Quantifying Contextuality

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September 1, 2016

Overview

- Unified, general framework for non-locality and contextuality
- Qualitative hierarchy of contextuality
- **Quantitative *measure of contextuality***

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- Qualitative hierarchy of contextuality
- **Quantitative *measure of contextuality***

Why?

- Compare degree of contextuality of empirical models
- . . . across different measurement scenarios
- Contextuality as a resource

Contextuality

Empirical Data (e.g. CHSH)

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	$1/2$	0	0	$1/2$
(a,b')	$3/8$	$1/8$	$1/8$	$3/8$
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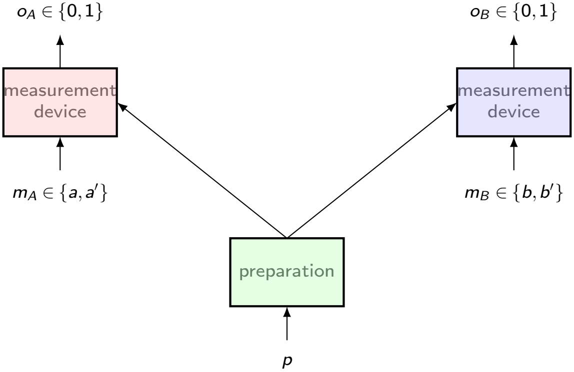

Measurement Scenarios: CHSH

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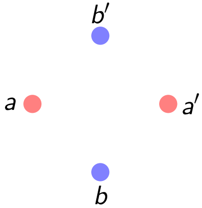

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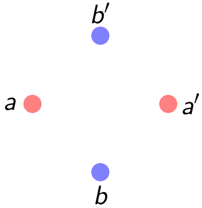

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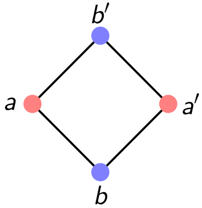

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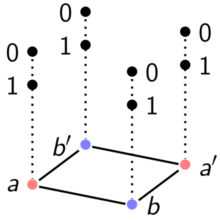

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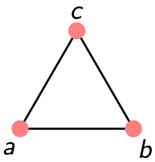
O a finite set — e.g.

$$O = \{0,1\}$$

Measurement Scenarios: 'Triangle'

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	0	$1/2$	$1/2$	0
(b,c)	0	$1/2$	$1/2$	0
(c,a)	0	$1/2$	$1/2$	0



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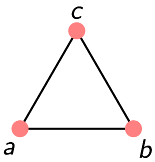

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$$X = \{a, b, c\}$$

Contexts:

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Outcomes:

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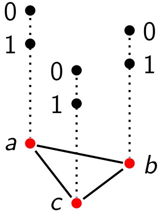

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Measurement Scenarios: 18-vector KS

- A set of 18 variables: $X = \{A, \ldots, O\}$
- A set of outcomes: $O = \{0, 1\}$
- A measurement cover: $\mathscr{M} = \{C_1, \ldots, C_9\}$
  whose contexts C_i correspond to the columns in the following table:

C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A	A	H	H	B	I	P	P	Q
B	E	I	K	E	K	Q	R	R
C	F	C	G	M	N	D	F	M
D	G	J	L	N	O	J	L	O

Empirical Models

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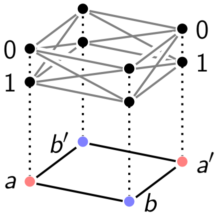

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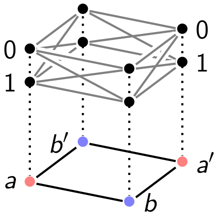

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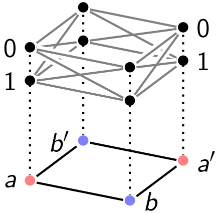

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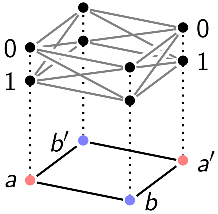

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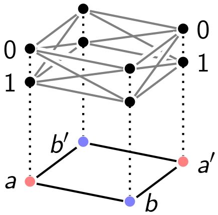

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NO-SIGNALLING

Contextuality

Classical data should arise as a convex combination of *global assignments*:

$(a, a', b, b') \mapsto (0,0,0,0)$, $(a, a', b, b') \mapsto (0,0,0,1)$, $\dots$, $(a, a', b, b') \mapsto (1,1,1,1)$

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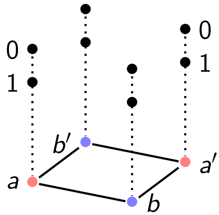

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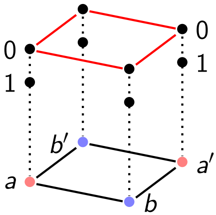

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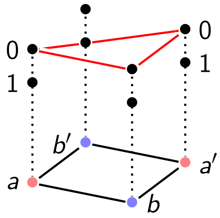

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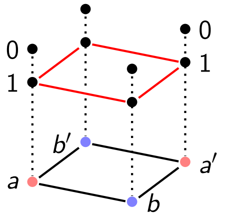

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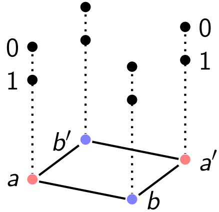

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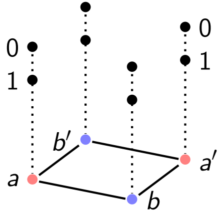

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(Contextuality rules out deterministic HVs; non-locality is a special case)

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a_1	b_2	✓	×	×	✓
a_2	b_1	✓	×	×	✓
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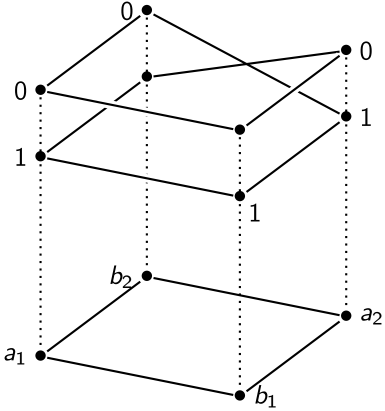

The Contextual Fraction

Proposition

Every empirical model admits a convex decomposition

$$e = \lambda e^{\mathsf{NC}} + (1-\lambda)e^{\mathsf{SC}}$$

into a non-contextual and a strongly contextual model. The maximum value λ for such decompositions, which is attained, is the non-contextual fraction of e, $\mathsf{NC}(e)$.

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Contextual fraction: $\mathsf{CF}(e) = 1 - \mathsf{NC}(e)$

- $\mathsf{CF}(e) \in [0,1]$
- e is *non-contextual* iff $\mathsf{CF}(e) = 0$
- e is *strongly contextual* iff $\mathsf{CF}(e) = 1$

Computing the Contextual Fraction

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Given a measurement scenario $\langle X, \mathcal{M}, O \rangle$, the *incidence matrix* **M** has

- rows indexed by $\langle C, s \rangle$, $C \in \mathcal{M}$, $s \in O^C$
- columns indexed by global assignments $g \in O^X$

$$\mathbf{M}[\langle C, s \rangle, g] := \begin{cases} 1 & \text{if } g|_C = s \\ 0 & \text{otherwise} \end{cases} .$$

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A probability distribution on (*i.e.* mixture of) deterministic NCHV models is given by a column vector $\mathbf{c}$; while an empirical model over the scenario can be flattened into a row vector $\mathbf{v}_e \in \mathbb{R}^m$, e.g.

$$\mathbf{v}_e = \{1/2, 0, 0, 1/2, \quad 3/8, 1/8, 1/8, 3/8, \quad 3/8, 1/8, 1/8, 3/8, \quad 1/8, 3/8, 3/8, 1/8\}$$

(Non-)Contextual Fraction via Linear Programming

Checking contextuality of e corresponds to solving

$$\begin{array}{ll} \texttt{Find} & \mathbf{d} \in \mathbb{R}^n \\ \texttt{such that} & \mathbf{M}\,\mathbf{d} = \mathbf{v}_e \\ \texttt{and} & \mathbf{d} \geq \mathbf{0} \end{array}$$

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Computing the non-contextual fraction corresponds to solving the following *linear program*:

$$\begin{array}{ll} \texttt{Find} & \mathbf{c} \in \mathbb{R}^n \\ \texttt{maximising} & \mathbf{1} \cdot \mathbf{c} \\ \texttt{subject to} & \mathbf{M}\,\mathbf{c} \leq \mathbf{v}_e \\ \texttt{and} & \mathbf{c} \geq \mathbf{0} \end{array}$$

Bell Inequality Violations

Generalised Bell Inequalities

An **inequality** for a scenario $\langle X, \mathscr{M}, O \rangle$ is given by:

- A set of coefficients $\alpha = \{\alpha_{(C,s)}\}_{C \in \mathscr{M}, s \in O^C}$
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- **Bell inequality** if it is satisfied by every NC model
- Bell inequality is **tight** if it is saturated by some NC model

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- The **normalised violation** of a Bell inequality $\langle \alpha, R \rangle$ by e is

$$\frac{\max\{0, \mathscr{B}_\alpha(e) - R\}}{\|\alpha\| - R} \quad \in [0,1]$$

Contextual Fraction & Bell Violations

Proposition

Let e be an empirical model

- *Normalised violation by e of any Bell inequality is at most* $\mathsf{CF}(e)$
- *There exists a Bell inequality for which this is attained*
- *This Bell inequality is tight at "the" non-contextual model* e^{NC}

$$e = \mathsf{NC}(e)\, e^{NC} + \mathsf{CF}(e)\, e^{SC}$$

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Quantifying Contextuality LP:

$$\begin{array}{ll} \texttt{Find} & \mathbf{c} \in \mathbb{R}^n \\ \texttt{maximising} & \mathbf{1} \cdot \mathbf{c} \\ \texttt{subject to} & \mathbf{M}\,\mathbf{c} \leq \mathbf{v}_e \\ \texttt{and} & \mathbf{c} \geq \mathbf{0} \end{array}$$

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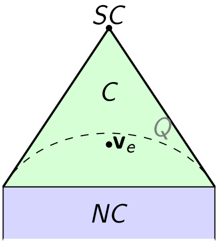

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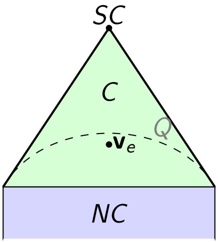

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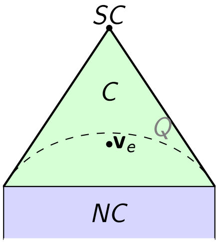

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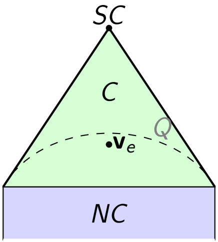

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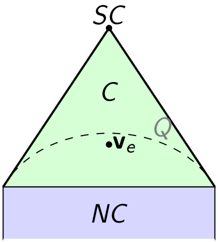

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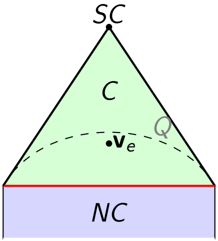

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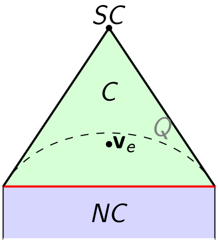

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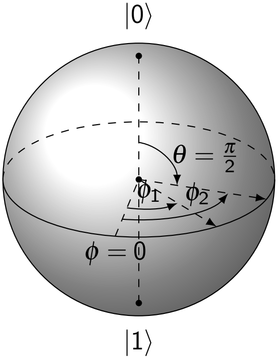

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- e.g. $(\phi_1, \phi_2) = (0, \pi/3)$ gives Bell–CHSH model

A	B	(0,0)	(0,1)	(1,0)	(1,1)
a_1	b_1	$1/2$	0	0	$1/2$
a_1	b_2	$3/8$	$1/8$	$1/8$	$3/8$
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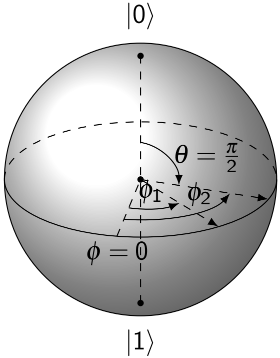

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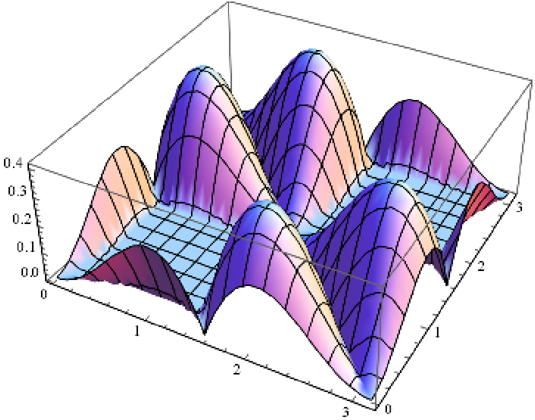

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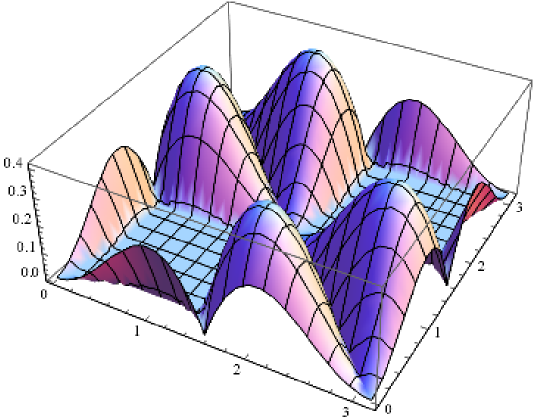

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Note that these achieve Tsirelson violation of the CHSH inequality.

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- Again, equatorial measurements on the Bloch sphere

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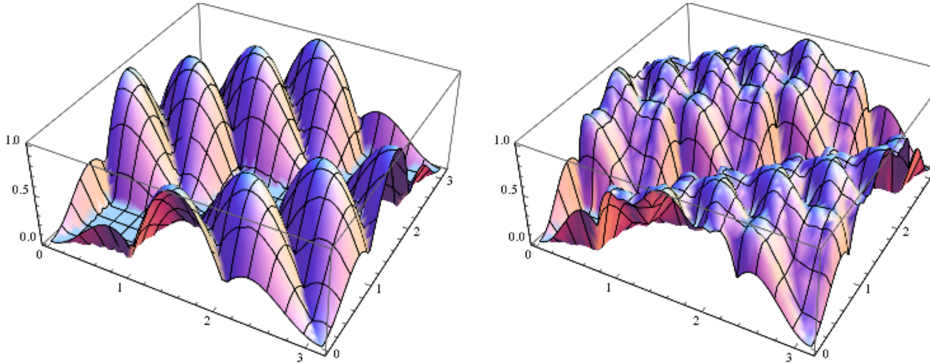

(a) (b)

Figure : CF(e) for equatorial measurements at ϕ_1 and ϕ_2 on each qubit of $\left|\psi_{\mathrm{GHZ}(n)}\right\rangle$ with: (a) $n = 3$; (b) $n = 4$.

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- General n: local equatorial measurements at

$$(\phi_1, \phi_2) \in \left\{ \left\{ \frac{(n+k)\pi}{2n}, \frac{k\pi}{2n} \right\} \middle| \, 0 \leq k < n \right\}$$

  on GHZ(n) state give rise to strong contextuality

Towards a Resource Theory of Contextuality

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- Algebra of empirical models, towards a process calculus?

Operations

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  $e : \langle X, \mathscr{M}, O\rangle,\ \alpha : (X, \mathscr{M}) \cong (X', \mathscr{M}') \ \rightsquigarrow\ e[\alpha] : \langle X', \mathscr{M}', O\rangle$

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For $C' \in M', s\colon\ C' \longrightarrow O,\ (e \restriction \mathscr{M}')_{C'}(s) := e_C\vert_{C'}(s)$ with any $C \in \mathscr{M}$ s.t. $C' \subseteq C$

Operations

- *relabelling*
  $e : \langle X, \mathscr{M}, O \rangle,\ \alpha : (X, \mathscr{M}) \cong (X', \mathscr{M}') \ \rightsquigarrow\ e[\alpha] : \langle X', \mathscr{M}', O \rangle$

For $C \in \mathscr{M}, s\colon \alpha(C) \longrightarrow O,\ e[\alpha]_{\alpha(C)}(s) := e_C(s \circ \alpha^{-1})$

- *restriction*
  $e : \langle X, \mathscr{M}, O \rangle,\ (X', \mathscr{M}') \leq (X, \mathscr{M}) \ \rightsquigarrow\ e \restriction \mathscr{M}' : \langle X', \mathscr{M}', O \rangle$

For $C' \in M', s\colon C' \longrightarrow O,\ (e \restriction \mathscr{M}')_{C'}(s) := e_C|_{C'}(s)$ with any $C \in \mathscr{M}$ s.t. $C' \subseteq C$

- *coarse-graining*
  $e : \langle X, \mathscr{M}, O \rangle,\ f\colon O \longrightarrow O' \ \rightsquigarrow\ e/f : \langle X, \mathscr{M}, O' \rangle$

For $C \in M, s\colon C \longrightarrow O',\ (e/f)_C(s) := \sum_{t\colon C \longrightarrow O, f \circ t = s} e_C(t)$

Operations

- *mixing*
  $e : \langle X, \mathscr{M}, O \rangle,\ e' : \langle X, \mathscr{M}, O \rangle, \lambda \in [0,1] \ \rightsquigarrow\ e +_\lambda e' : \langle X, \mathscr{M}, O \rangle$

For $C \in M, s\colon C \longrightarrow O'$,
$$(e +_\lambda e')_C(s) := \lambda e_C(s) + (1-\lambda) e'_C(s)$$

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- *choice*
  $e : \langle X, \mathscr{M}, O\rangle,\ e' : \langle X', \mathscr{M}', O\rangle \rightsquigarrow e\&e' : \langle X \sqcup X', \mathscr{M} \sqcup \mathscr{M}', O\rangle$

  > For $C \in M$, $(e\&e')_C := e_C$
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- *tensor*
  $e : \langle X, \mathscr{M}, O\rangle,\ e' : \langle X', \mathscr{M}', O\rangle \ \rightsquigarrow\ e \otimes e' : \langle X \sqcup X', \mathscr{M} \star \mathscr{M}', O\rangle$

  $\mathscr{M} \star \mathscr{M}' := \{C \sqcup D \mid C \in \mathscr{M}, D \in \mathscr{M}'\}$
  For $C \in \mathscr{M}, D \in \mathscr{M}', s = \langle s_1, s_2\rangle\colon C \sqcup D \longrightarrow O$,
  $$(e \otimes e')_{C \sqcup D}\langle s_1, s_2\rangle := e_C(s_1)\, e'_D(s_2)$$

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  $\mathsf{CF}(e_1 \otimes e_2) =$
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Quantifying Quantum Advantage

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> *The greater the violation of the classical bound we want, the more contextuality there has to be.*

We shall look at one such result in terms of games. The class of games we will consider are a (vast) generalization of XOR games (but can be generalized much further). They subsume what are sometimes called "pseudo-telepathy games".

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The classical bound for the game is the maximum success probability for any non-contextual strategy.

Logical Bell inequalities give the classical bound

Say that a game $\{W_C\}$ is *K-consistent* if the maximum cardinality of a consistent sub-family of $\{W_C\}$ is K.

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The classical bound for a K*-consistent game is* $\frac{1}{|\mathscr{M}|}K$.

A suitable measure of the non-classicality (or "hardness") of a K-consistent game G is $\mu_G := |\mathscr{M}| - K$.

Relating the contextual fraction to hardness of a task

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Theorem

Consider a game G, and a strategy (empirical model) e, with success probability $p_S(e)$, and failure probability $p_F(e) := 1 - p_S(e)$. Then we have

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This says that for any game with a given level of difficulty μ_G, the higher we want the success probability for a strategy e to be, the more contextual e has to be.

An analogous result for quantum computation

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A similar result can be proved for the measurement-based quantum computation paradigm, refining a result by Robert Raussendorf:

Theorem

Given a boolean function f with a level of difficulty ν_f measured by how far it is from being mod 2 linear, then

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These results are early steps towards developing a quantitative theory of contextuality as a resource for exceeding classical bounds on information processing tasks.

Contextuality in the presence of signalling

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Given a possibly signalling empirical model e (*i.e.* we are not assuming compatibility), we can consider maximal convex decompositions

$$e = \lambda e^{\mathsf{NS}} + (1 - \lambda) e^{\mathsf{SS}}.$$

where e^{NS} is no-signalling, and e^{SS} is "strongly signalling", *i.e.* with no no-signalling fraction.

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Note that $\mathsf{NS}(e) = 1$ if and only if e is no-signalling.

Computing the No-Signalling Fraction

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This can be computed by the following linear program:

$$\begin{array}{ll} \texttt{Find} & \mathbf{w} \in \mathbb{R}^n \\ \texttt{maximising} & \dfrac{1}{|\mathscr{M}|}\, \mathbf{1} \cdot \mathbf{w} \\ \texttt{subject to} & \mathbf{N}\,\mathbf{w} = \mathbf{0} \\ \texttt{and} & \mathbf{w} \leq \mathbf{v}^e \\ \texttt{and} & \mathbf{w} \geq \mathbf{0} \end{array} \quad . \tag{1}$$

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Here **N** is the No-Signalling matrix.

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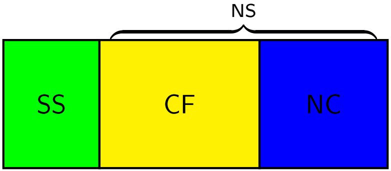

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	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	23	3	4	23
(a,b')	33	11	5	30
(a',b)	22	10	6	24
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- NIST:

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	6378	3289	3147	44336240
(a,b')	6794	2825	23230	44311018
(a',b)	6486	21358	2818	44302570
(a',b')	106	27562	30000	44274530

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NO-SIGNALLING

- Experimental data does not perfectly satisfy no-signalling...

Quantifying Signalling

e is *no-signalling* iff

$$\mathbf{N}\,\mathbf{v}_e = \mathbf{0}$$

where

$$\mathbf{N}[i,j] := \begin{cases} 1 & \text{if } s_j \in O^{C_i} \text{ and } \; s_j|_{C_i'} = t_i \\ -1 & \text{if } s_j \in O^{C_i'} \text{ and } \; s_j|_{C_i} = t_i \\ 0 & \text{otherwise} \end{cases}$$

- $(\langle t, C, C' \rangle_i)$ an enumeration of $\{\langle t, C, C' \rangle \mid t \in O^{C \cap C'} \text{ and } (C, C') \in \mathscr{M}^2\}$
- (s_j) an enumeration of $\{s \mid t \in O^C \text{ and } C \in \mathscr{M}^2\}$

Quantifying Signalling

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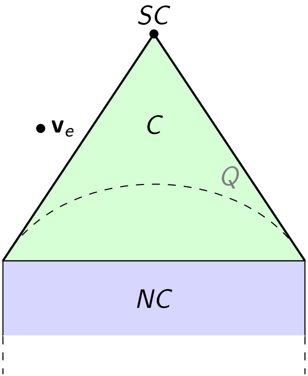
Otherwise we can obtain the *no-signalling fraction* with the LP

$$\begin{array}{ll} \texttt{maximise} & \mathbf{1}\cdot\mathbf{z} \\ \texttt{subject to} & \mathbf{N}\,\mathbf{z} = \mathbf{0} \\ \texttt{and} & \mathbf{z} \leq \mathbf{v}_e \\ \texttt{and} & \mathbf{z} \geq \mathbf{0} \end{array}$$

Quantifying Signalling & Contextuality



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$$e = \mu\, e_{\mathsf{NS}} + (1 - \mu) e_{\mathsf{SS}}$$

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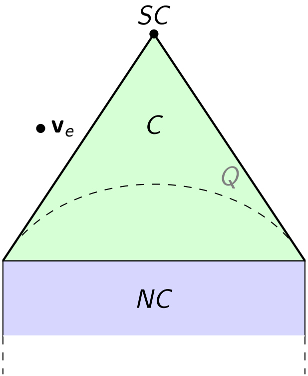

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\texttt{and} & \mathbf{x} \geq \mathbf{0}
\end{array}
$$

Setting $\lambda = \mathbf{1}\cdot\mathbf{x}^*$

$$\boxed{e = \mu\lambda\, e_{\mathsf{NC}} + \mu(1-\lambda) e_{\mathsf{SC}} + (1-\mu) e_{\mathsf{SS}}}$$

Analysis of Real Data (Delft)

Decomposition of data:

$$e_{\text{Delft}} \approx 0.0664\, e_{\text{SS}} + \mathbf{0.4073}\, e_{\text{SC}} + 0.5263\, e_{\text{NC}}$$

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Ratio of signalling to genuine contextuality:

$$0.163$$

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Reported Bell violation:

$$0.0000116$$

(Different data?)

Analysis of Real Data (NIST)

Decomposition of data:

$$e_{\mathsf{NIST}} \approx 0.0000049\, e_{\mathsf{SS}} + \mathbf{0.0000281}\, e_{\mathsf{SC}} + 0.9999670\, e_{\mathsf{NC}}$$

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(Different data?)

Ratio of signalling to genuine contextuality:

$$0.175$$

Negative Probabilities

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If e is any compatible empirical model, there is a signed measure $d : X \to \mathbb{R}$ *with* $\sum_{x \in X} d(x) = 1$ *such that* $d|_C = e_C$ *for all* $C \in \mathscr{M}$.

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Question: How does NP relate to CF?