Logic and Quantum information
Lecture III: Quantum Realizability

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If we represent qubit space with a standard basis $\{|0\rangle, |1\rangle\}$, then n-qubit space has basis

$$\{|s\rangle \,:\, s \in \{0,1\}^n\}$$

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- The possible outcomes of an observable $A = \sum_i \lambda_i e_i$ are given by the eigenvalues λ_i.
- The probability of getting the outcome λ_i when measuring A on the state $|\psi\rangle$ is given by the **Born rule**:

$$|\langle e_i \mid \psi\rangle|^2.$$

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We shall stick to the simplest level of presentation ...

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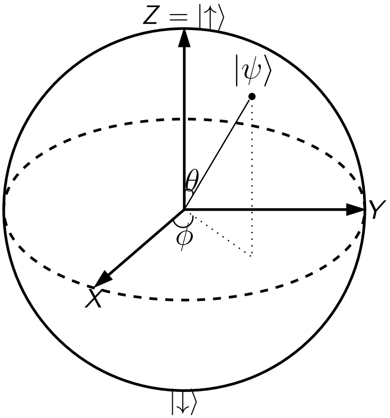
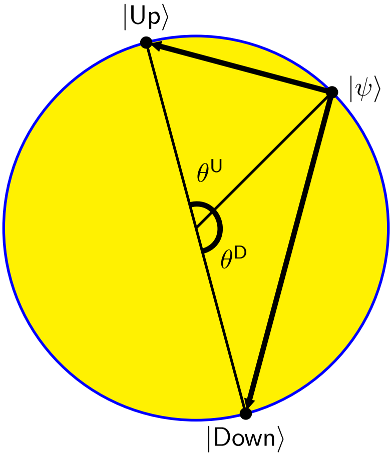
Developments such as **device-independent QKD**.

The Bloch sphere representation of qubits



Truth makes an angle with reality



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- When we subject a qubit to a measurement $(\mathsf{Up}, \mathsf{Down})$, the state of the qubit determines a probability distribution on the two possible outcomes. The probabilities are determined by the **angles** between the qubit state $|\psi\rangle$ and the points $(|\mathsf{Up}\rangle, |\mathsf{Down}\rangle)$ which specify the measurement. In algebraic terms, $|\psi\rangle$, $|\mathsf{Up}\rangle$ and $|\mathsf{Down}\rangle$ are unit vectors in the complex vector space $\mathbb{C}^2$, and the probability of observing Up when in state $|\psi\rangle$ is given by the square modulus of the inner product:

  $$|\langle\psi|\mathsf{Up}\rangle|^2.$$

  This is known as the **Born rule**. It gives the basic predictive content of quantum mechanics.

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The sense in which the qubit generalises the classical bit is that, for each question we can ask — *i.e.* for each measurement — there are just two possible answers. We can view the states of the qubit as superpositions of the classical states 0 and 1, so that we have a probability of getting each of the answers for any given state.

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But in addition, we have the important feature that there are a continuum of possible questions we can ask. However, note that on each run of the system, we can only ask **one** of these questions. We cannot simultaneously observe Up or Down in two different directions. Note that this corresponds to the feature of the scenario we discussed, that Alice and Bob could only look at one their local registers on each round.

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Note in addition that a measurement has an **effect** on the state, which will no longer be the original state $|\psi\rangle$, but rather one of the states Up or Down, in accordance with the measured value.

Quantum Entanglement

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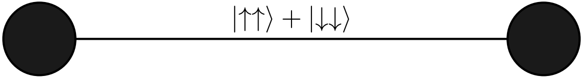
Bell state:



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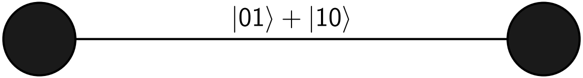

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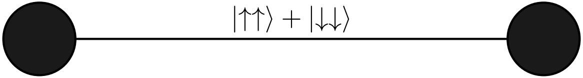

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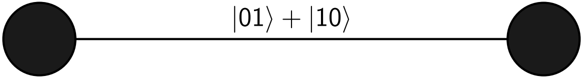

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$$\sum_i \lambda_i \cdot \phi_i \otimes \psi_i$$

Superposition encodes **correlation**.

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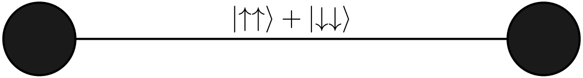

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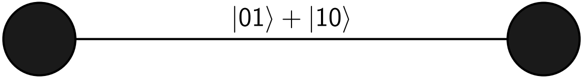

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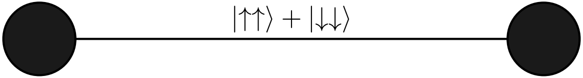

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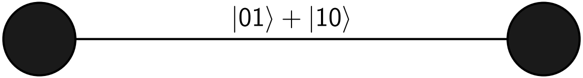

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Bell's theorem: QM is **essentially non-local**.

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Example: The Bell Model

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	$1/2$	0	0	$1/2$
a_1	b_2	$3/8$	$1/8$	$1/8$	$3/8$
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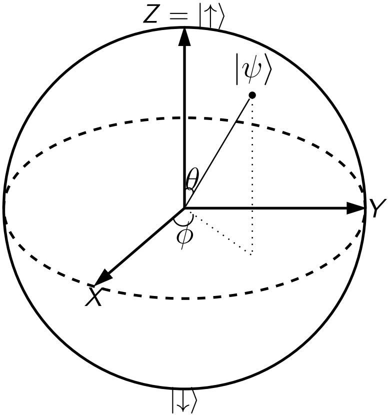
Extensively tested experimentally.

Computing the Bell table



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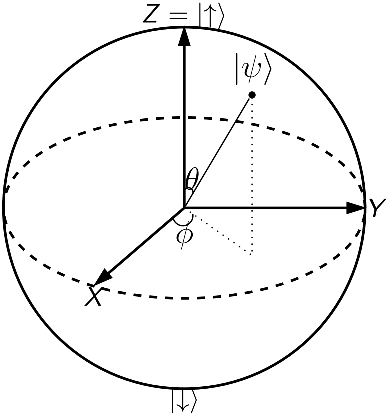

Spin measurements lying in the equatorial plane of the Bloch sphere
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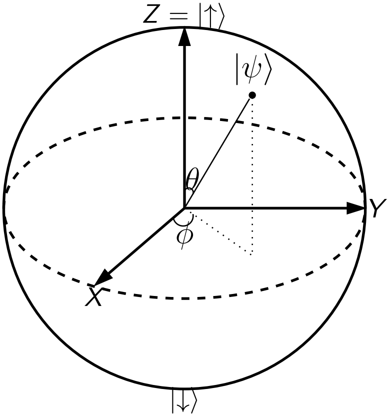

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X itself, $\phi = 0$:
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Probability of this event M when measuring (a, b') on $B = (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)/\sqrt{2}$ is given by Born rule:

$$|\langle B|M\rangle|^2.$$

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Using the Euler identity $e^{i\theta} = \cos\theta + i\sin\theta$, we have

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The other entries can be computed similarly.

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Other attempts by Masanes and Mueller, Brukner and Dakic, the Pavia group (D’Ariano, Chiribella and Perinotti), . . .

Empirical Models

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Example: The Bell Model

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	$1/2$	0	0	$1/2$
a_1	b_2	$3/8$	$1/8$	$1/8$	$3/8$
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Important note: this is **quantum realizable**.

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Important note: this is **quantum realizable**.

Generated by Bell state

$$\frac{|00\rangle \ + \ |11\rangle}{\sqrt{2}},$$

subjected to measurements in the XY-plane, at relative angle $\pi/3$.

The PR Box

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	1	0	0	1
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The PR Box

This satisfies No-Signalling, so is consistent with SR, but it is **not** quantum realisable.

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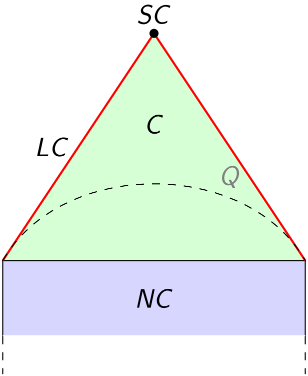

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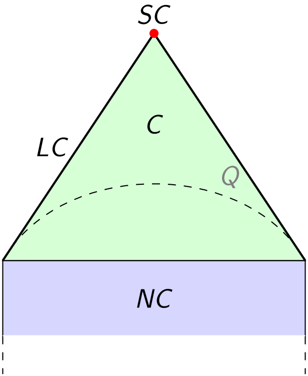

Key question: find compelling principles to explain why Nature picks out the quantum set.

Quantum Realizations of Relational Models

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We take QM(d) to be the sub-class of models realisable in a Hilbert space of **finite** dimension d.

Quantum Realization of the Hardy Model

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We consider the two-qubit system, with X_2 and Y_2 measurement in the computational basis. The eigenvectors for X_1 are taken to be

$$\sqrt{\frac{3}{5}}|0\rangle + \sqrt{\frac{2}{5}}|1\rangle, \qquad -\sqrt{\frac{2}{5}}|0\rangle + \sqrt{\frac{3}{5}}|1\rangle$$

and similarly for Y_1. The state is taken to be

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The possibilistic collapse of this model is thus a Hardy model.

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The class QM(d) *is in* PSPACE*. That is, there is a* PSPACE *algorithm to decide, given an empirical model, if it arises from a quantum system of dimension* d*.*

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This fragment has PSPACE complexity (Canny). Moreover, the sentence can be constructed in polynomial time from the given empirical model. Hence membership of QM(d) is in PSPACE.

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In particular, the proof that there is a quantum realisation in the limit uses an **infinite-dimensional** Hilbert space.

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Other questions: e.g. generalise the NPA hierarchy to arbitrary measurement scenarios.

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Proposition

For each n, HV(n) *is in* NP*.*

Proof

From the previous Proposition, it is clear that HV(n) is defined by the following second-order formula interpreted over finite structures (U, e):

$$\forall \vec{x}. \exists \vec{y}. R(\vec{x}, \vec{y}) \ \wedge \ \forall \vec{x}, \vec{y}.\, R(\vec{x}, \vec{y}) \ \rightarrow \ \exists f_1, \ldots, f_n. \textstyle\bigwedge_i f_i(x_i) = y_i \ \wedge \ \forall \vec{v}. R(\vec{v}, f(\vec{v})).$$

LHV is in NP

In order to simplify notation, we shall consider possibilistic empirical models of the form (U, e), where $e \subseteq U^n \times U^n$. Thus we use the same underlying set U for both measurements and outcomes at each site.

We shall write $\mathrm{HV}(n)$ for the class of models of this form which has a local hidden variable realisation (*i.e.* a boolean global section). We are interested in the algorithmic problem of determining if a structure (U, e) of arity n is in $\mathrm{HV}(n)$.

Proposition

For each n, $\mathrm{HV}(n)$ is in NP.

Proof

From the previous Proposition, it is clear that $\mathrm{HV}(n)$ is defined by the following second-order formula interpreted over finite structures (U, e):

$$\forall \vec{x}.\exists \vec{y}.R(\vec{x}, \vec{y}) \ \wedge \ \forall \vec{x}, \vec{y}.\, R(\vec{x}, \vec{y}) \ \rightarrow \ \exists f_1, \ldots, f_n. \bigwedge_i f_i(x_i) = y_i \ \wedge \ \forall \vec{v}.R(\vec{v}, f(\vec{v})).$$

By standard quantifier manipulations, this can be brought into an equivalent Σ^1_1 form, and hence $\mathrm{HV}(n)$ is in NP. □

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- These are used to show that HV(n), $n > 2$, is NP-complete; smaller instances are in PTIME.
- The robust paradigm is an interesting and non-trivial extension of current theory, and worthy of further study.