & l = o, & d = n.
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$$

- All variables must be equated.
- Minimality: set any variable to zero, then all must be zero.
- Only remaining non-trivial equation is $a = a + a$.
- No non-zero, real solution!

A Bell-type example

$$X_{\text{Bell}} = \{A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2\}$$
$$\mathcal{M}_{\text{Bell}} = \{A_1, B_1, C_1, D_1\} \times \{A_2, B_2, C_2, D_2\}$$
$$O = \{0, 1, 2\}$$

Possible sections:

$A_1 A_2$			$\mapsto$	00,	11,	22
$B_1 B_2$,	$C_1 C_2$,	$D_1 D_2$	$\mapsto$	00,	11	
$A_1 B_2$,	$A_2 B_1$		$\mapsto$	00,	10,	21
$A_1 C_2$,	$A_2 C_1$		$\mapsto$	00,	11,	21
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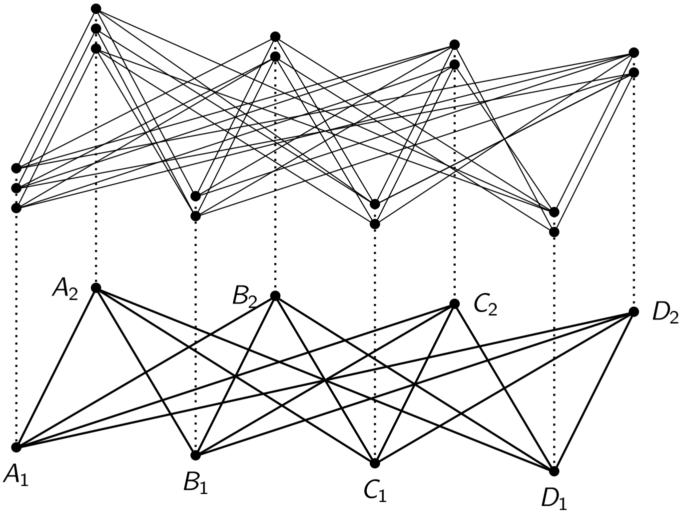

Still an open question

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The Kochen-Specker argument rests on properties of certain families of measurements which hold for **any** quantum state.

A trade-off: Bell's theorem has weaker conclusions, but also weaker assumptions.

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Note that, if we regard the elements of X as propositional variables, we can think of $s \in O^C$ as a truth-value assignment.

Then the KS property for an assignment s is equivalent to s satisfying the following formula:

$$\mathrm{ONE}(C) := \bigvee_{x \in C} (x \wedge \bigwedge_{x' \in C \setminus \{x\}} \neg x')$$

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N.B. Generalization to arbitrary O, unsatisfiability of a CSP.

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Is this a K-S construction?

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Subsumed by our cohomology results.

Quantum Representations

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This yields one of the most economical known examples of a KS construction.

By contrast, the Specker triangle is **not** quantum realizable.

From vectors to observables

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Hence we get a state-independent proof of strong contextuality in QM.

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Computational work by Arends and Ouaknine established a lower bound of 18, recently improved to 22 by Westerbaan and Uijlen.

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For an accessible overview of Contextual Semantics, see the article in the *Logic in Computer Science* Column, Bulletin of EATCS No. 113, June 2014 (and `arXiv`).

People

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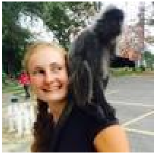


People

Adam Brandenburger, Lucien Hardy, Shane Mansfield, Rui Soares Barbosa, Ray Lal, Mehrnoosh Sadrzadeh, Phokion Kolaitis, Georg Gottlob, Carmen Constantin, Kohei Kishida, Linde Wester, Giovanni Caru

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