Logic and Quantum Information
Lecture II: The Topology of Paradox

Samson Abramsky

Department of Computer Science
The University of Oxford

What Do ‘Observables’ Observe?

What Do 'Observables' Observe?

Surely **objective properties** of a physical system, which are independent of our choice of which measurements to perform — of our **measurement context**.

What Do ‘Observables’ Observe?

Surely **objective properties** of a physical system, which are independent of our choice of which measurements to perform — of our **measurement context**.

More precisely, this would say that for each possible state of the system, there is a function λ which for each measurement m specifies an outcome $\lambda(m)$, **independently of which other measurements may be performed**.

What Do ‘Observables’ Observe?

Surely **objective properties** of a physical system, which are independent of our choice of which measurements to perform — of our **measurement context**.

More precisely, this would say that for each possible state of the system, there is a function λ which for each measurement m specifies an outcome $\lambda(m)$, **independently of which other measurements may be performed**.

This point of view is called **non-contextuality**. It is equivalent to the assumption of a classical source.

What Do 'Observables' Observe?

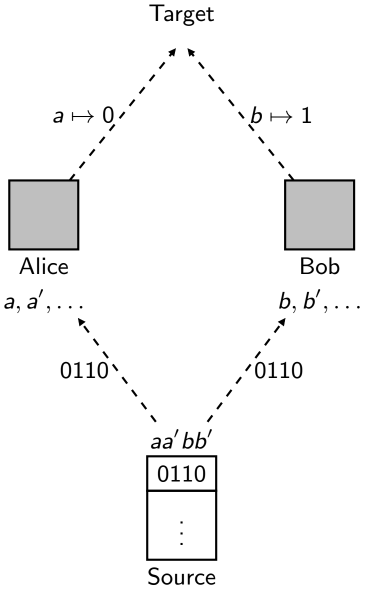
Surely **objective properties** of a physical system, which are independent of our choice of which measurements to perform — of our **measurement context**.

More precisely, this would say that for each possible state of the system, there is a function λ which for each measurement m specifies an outcome $\lambda(m)$, **independently of which other measurements may be performed**.

This point of view is called **non-contextuality**. It is equivalent to the assumption of a classical source.

However, this view is **impossible to sustain** in the light of our **actual observations of (micro)-physical reality**.

Hidden Variables: The Mermin instruction set picture



The ‘Hardy Paradox’: Bell’s theorem without inequalities

The ‘Hardy Paradox’: Bell’s theorem without inequalities

Hardy models: those whose **support** satisfies

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_1, b_1)	1			
(a_1, b_2)	0			
(a_2, b_1)	0			
(a_2, b_2)				0

Which ‘instruction set’ λ could the outcomes $(0,0)$ for measurements (a_1, b_1) could come? Clearly, we must have

$$\lambda : a_1 \mapsto 0, \quad b_1 \mapsto 0.$$

The ‘Hardy Paradox’: Bell’s theorem without inequalities

Hardy models: those whose **support** satisfies

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_1, b_1)	1			
(a_2, b_2)	$\boxed{0}$	$\boxed{?}$		
(a_2, b_1)	0			
(a_2, b_2)				0

Which ‘instruction set’ λ could the outcomes $(0,0)$ for measurements (a_1, b_1) could come? Clearly, we must have

$$\lambda : a_1 \mapsto 0, \quad b_1 \mapsto 0.$$

The 'Hardy Paradox': Bell's theorem without inequalities

Hardy models: those whose **support** satisfies

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_1, b_1)	1			
(a_1, b_2)	0	1		
(a_2, b_1)	0			
(a_2, b_2)				0

Which 'instruction set' λ could the outcomes $(0,0)$ for measurements (a_1, b_1) could come? Clearly, we must have

$$\lambda : a_1 \mapsto 0, \quad b_1 \mapsto 0.$$

The 'Hardy Paradox': Bell's theorem without inequalities

Hardy models: those whose **support** satisfies

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_1, b_1)	1			
(a_1, b_2)	0	1		
(a_2, b_1)	0			
(a_2, b_2)		?		0

Which 'instruction set' λ could the outcomes $(0,0)$ for measurements (a_1, b_1) could come? Clearly, we must have

$$\lambda : a_1 \mapsto 0, \quad b_1 \mapsto 0.$$

The ‘Hardy Paradox’: Bell’s theorem without inequalities

Hardy models: those whose **support** satisfies

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_1, b_1)	1			
(a_1, b_2)	0	1		
(a_2, b_1)	0			
(a_2, b_2)		1		0

Which ‘instruction set’ λ could the outcomes $(0,0)$ for measurements (a_1, b_1) could come? Clearly, we must have

$$\lambda : a_1 \mapsto 0, \quad b_1 \mapsto 0.$$

The 'Hardy Paradox': Bell's theorem without inequalities

Hardy models: those whose **support** satisfies

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_1, b_1)	1			
(a_1, b_2)	0	1		
(a_2, b_1)	0			
(a_2, b_2)		1		0

So there is a unique 'instruction set' λ that outcomes $(0,0)$ for measurements (a_1, b_1) could come from:

$$\lambda : a_1 \mapsto 0, \quad a_2 \mapsto 0, \quad b_1 \mapsto 0, \quad b_2 \mapsto 1.$$

The 'Hardy Paradox': Bell's theorem without inequalities

Hardy models: those whose **support** satisfies

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_1, b_1)	1			
(a_1, b_2)	0	1		
(a_2, b_1)	$\boxed{0}$			
(a_2, b_2)		1		0

So there is a unique 'instruction set' λ that outcomes $(0,0)$ for measurements (a_1, b_1) could come from:

$$\lambda : a_1 \mapsto 0, \quad a_2 \mapsto 0, \quad b_1 \mapsto 0, \quad b_2 \mapsto 1.$$

However, this would require the outcome $(0,0)$ for measurements (a_2, b_1) to be possible, and this is **precluded**.

The 'Hardy Paradox': Bell's theorem without inequalities

Hardy models: those whose **support** satisfies

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_1, b_1)	1			
(a_1, b_2)	0	1		
(a_2, b_1)	$\boxed{0}$			
(a_2, b_2)		1		0

So there is a unique 'instruction set' λ that outcomes $(0,0)$ for measurements (a_1, b_1) could come from:

$$\lambda : a_1 \mapsto 0, \quad a_2 \mapsto 0, \quad b_1 \mapsto 0, \quad b_2 \mapsto 1.$$

However, this would require the outcome $(0,0)$ for measurements (a_2, b_1) to be possible, and this is **precluded**.

Thus Hardy models are **contextual**. They cannot be explained by a classical source.

Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

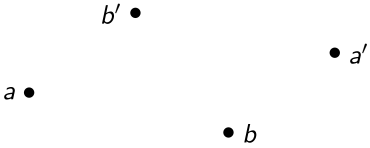
	00	01	10	11
ab	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
ab'	$\times$	$\checkmark$	$\checkmark$	$\checkmark$
$a'b$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$
$a'b'$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$

Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

	00	01	10	11
ab	✓	✓	✓	✓
ab'	×	✓	✓	✓
$a'b$	×	✓	✓	✓
$a'b'$	✓	✓	✓	×

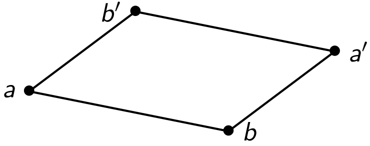


Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

	00	01	10	11
ab	✓	✓	✓	✓
ab'	×	✓	✓	✓
$a'b$	×	✓	✓	✓
$a'b'$	✓	✓	✓	×

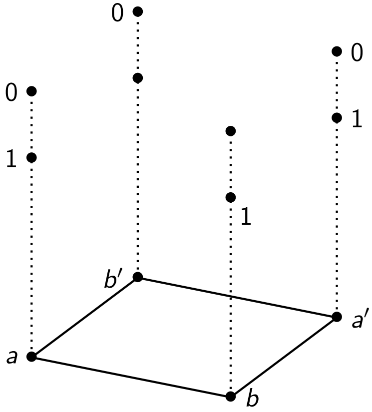


Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

	00	01	10	11
ab	✓	✓	✓	✓
ab'	×	✓	✓	✓
$a'b$	×	✓	✓	✓
$a'b'$	✓	✓	✓	×

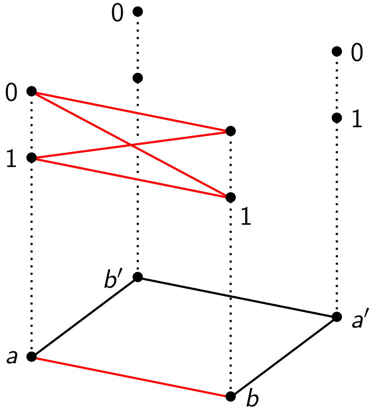


Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

	00	01	10	11
ab	✓	✓	✓	✓
ab'	×	✓	✓	✓
$a'b$	×	✓	✓	✓
$a'b'$	✓	✓	✓	×

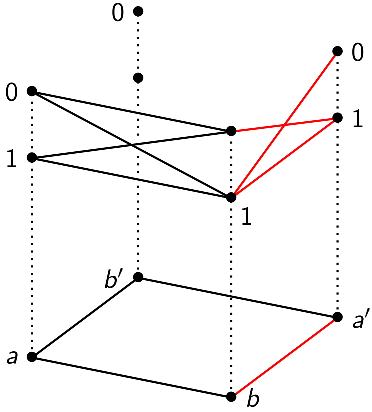


Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

	00	01	10	11
ab	✓	✓	✓	✓
ab'	×	✓	✓	✓
$a'b$	×	✓	✓	✓
$a'b'$	✓	✓	✓	×

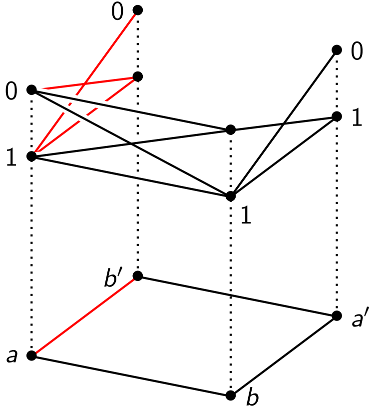


Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

	00	01	10	11
ab	✓	✓	✓	✓
ab'	×	✓	✓	✓
$a'b$	×	✓	✓	✓
$a'b'$	✓	✓	✓	×

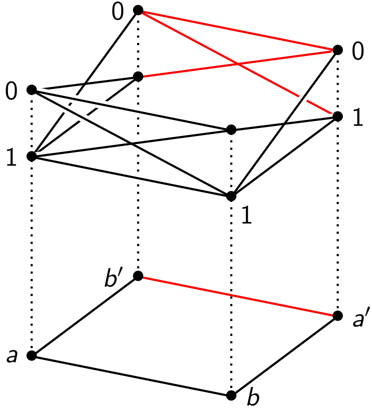


Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

	00	01	10	11
ab	✓	✓	✓	✓
ab'	×	✓	✓	✓
$a'b$	×	✓	✓	✓
$a'b'$	✓	✓	✓	×

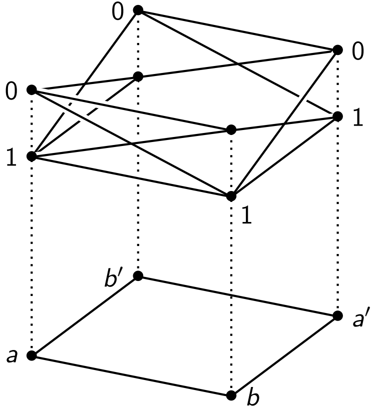


Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

	00	01	10	11
ab	✓	✓	✓	✓
ab'	×	✓	✓	✓
$a'b$	×	✓	✓	✓
$a'b'$	✓	✓	✓	×

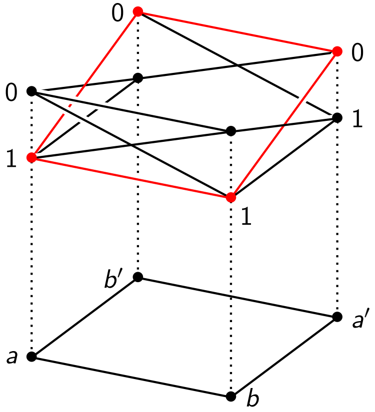


Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

	00	01	10	11
ab	✓	✓	✓	✓
ab'	×	✓	✓	✓
$a'b$	×	✓	✓	✓
$a'b'$	✓	✓	✓	×

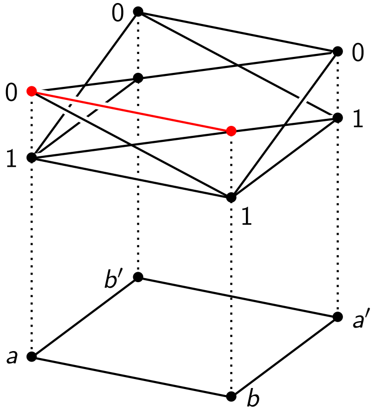


Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

	00	01	10	11
ab	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
ab'	$\times$	$\checkmark$	$\checkmark$	$\checkmark$
$a'b$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$
$a'b'$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$

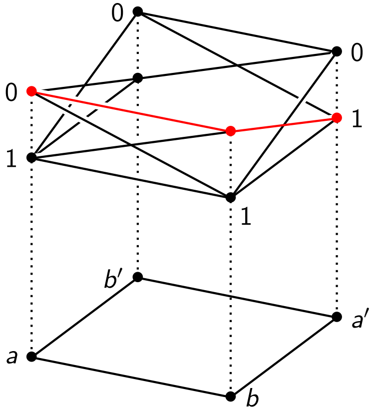


Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

	00	01	10	11
ab	✓	✓	✓	✓
ab'	×	✓	✓	✓
$a'b$	×	✓	✓	✓
$a'b'$	✓	✓	✓	×

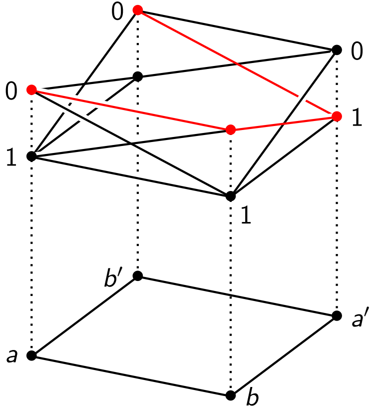


Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

	00	01	10	11
ab	✓	✓	✓	✓
ab'	×	✓	✓	✓
$a'b$	×	✓	✓	✓
$a'b'$	✓	✓	✓	×

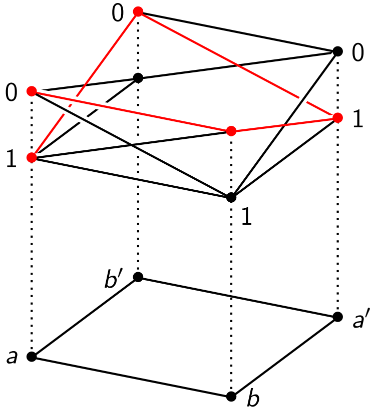


Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

	00	01	10	11
ab	✓	✓	✓	✓
ab'	×	✓	✓	✓
$a'b$	×	✓	✓	✓
$a'b'$	✓	✓	✓	×

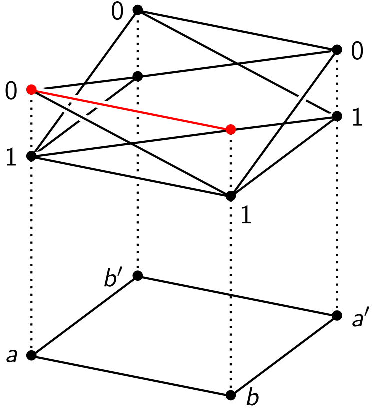


Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

	00	01	10	11
ab	✓	✓	✓	✓
ab'	×	✓	✓	✓
$a'b$	×	✓	✓	✓
$a'b'$	✓	✓	✓	×

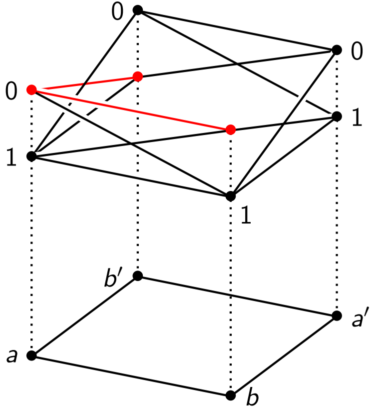


Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

	00	01	10	11
ab	✓	✓	✓	✓
ab'	×	✓	✓	✓
$a'b$	×	✓	✓	✓
$a'b'$	✓	✓	✓	×

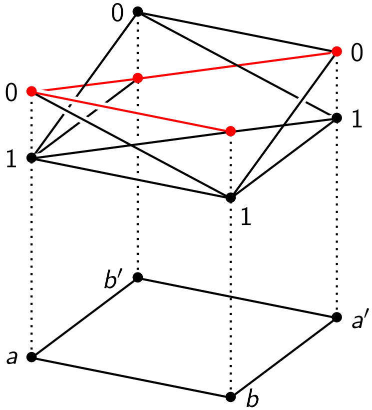


Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

	00	01	10	11
ab	✓	✓	✓	✓
ab'	×	✓	✓	✓
$a'b$	×	✓	✓	✓
$a'b'$	✓	✓	✓	×

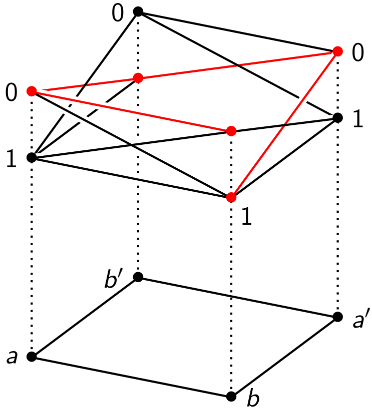


Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

	00	01	10	11
ab	✓	✓	✓	✓
ab'	×	✓	✓	✓
$a'b$	×	✓	✓	✓
$a'b'$	✓	✓	✓	×



Strong Contextuality

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	1	0	0	1
a_1	b_2	1	0	0	1
a_2	b_1	1	0	0	1
a_2	b_2	0	1	1	0

The PR Box

Strong Contextuality

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	1	0	0	1
a_1	b_2	1	0	0	1
a_2	b_1	1	0	0	1
a_2	b_2	0	1	1	0

The PR Box

Note this achieves the algebraic maximum of 4 for our logical Bell inequality.

Strong Contextuality

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	1	0	0	1
a_1	b_2	1	0	0	1
a_2	b_1	1	0	0	1
a_2	b_2	0	1	1	0

The PR Box

Note this achieves the algebraic maximum of 4 for our logical Bell inequality.

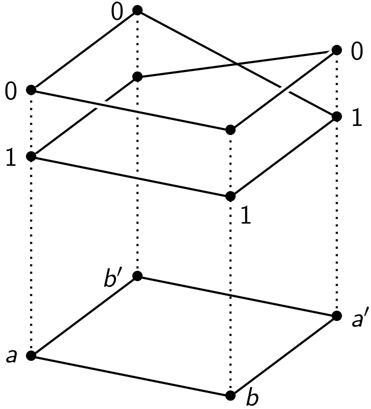
In terms of the XOR game, it is a **winning strategy**.

Bundle Pictures

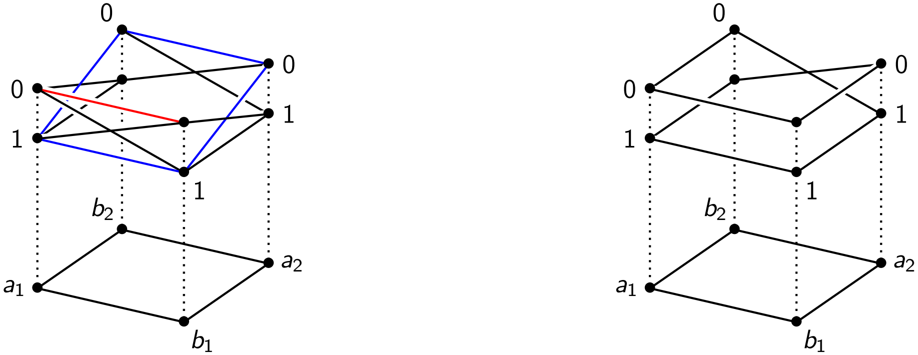
Strong Contextuality

- E.g. the PR box:

	00	01	10	11
ab	✓	×	×	✓
ab'	✓	×	×	✓
$a'b$	✓	×	×	✓
$a'b'$	×	✓	✓	×

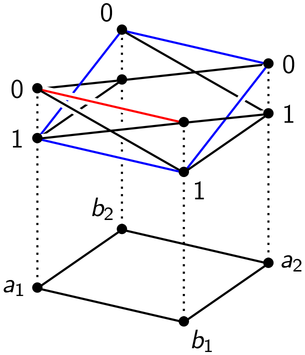


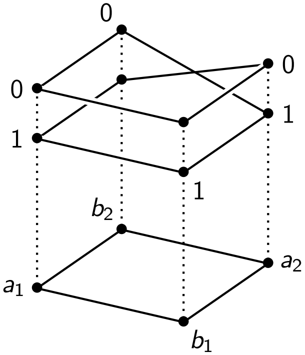
Visualizing Contextuality



The Hardy table and the PR box as bundles

Visualizing Contextuality





The Hardy table and the PR box as bundles

A hierarchy of degrees of contextuality:

$$\text{Bell} < \text{Hardy} < \text{GHZ}$$

Contextuality, Logic and Paradoxes

Contextuality, Logic and Paradoxes

Liar cycles. A Liar cycle of length N is a sequence of statements

$$\begin{aligned} S_1 &: \; S_2 \text{ is true,} \\ S_2 &: \; S_3 \text{ is true,} \\ &\vdots \\ S_{N-1} &: \; S_N \text{ is true,} \\ S_N &: \; S_1 \text{ is false.} \end{aligned}$$

For $N = 1$, this is the classic Liar sentence

$$S : \; S \text{ is false.}$$

Contextuality, Logic and Paradoxes

Liar cycles. A Liar cycle of length N is a sequence of statements

$$\begin{aligned} S_1 &: \ S_2 \text{ is true,} \\ S_2 &: \ S_3 \text{ is true,} \\ &\vdots \\ S_{N-1} &: \ S_N \text{ is true,} \\ S_N &: \ S_1 \text{ is false.} \end{aligned}$$

For $N = 1$, this is the classic Liar sentence

$$S : \ S \text{ is false.}$$

Following Cook, Walicki et al. we can model the situation by boolean equations:

$$x_1 = x_2, \ \ldots, \ x_{n-1} = x_n, \ x_n = \neg x_1$$

Contextuality, Logic and Paradoxes

Liar cycles. A Liar cycle of length N is a sequence of statements

$$\begin{aligned} S_1 &: \; S_2 \text{ is true,} \\ S_2 &: \; S_3 \text{ is true,} \\ &\vdots \\ S_{N-1} &: \; S_N \text{ is true,} \\ S_N &: \; S_1 \text{ is false.} \end{aligned}$$

For $N = 1$, this is the classic Liar sentence

$$S : \; S \text{ is false.}$$

Following Cook, Walicki et al. we can model the situation by boolean equations:

$$x_1 = x_2, \quad \ldots, \quad x_{n-1} = x_n, \quad x_n = \neg x_1$$

The "paradoxical" nature of the original statements is now captured by the inconsistency of these equations.

Contextuality in the Liar; Liar cycles in the PR Box

Contextuality in the Liar; Liar cycles in the PR Box

We can regard each of these equations as fibered over the set of variables which occur in it:

$$
\begin{array}{rrcl}
\{x_1, x_2\}: & x_1 & = & x_2 \\
\{x_2, x_3\}: & x_2 & = & x_3 \\
 & \vdots & & \\
\{x_{n-1}, x_n\}: & x_{n-1} & = & x_n \\
\{x_n, x_1\}: & x_n & = & \neg x_1
\end{array}
$$

Contextuality in the Liar; Liar cycles in the PR Box

We can regard each of these equations as fibered over the set of variables which occur in it:

$$\begin{array}{rrcl}
\{x_1, x_2\}: & x_1 & = & x_2 \\
\{x_2, x_3\}: & x_2 & = & x_3 \\
& \vdots & & \\
\{x_{n-1}, x_n\}: & x_{n-1} & = & x_n \\
\{x_n, x_1\}: & x_n & = & \neg x_1
\end{array}$$

Any subset of up to $n-1$ of these equations is consistent; while the whole set is inconsistent.

Contextuality in the Liar; Liar cycles in the PR Box

We can regard each of these equations as fibered over the set of variables which occur in it:

$$
\begin{aligned}
\{x_1, x_2\} : \quad x_1 &= x_2 \\
\{x_2, x_3\} : \quad x_2 &= x_3 \\
&\vdots \\
\{x_{n-1}, x_n\} : \quad x_{n-1} &= x_n \\
\{x_n, x_1\} : \quad x_n &= \neg x_1
\end{aligned}
$$

Any subset of up to $n-1$ of these equations is consistent; while the whole set is inconsistent.

Up to rearrangement, **the Liar cycle of length 4 corresponds exactly to the PR box**.

Contextuality in the Liar; Liar cycles in the PR Box

We can regard each of these equations as fibered over the set of variables which occur in it:

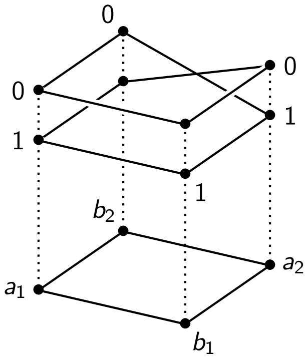
$$\begin{array}{rrcl}
\{x_1, x_2\}: & x_1 & = & x_2 \\
\{x_2, x_3\}: & x_2 & = & x_3 \\
& & \vdots & \\
\{x_{n-1}, x_n\}: & x_{n-1} & = & x_n \\
\{x_n, x_1\}: & x_n & = & \neg x_1
\end{array}$$

Any subset of up to $n-1$ of these equations is consistent; while the whole set is inconsistent.

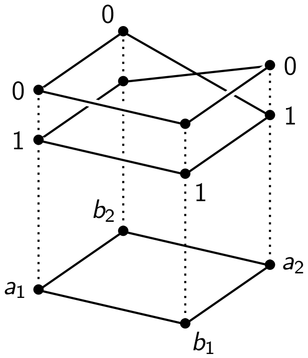
Up to rearrangement, **the Liar cycle of length 4 corresponds exactly to the PR box**.

The usual reasoning to derive a contradiction from the Liar cycle corresponds precisely to the attempt to find a univocal path in the bundle diagram.

Paths to contradiction



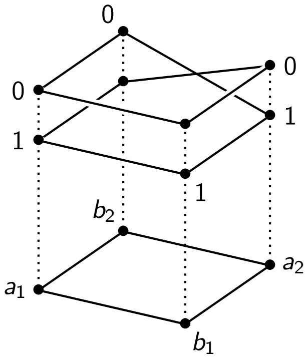
Paths to contradiction



Suppose that we try to set a_2 to 1. Following the path on the right leads to the following local propagation of values:

$$a_2 = 1 \rightsquigarrow b_1 = 1 \rightsquigarrow a_1 = 1 \rightsquigarrow b_2 = 1 \rightsquigarrow a_2 = 0$$
$$a_2 = 0 \rightsquigarrow b_1 = 0 \rightsquigarrow a_1 = 0 \rightsquigarrow b_2 = 0 \rightsquigarrow a_2 = 1$$

Paths to contradiction



Suppose that we try to set a_2 to 1. Following the path on the right leads to the following local propagation of values:

$$a_2 = 1 \rightsquigarrow b_1 = 1 \rightsquigarrow a_1 = 1 \rightsquigarrow b_2 = 1 \rightsquigarrow a_2 = 0$$
$$a_2 = 0 \rightsquigarrow b_1 = 0 \rightsquigarrow a_1 = 0 \rightsquigarrow b_2 = 0 \rightsquigarrow a_2 = 1$$

We have discussed a specific case here, but the analysis can be generalised to a large class of examples.

The Robinson Consistency Theorem

The Robinson Consistency Theorem

A classic result:

Theorem (Robinson Joint Consistency Theorem)

Let T_i be a theory over the language L_i, $i = 1, 2$. If there is no sentence ϕ in $L_1 \cap L_2$ with $T_1 \vdash \phi$ and $T_2 \vdash \neg\phi$, then $T_1 \cup T_2$ is consistent.

The Robinson Consistency Theorem

A classic result:

Theorem (Robinson Joint Consistency Theorem)

Let T_i be a theory over the language L_i, $i = 1, 2$. If there is no sentence ϕ in $L_1 \cap L_2$ with $T_1 \vdash \phi$ and $T_2 \vdash \neg\phi$, then $T_1 \cup T_2$ is consistent.

Thus this theorem says that two compatible theories can be glued together. In this binary case, local consistency implies global consistency.

The Robinson Consistency Theorem

A classic result:

Theorem (Robinson Joint Consistency Theorem)

Let T_i be a theory over the language L_i, $i = 1, 2$. If there is no sentence ϕ in $L_1 \cap L_2$ with $T_1 \vdash \phi$ and $T_2 \vdash \neg\phi$, then $T_1 \cup T_2$ is consistent.

Thus this theorem says that two compatible theories can be glued together. In this binary case, local consistency implies global consistency.

Note, however, that an extension of the theorem beyond the binary case **fails**. That is, if we have three theories which are pairwise compatible, it need not be the case that they can be glued together consistently.

The Robinson Consistency Theorem

A classic result:

Theorem (Robinson Joint Consistency Theorem)

Let T_i be a theory over the language L_i, $i = 1, 2$. If there is no sentence ϕ in $L_1 \cap L_2$ with $T_1 \vdash \phi$ and $T_2 \vdash \neg\phi$, then $T_1 \cup T_2$ is consistent.

Thus this theorem says that two compatible theories can be glued together. In this binary case, local consistency implies global consistency.

Note, however, that an extension of the theorem beyond the binary case **fails**. That is, if we have three theories which are pairwise compatible, it need not be the case that they can be glued together consistently.

A minimal counter-example is provided at the propositional level by the following "triangle":

$$T_1 = \{x_1 \longleftrightarrow \neg x_2\}, \; T_2 = \{x_2 \longleftrightarrow \neg x_3\}, \; T_3 = \{x_3 \longleftrightarrow \neg x_1\}.$$

The Robinson Consistency Theorem

A classic result:

Theorem (Robinson Joint Consistency Theorem)

Let T_i be a theory over the language L_i, $i = 1, 2$. If there is no sentence ϕ in $L_1 \cap L_2$ with $T_1 \vdash \phi$ and $T_2 \vdash \neg\phi$, then $T_1 \cup T_2$ is consistent.

Thus this theorem says that two compatible theories can be glued together. In this binary case, local consistency implies global consistency.

Note, however, that an extension of the theorem beyond the binary case **fails**. That is, if we have three theories which are pairwise compatible, it need not be the case that they can be glued together consistently.

A minimal counter-example is provided at the propositional level by the following "triangle":

$$T_1 = \{x_1 \longleftrightarrow \neg x_2\}, \; T_2 = \{x_2 \longleftrightarrow \neg x_3\}, \; T_3 = \{x_3 \longleftrightarrow \neg x_1\}.$$

This example is well-known in the quantum contextuality literature as the **Specker triangle**.

Formalizing Contextuality: Measurement Scenarios

A **measurement scenario** is a triple $(X, \mathcal{M}, O)$ where:

Formalizing Contextuality: Measurement Scenarios

A **measurement scenario** is a triple $(X, \mathcal{M}, O)$ where:

- X is a set of variables which can be measured, observed or evaluated

Formalizing Contextuality: Measurement Scenarios

A **measurement scenario** is a triple $(X, \mathcal{M}, O)$ where:

- X is a set of variables which can be measured, observed or evaluated
- $\mathcal{M}$ is a family of sets of variables, those which can be measured together. These form the **contexts**.

Formalizing Contextuality: Measurement Scenarios

A **measurement scenario** is a triple $(X, \mathcal{M}, O)$ where:

- X is a set of variables which can be measured, observed or evaluated
- $\mathcal{M}$ is a family of sets of variables, those which can be measured together. These form the **contexts**.
- O is a set of possible outcomes or values for the variables.

Formalizing Contextuality: Measurement Scenarios

A **measurement scenario** is a triple $(X, \mathcal{M}, O)$ where:

- X is a set of variables which can be measured, observed or evaluated
- $\mathcal{M}$ is a family of sets of variables, those which can be measured together. These form the **contexts**.
- O is a set of possible outcomes or values for the variables.

Example:

Formalizing Contextuality: Measurement Scenarios

A **measurement scenario** is a triple $(X, \mathcal{M}, O)$ where:

- X is a set of variables which can be measured, observed or evaluated
- $\mathcal{M}$ is a family of sets of variables, those which can be measured together. These form the **contexts**.
- O is a set of possible outcomes or values for the variables.

Example:

In our tables, the set of variables is $X = \{a, a', b, b'\}$.

Formalizing Contextuality: Measurement Scenarios

A **measurement scenario** is a triple $(X, \mathcal{M}, O)$ where:

- X is a set of variables which can be measured, observed or evaluated
- $\mathcal{M}$ is a family of sets of variables, those which can be measured together. These form the **contexts**.
- O is a set of possible outcomes or values for the variables.

Example:

In our tables, the set of variables is $X = \{a, a', b, b'\}$.
The measurement contexts are:

$$\{\{a_1, b_1\}, \quad \{a_2, b_1\}, \quad \{a_1, b_2\}, \quad \{a_2, b_2\}\}$$

Formalizing Contextuality: Measurement Scenarios

A **measurement scenario** is a triple $(X, \mathcal{M}, O)$ where:

- X is a set of variables which can be measured, observed or evaluated
- $\mathcal{M}$ is a family of sets of variables, those which can be measured together. These form the **contexts**.
- O is a set of possible outcomes or values for the variables.

Example:

In our tables, the set of variables is $X = \{a, a', b, b'\}$.
The measurement contexts are:

$$\{\{a_1, b_1\}, \quad \{a_2, b_1\}, \quad \{a_1, b_2\}, \quad \{a_2, b_2\}\}$$

The outcomes are

$$O \; = \; \{0, 1\}$$

Formalizing Contextuality: Measurement Scenarios

A **measurement scenario** is a triple $(X, \mathcal{M}, O)$ where:

- X is a set of variables which can be measured, observed or evaluated
- $\mathcal{M}$ is a family of sets of variables, those which can be measured together. These form the **contexts**.
- O is a set of possible outcomes or values for the variables.

Example:

In our tables, the set of variables is $X = \{a, a', b, b'\}$.
The measurement contexts are:

$$\{\{a_1, b_1\}, \quad \{a_2, b_1\}, \quad \{a_1, b_2\}, \quad \{a_2, b_2\}\}$$

The outcomes are

$$O \ = \ \{0, 1\}$$

A joint outcome or **event** in a context C is $s \in O^C$, e.g. $s = \{a \mapsto 0, b \mapsto 1\}$.

A Kochen-Specker construction

A Kochen-Specker construction

This uses

A Kochen-Specker construction

This uses

- A set X of 18 variables, $\{A, \ldots, O\}$

A Kochen-Specker construction

This uses

- A set X of 18 variables, $\{A, \ldots, O\}$
- A measurement cover $\mathcal{U} = \{U_1, \ldots, U_9\}$, where the columns U_i are the sets in the cover:

A Kochen-Specker construction

This uses

- A set X of 18 variables, $\{A, \ldots, O\}$
- A measurement cover $\mathcal{U} = \{U_1, \ldots, U_9\}$, where the columns U_i are the sets in the cover:

U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9
A	A	H	H	B	I	P	P	Q
B	E	I	K	E	K	Q	R	R
C	F	C	G	M	N	D	F	M
D	G	J	L	N	O	J	L	O

A Kochen-Specker construction

This uses

- A set X of 18 variables, $\{A, \ldots, O\}$
- A measurement cover $\mathcal{U} = \{U_1, \ldots, U_9\}$, where the columns U_i are the sets in the cover:

U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9
A	A	H	H	B	I	P	P	Q
B	E	I	K	E	K	Q	R	R
C	F	C	G	M	N	D	F	M
D	G	J	L	N	O	J	L	O

The original K-S construction used 117 variables!

Empirical Models

Empirical Models

Let $(X, \mathcal{M}, O)$ be a measurement scenario. An **empirical model** for this scenario is a family

$$\{d_C\}_{C \in \mathcal{M}}$$

where $d_C \in \mathrm{Prob}(O^C)$ for $C \in \mathcal{M}$.

Empirical Models

Let $(X, \mathcal{M}, O)$ be a measurement scenario. An **empirical model** for this scenario is a family

$$\{d_C\}_{C \in \mathcal{M}}$$

where $d_C \in \mathrm{Prob}(O^C)$ for $C \in \mathcal{M}$.

In other words, the empirical model specifies a probability distribution over the events in each context.

Empirical Models

Let $(X, \mathcal{M}, O)$ be a measurement scenario. An **empirical model** for this scenario is a family

$$\{d_C\}_{C \in \mathcal{M}}$$

where $d_C \in \mathrm{Prob}(O^C)$ for $C \in \mathcal{M}$.

In other words, the empirical model specifies a probability distribution over the events in each context.

These distributions are the rows of our probability tables.

Mathematical Structure of Probability Tables

Mathematical Structure of Probability Tables

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a	b	0	$1/2$	$1/2$	0
a'	b	$3/8$	$1/8$	$1/8$	$3/8$
a	b'	$3/8$	$1/8$	$1/8$	$3/8$
a'	b'	$3/8$	$1/8$	$1/8$	$3/8$

Mathematical Structure of Probability Tables

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a	b	0	$1/2$	$1/2$	0
a'	b	$3/8$	$1/8$	$1/8$	$3/8$
a	b'	$3/8$	$1/8$	$1/8$	$3/8$
a'	b'	$3/8$	$1/8$	$1/8$	$3/8$

The **measurement contexts** are

$$\{a,b\}, \quad \{a',b\}, \quad \{a,b'\}, \quad \{a',b'\}.$$

Mathematical Structure of Probability Tables

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a	b	0	$1/2$	$1/2$	0
a'	b	$3/8$	$1/8$	$1/8$	$3/8$
a	b'	$3/8$	$1/8$	$1/8$	$3/8$
a'	b'	$3/8$	$1/8$	$1/8$	$3/8$

The **measurement contexts** are

$$\{a, b\}, \quad \{a', b\}, \quad \{a, b'\}, \quad \{a', b'\}.$$

Each measurement has possible outcomes 0 or 1. The matrix entry at row (a', b) and column $(0, 1)$ indicates the **event**

$$\{a' \mapsto 0, \; b \mapsto 1\}.$$

Mathematical Structure of Probability Tables

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a	b	0	1/2	1/2	0
a'	b	3/8	1/8	1/8	3/8
a	b'	3/8	1/8	1/8	3/8
a'	b'	3/8	1/8	1/8	3/8

The **measurement contexts** are

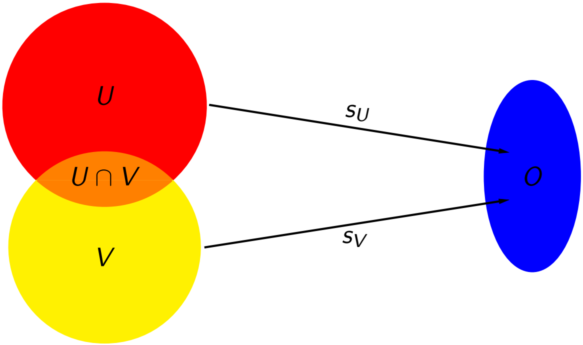
$$\{a, b\}, \quad \{a', b\}, \quad \{a, b'\}, \quad \{a', b'\}.$$

Each measurement has possible outcomes 0 or 1. The matrix entry at row (a', b) and column $(0, 1)$ indicates the **event**

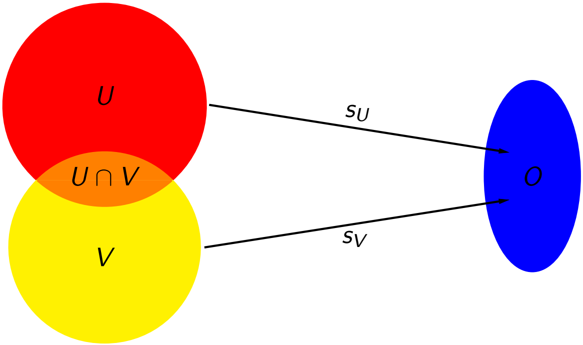
$$\{a' \mapsto 0, \ b \mapsto 1\}.$$

Each row of the table specifies a **probability distribution** on events O^C for a given choice of measurements C.

Gluing functional sections



Gluing functional sections



If $s_U|_{U \cap V} = s_V|_{U \cap V}$, they can be glued to form

$$s : U \cup V \longrightarrow O$$

such that $s|_U = s_U$ and $s|_V = s_V$.

The need for restriction

The need for restriction

We would like to express the condition that an empirical model is **compatible**, *i.e.* "locally consistent".

The need for restriction

We would like to express the condition that an empirical model is **compatible**, *i.e.* "locally consistent".

We want to do this by saying that the distributions "agree on overlaps". For all $C, C' \in \mathcal{M}$:

$$d_C|_{C \cap C'} \ = \ d_{C'}|_{C \cap C'}.$$

The need for restriction

We would like to express the condition that an empirical model is **compatible**, *i.e.* “locally consistent”.

We want to do this by saying that the distributions “agree on overlaps”. For all $C, C' \in \mathcal{M}$:

$$d_C|_{C \cap C'} \; = \; d_{C'}|_{C \cap C'}.$$

Cf. the usual notion of compatibility of a family of functions defined on subsets.

The need for restriction

We would like to express the condition that an empirical model is **compatible**, *i.e.* "locally consistent".

We want to do this by saying that the distributions "agree on overlaps". For all $C, C' \in \mathcal{M}$:

$$d_C|_{C \cap C'} \; = \; d_{C'}|_{C \cap C'}.$$

Cf. the usual notion of compatibility of a family of functions defined on subsets.

A formula for restriction of distributions: if $C' \subseteq C$, $d \in \mathrm{Prob}(O^C)$,

$$d|_{C'}(s) \; := \sum_{t \in O^C,\, t|_C = s} d(t)$$

The need for restriction

We would like to express the condition that an empirical model is **compatible**, *i.e.* "locally consistent".

We want to do this by saying that the distributions "agree on overlaps". For all $C, C' \in \mathcal{M}$:

$$d_C|_{C \cap C'} \; = \; d_{C'}|_{C \cap C'}.$$

Cf. the usual notion of compatibility of a family of functions defined on subsets.

A formula for restriction of distributions: if $C' \subseteq C$, $d \in \mathrm{Prob}(O^C)$,

$$d|_{C'}(s) \; := \sum_{t \in O^C,\, t|_C = s} d(t)$$

This is just **marginalization**: if $C = C' \sqcup C''$, then $O^C = O^{C'} \times O^{C''}$.

The need for restriction

We would like to express the condition that an empirical model is **compatible**, *i.e.* "locally consistent".

We want to do this by saying that the distributions "agree on overlaps". For all $C, C' \in \mathcal{M}$:

$$d_C|_{C\cap C'} \;=\; d_{C'}|_{C\cap C'}.$$

Cf. the usual notion of compatibility of a family of functions defined on subsets.

A formula for restriction of distributions: if $C' \subseteq C$, $d \in \mathrm{Prob}(O^C)$,

$$d|_{C'}(s) \;:=\; \sum_{t \in O^C,\, t|_C = s} d(t)$$

This is just **marginalization**: if $C = C' \sqcup C''$, then $O^C = O^{C'} \times O^{C''}$.

So compatibility says that the distributions on different contexts have consistent marginals.

Compatibility and No-Signalling

There is an important physical principle of **No-Signalling**:

Compatibility and No-Signalling

There is an important physical principle of **No-Signalling**:

- Suppose that $C = \{a, b\}$, and $C' = \{a, b'\}$, where a is a variable measured by an agent Alice, while b and b' are variables measured by Bob, who may be spacelike separated from Alice.

Compatibility and No-Signalling

There is an important physical principle of **No-Signalling**:

- Suppose that $C = \{a, b\}$, and $C' = \{a, b'\}$, where a is a variable measured by an agent Alice, while b and b' are variables measured by Bob, who may be spacelike separated from Alice.
- Then under relativistic constraints, Bob's choice of measurement — b or b' — should not be able to affect the distribution Alice observes on the outcomes from her measurement of a.

Compatibility and No-Signalling

There is an important physical principle of **No-Signalling**:

- Suppose that $C = \{a, b\}$, and $C' = \{a, b'\}$, where a is a variable measured by an agent Alice, while b and b' are variables measured by Bob, who may be spacelike separated from Alice.
- Then under relativistic constraints, Bob's choice of measurement — b or b' — should not be able to affect the distribution Alice observes on the outcomes from her measurement of a.
- This is captured by saying that the distribution on $\{a\} = \{a, b\} \cap \{a, b'\}$ is the same whether we marginalize from the distribution e_C, or the distribution $e_{C'}$.

Compatibility and No-Signalling

There is an important physical principle of **No-Signalling**:

- Suppose that $C = \{a, b\}$, and $C' = \{a, b'\}$, where a is a variable measured by an agent Alice, while b and b' are variables measured by Bob, who may be spacelike separated from Alice.
- Then under relativistic constraints, Bob's choice of measurement — b or b' — should not be able to affect the distribution Alice observes on the outcomes from her measurement of a.
- This is captured by saying that the distribution on $\{a\} = \{a, b\} \cap \{a, b'\}$ is the same whether we marginalize from the distribution e_C, or the distribution $e_{C'}$.
- This condition is generalized by compatibility – and this general form is satisfied by quantum systems.

No-Signalling for Alice-Bob Tables

No-Signalling for Alice-Bob Tables

Consider the following schematic representation of an Alice-Bob table:

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a	b	c	d	e	f
a'	b	g	h	i	j
a	b'	k	l	m	n
a'	b'	o	p	q	r

where we have labelled the entries with the letters $c, \ldots, r$.

No-Signalling for Alice-Bob Tables

Consider the following schematic representation of an Alice-Bob table:

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a	b	c	d	e	f
a'	b	g	h	i	j
a	b'	k	l	m	n
a'	b'	o	p	q	r

where we have labelled the entries with the letters $c, \ldots, r$.

The no-signalling conditions for the non-empty intersections of contexts are given by the following equations:

$$c + e = k + m, \qquad d + f = l + n, \qquad g + i = o + q, \qquad h + j = p + r$$

$$c + d = g + h, \qquad e + f = i + j, \qquad k + l = o + p, \qquad m + n = q + r$$

No-Signalling for Alice-Bob Tables

Consider the following schematic representation of an Alice-Bob table:

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a	b	c	d	e	f
a'	b	g	h	i	j
a	b'	k	l	m	n
a'	b'	o	p	q	r

where we have labelled the entries with the letters $c, \ldots, r$.

The no-signalling conditions for the non-empty intersections of contexts are given by the following equations:

$$c + e = k + m, \qquad d + f = l + n, \qquad g + i = o + q, \qquad h + j = p + r$$

$$c + d = g + h, \qquad e + f = i + j, \qquad k + l = o + p, \qquad m + n = q + r$$

You can check that these conditions are satisfied by the Bell table.

No-Signalling for Alice-Bob Tables

Consider the following schematic representation of an Alice-Bob table:

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a	b	c	d	e	f
a'	b	g	h	i	j
a	b'	k	l	m	n
a'	b'	o	p	q	r

where we have labelled the entries with the letters $c, \ldots, r$.

The no-signalling conditions for the non-empty intersections of contexts are given by the following equations:

$$c + e = k + m, \qquad d + f = l + n, \qquad g + i = o + q, \qquad h + j = p + r$$

$$c + d = g + h, \qquad e + f = i + j, \qquad k + l = o + p, \qquad m + n = q + r$$

You can check that these conditions are satisfied by the Bell table.

Moreover, the PR box has a **unique family of distributions** which satisfy these conditions.

Contextuality defined

Contextuality defined

An empirical model $\{d_C\}_{C \in \mathcal{M}}$ on a measurement scenario $(X, \mathcal{M}, O)$ is **non-contextual** if there is a distribution $d \in \mathrm{Prob}(O^X)$ such that, for all $C \in \mathcal{M}$:

$$d|_C \; = \; d_C.$$

Contextuality defined

An empirical model $\{d_C\}_{C \in \mathcal{M}}$ on a measurement scenario $(X, \mathcal{M}, O)$ is **non-contextual** if there is a distribution $d \in \mathsf{Prob}(O^X)$ such that, for all $C \in \mathcal{M}$:

$$d|_C \; = \; d_C.$$

That is, we can glue all the local information together into a global consistent description from which the local information can be recovered.

Contextuality defined

An empirical model $\{d_C\}_{C \in \mathcal{M}}$ on a measurement scenario $(X, \mathcal{M}, O)$ is **non-contextual** if there is a distribution $d \in \mathrm{Prob}(O^X)$ such that, for all $C \in \mathcal{M}$:

$$d|_C = d_C.$$

That is, we can glue all the local information together into a global consistent description from which the local information can be recovered.

We call such a d a **global section**.

Contextuality defined

An empirical model $\{d_C\}_{C \in \mathcal{M}}$ on a measurement scenario $(X, \mathcal{M}, O)$ is **non-contextual** if there is a distribution $d \in \mathrm{Prob}(O^X)$ such that, for all $C \in \mathcal{M}$:

$$d|_C \ = \ d_C.$$

That is, we can glue all the local information together into a global consistent description from which the local information can be recovered.

We call such a d a **global section**.

If no such global section exists, the empirical model is **contextual**.

Contextuality defined

An empirical model $\{d_C\}_{C \in \mathcal{M}}$ on a measurement scenario $(X, \mathcal{M}, O)$ is **non-contextual** if there is a distribution $d \in \mathrm{Prob}(O^X)$ such that, for all $C \in \mathcal{M}$:

$$d|_C \;=\; d_C.$$

That is, we can glue all the local information together into a global consistent description from which the local information can be recovered.

We call such a d a **global section**.

If no such global section exists, the empirical model is **contextual**.

The import of Bell's theorem and similar results is that there are empirical models arising from quantum mechanics which are contextual.

Classes of Empirical Models

Classes of Empirical Models

There is a class of empirical models, for each measurement scenario $(X, \mathcal{M}, O)$, which are **quantum realizable**.

Classes of Empirical Models

There is a class of empirical models, for each measurement scenario $(X, \mathcal{M}, O)$, which are **quantum realizable**.

That is, we can find quantum states and local observables which generate the family of distributions $\{d_C\}_{C \in \mathcal{M}}$.

Classes of Empirical Models

There is a class of empirical models, for each measurement scenario $(X, \mathcal{M}, O)$, which are **quantum realizable**.

That is, we can find quantum states and local observables which generate the family of distributions $\{d_C\}_{C \in \mathcal{M}}$.

It turns out that **all quantum realizable models are compatible**.

Classes of Empirical Models

There is a class of empirical models, for each measurement scenario $(X, \mathcal{M}, O)$, which are **quantum realizable**.

That is, we can find quantum states and local observables which generate the family of distributions $\{d_C\}_{C \in \mathcal{M}}$.

It turns out that **all quantum realizable models are compatible**.

Compatibility is in fact the general form of an important physical principle known as No-Signalling, which ensures the consistency of quantum mechanics with Special Relativity.

Classes of Empirical Models

There is a class of empirical models, for each measurement scenario $(X, \mathcal{M}, O)$, which are **quantum realizable**.

That is, we can find quantum states and local observables which generate the family of distributions $\{d_C\}_{C \in \mathcal{M}}$.

It turns out that **all quantum realizable models are compatible**.

Compatibility is in fact the general form of an important physical principle known as No-Signalling, which ensures the consistency of quantum mechanics with Special Relativity.

However, there are compatible (*i.e.* No-Signalling) empirical models which are **not** quantum realizable.

Classes of Empirical Models

There is a class of empirical models, for each measurement scenario $(X, \mathcal{M}, O)$, which are **quantum realizable**.

That is, we can find quantum states and local observables which generate the family of distributions $\{d_C\}_{C \in \mathcal{M}}$.

It turns out that **all quantum realizable models are compatible**.

Compatibility is in fact the general form of an important physical principle known as No-Signalling, which ensures the consistency of quantum mechanics with Special Relativity.

However, there are compatible (*i.e.* No-Signalling) empirical models which are **not** quantum realizable.

We thus get a strict hierarchy of empirical models:

$$\mathsf{NC} \;\subset\; \mathsf{QM} \;\subset\; \mathsf{NS}$$

The PR Box

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	1	0	0	1
a_1	b_2	1	0	0	1
a_2	b_1	1	0	0	1
a_2	b_2	0	1	1	0

The PR Box

The PR Box

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	1	0	0	1
a_1	b_2	1	0	0	1
a_2	b_1	1	0	0	1
a_2	b_2	0	1	1	0

The PR Box

This satisfies No-Signalling, so is consistent with SR, but it is **not** quantum realisable.

Empirical Models as Vectors

Empirical Models as Vectors

We can regard an empirical model $\{d_C\}_{C \in \mathcal{M}}$ as a vector

$$\mathbf{v} = (\mathbf{v}_{C,s})_{C \in \mathcal{M}, s \in \mathcal{O}^C}, \qquad \mathbf{v}_{C,s} \ := \ d_C(s)$$

in a high-dimensional real vector space.

Empirical Models as Vectors

We can regard an empirical model $\{d_C\}_{C \in \mathcal{M}}$ as a vector

$$\mathbf{v} = (\mathbf{v}_{C,s})_{C \in \mathcal{M}, s \in O^C}, \qquad \mathbf{v}_{C,s} \ := \ d_C(s)$$

in a high-dimensional real vector space.

Note that, in a Bell-type scenario with n parties, k measurement choices at each site, and ℓ possible outcomes for each measurement, the dimension is $k^n \ell^n$.

Empirical Models as Vectors

We can regard an empirical model $\{d_C\}_{C \in \mathcal{M}}$ as a vector

$$\mathbf{v} = (\mathbf{v}_{C,s})_{C \in \mathcal{M}, s \in O^C}, \qquad \mathbf{v}_{C,s} \; := \; d_C(s)$$

in a high-dimensional real vector space.

Note that, in a Bell-type scenario with n parties, k measurement choices at each site, and ℓ possible outcomes for each measurement, the dimension is $k^n \ell^n$.

Note that empirical models over a given measurement scenario are closed under convex combinations:

$$(\mu d + (1-\mu)d')_C(s) \; := \; \mu d_C(s) + (1-\mu)d'_C(s).$$

Empirical Models as Vectors

We can regard an empirical model $\{d_C\}_{C \in \mathcal{M}}$ as a vector

$$\mathbf{v} = (\mathbf{v}_{C,s})_{C \in \mathcal{M}, s \in O^C}, \qquad \mathbf{v}_{C,s} \ := \ d_C(s)$$

in a high-dimensional real vector space.

Note that, in a Bell-type scenario with n parties, k measurement choices at each site, and ℓ possible outcomes for each measurement, the dimension is $k^n \ell^n$.

Note that empirical models over a given measurement scenario are closed under convex combinations:

$$(\mu d + (1 - \mu) d')_C(s) \ := \ \mu d_C(s) + (1 - \mu) d'_C(s).$$

Moreover, convex combinations of compatible models are compatible.

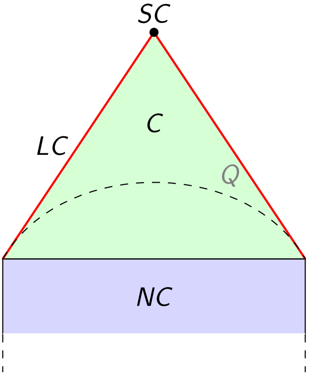
The Quantum Set

The Quantum Set

A subtle convex set sandwiched between two polytopes.

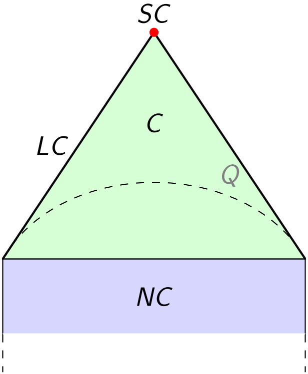
The Quantum Set

A subtle convex set sandwiched between two polytopes.



The Quantum Set

A subtle convex set sandwiched between two polytopes.



Key question: find compelling principles to explain why Nature picks out the quantum set.

The Support of a Model

The Support of a Model

The **support** of an empirical model $\{d_C\}_{C \in \mathcal{M}}$ is defined as follows. For each $C \in \mathcal{M}$, we define $\mathcal{S}(C) \subseteq O^C$:

$$\mathcal{S}(C) \; := \; \{s \in O^C \mid d_C(s) \neq 0\}$$

The Support of a Model

The **support** of an empirical model $\{d_C\}_{C \in \mathcal{M}}$ is defined as follows. For each $C \in \mathcal{M}$, we define $\mathcal{S}(C) \subseteq O^C$:

$$\mathcal{S}(C) \;:=\; \{s \in O^C \mid d_C(s) \neq 0\}$$

If the empirical model is compatible, so is the support in the following sense: for all $C, C' \in \mathcal{M}$

$$\{s|_{C \cap C'} \,:\, s \in \mathcal{S}(C)\} \;=\; \{s'|_{C \cap C'} \,:\, s' \in \mathcal{S}(C')\}$$

The Support of a Model

The **support** of an empirical model $\{d_C\}_{C \in \mathcal{M}}$ is defined as follows. For each $C \in \mathcal{M}$, we define $\mathcal{S}(C) \subseteq O^C$:

$$\mathcal{S}(C) \;:=\; \{s \in O^C \mid d_C(s) \neq 0\}$$

If the empirical model is compatible, so is the support in the following sense: for all $C, C' \in \mathcal{M}$

$$\{s|_{C \cap C'} \,:\, s \in \mathcal{S}(C)\} \;=\; \{s'|_{C \cap C'} \,:\, s' \in \mathcal{S}(C')\}$$

Thus the support satisfies No-Signalling at the level of **possibilities**.

The Support of a Model

The **support** of an empirical model $\{d_C\}_{C \in \mathcal{M}}$ is defined as follows. For each $C \in \mathcal{M}$, we define $\mathcal{S}(C) \subseteq O^C$:

$$\mathcal{S}(C) \; := \; \{s \in O^C \mid d_C(s) \neq 0\}$$

If the empirical model is compatible, so is the support in the following sense: for all $C, C' \in \mathcal{M}$

$$\{s|_{C \cap C'} \, : \, s \in \mathcal{S}(C)\} \; = \; \{s'|_{C \cap C'} \, : \, s' \in \mathcal{S}(C')\}$$

Thus the support satisfies No-Signalling at the level of **possibilities**.

This is equivalent to saying that, for all $C \subseteq C'$, the restriction map

$$\rho^{C'}_C : \mathcal{S}(C') \longrightarrow \mathcal{S}(C) \quad :: s \mapsto s|_C$$

is surjective.

Degrees of contextuality

Degrees of contextuality

Firstly, we say that a global assignment $t \in O^X$ is **consistent with the support** of a model if for all $C' \in \mathcal{M}$, $t|_{C'}$ is in the support at C'.

Degrees of contextuality

Firstly, we say that a global assignment $t \in O^X$ is **consistent with the support** of a model if for all $C' \in \mathcal{M}$, $t|_{C'}$ is in the support at C'.

An empirical model is

Degrees of contextuality

Firstly, we say that a global assignment $t \in O^X$ is **consistent with the support** of a model if for all $C' \in \mathcal{M}$, $t|_{C'}$ is in the support at C'.

An empirical model is

- **logically contextual** if some possible joint outcome $s \in O^C$ in the support is not accounted for by any global assignment $t \in O^X$ which is consistent with the support of the model. That is, for no such t do we have $t|C = s$.

Degrees of contextuality

Firstly, we say that a global assignment $t \in O^X$ is **consistent with the support** of a model if for all $C' \in \mathcal{M}$, $t|_{C'}$ is in the support at C'.

An empirical model is

- **logically contextual** if some possible joint outcome $s \in O^C$ in the support is not accounted for by any global assignment $t \in O^X$ which is consistent with the support of the model. That is, for no such t do we have $t|C = s$.

  Geometrically, this is saying that some local section cannot be extended to a global one. Equivalently, that the support of the model cannot be covered by the consistent global assignments.

Degrees of contextuality

Firstly, we say that a global assignment $t \in O^X$ is **consistent with the support** of a model if for all $C' \in \mathcal{M}$, $t|_{C'}$ is in the support at C'.

An empirical model is

- **logically contextual** if some possible joint outcome $s \in O^C$ in the support is not accounted for by any global assignment $t \in O^X$ which is consistent with the support of the model. That is, for no such t do we have $t|C = s$.

  Geometrically, this is saying that some local section cannot be extended to a global one. Equivalently, that the support of the model cannot be covered by the consistent global assignments.

- It is **strongly contextual** if its support has **no global section**; that is, there is no consistent global assignment.

Degrees of contextuality

Firstly, we say that a global assignment $t \in O^X$ is **consistent with the support** of a model if for all $C' \in \mathcal{M}$, $t|_{C'}$ is in the support at C'.

An empirical model is

- **logically contextual** if some possible joint outcome $s \in O^C$ in the support is not accounted for by any global assignment $t \in O^X$ which is consistent with the support of the model. That is, for no such t do we have $t|C = s$.

  Geometrically, this is saying that some local section cannot be extended to a global one. Equivalently, that the support of the model cannot be covered by the consistent global assignments.

- It is **strongly contextual** if its support has **no global section**; that is, there is no consistent global assignment.

  This says that **no** possible joint outcome is accounted for by **any** global section!

Degrees of contextuality

Firstly, we say that a global assignment $t \in O^X$ is **consistent with the support** of a model if for all $C' \in \mathcal{M}$, $t|_{C'}$ is in the support at C'.

An empirical model is

- **logically contextual** if some possible joint outcome $s \in O^C$ in the support is not accounted for by any global assignment $t \in O^X$ which is consistent with the support of the model. That is, for no such t do we have $t|C = s$.

  Geometrically, this is saying that some local section cannot be extended to a global one. Equivalently, that the support of the model cannot be covered by the consistent global assignments.

- It is **strongly contextual** if its support has **no global section**; that is, there is no consistent global assignment.

  This says that **no** possible joint outcome is accounted for by **any** global section!

  Obviously, strong contextuality implies logical contextuality.

A Hierarchy

A Hierarchy

We can distinguish three degrees of contextuality among models:

A Hierarchy

We can distinguish three degrees of contextuality among models:

- Strong contextuality implies logical contextuality, which implies (probabilistic) contextuality.

A Hierarchy

We can distinguish three degrees of contextuality among models:

- Strong contextuality implies logical contextuality, which implies (probabilistic) contextuality.
- The Bell model is contextual, but not logically contextual.

A Hierarchy

We can distinguish three degrees of contextuality among models:

- Strong contextuality implies logical contextuality, which implies (probabilistic) contextuality.
- The Bell model is contextual, but not logically contextual.
- The Hardy model is logically contextual, but not strongly contextual.

A Hierarchy

We can distinguish three degrees of contextuality among models:

- Strong contextuality implies logical contextuality, which implies (probabilistic) contextuality.
- The Bell model is contextual, but not logically contextual.
- The Hardy model is logically contextual, but not strongly contextual.
- The PR box is strongly contextual.

A Hierarchy

We can distinguish three degrees of contextuality among models:

- Strong contextuality implies logical contextuality, which implies (probabilistic) contextuality.
- The Bell model is contextual, but not logically contextual.
- The Hardy model is logically contextual, but not strongly contextual.
- The PR box is strongly contextual.

Thus we have a strict hierarchy

$$\text{probabilistic contextuality} \; < \; \text{logical contextuality} \; < \; \text{strong contextuality}$$

A Hierarchy

We can distinguish three degrees of contextuality among models:

- Strong contextuality implies logical contextuality, which implies (probabilistic) contextuality.
- The Bell model is contextual, but not logically contextual.
- The Hardy model is logically contextual, but not strongly contextual.
- The PR box is strongly contextual.

Thus we have a strict hierarchy

$$\text{probabilistic contextuality} \; < \; \text{logical contextuality} \; < \; \text{strong contextuality}$$

The model arising from the GHZ quantum state (with 3 or more parties) with X, Y measurements at each site is strongly contextual.

A Hierarchy

We can distinguish three degrees of contextuality among models:

- Strong contextuality implies logical contextuality, which implies (probabilistic) contextuality.
- The Bell model is contextual, but not logically contextual.
- The Hardy model is logically contextual, but not strongly contextual.
- The PR box is strongly contextual.

Thus we have a strict hierarchy

$$\text{probabilistic contextuality} \; < \; \text{logical contextuality} \; < \; \text{strong contextuality}$$

The model arising from the GHZ quantum state (with 3 or more parties) with X, Y measurements at each site is strongly contextual.

Thus in terms of well-known quantum examples, we have

$$\text{Bell} < \text{Hardy} < \text{GHZ}$$

Degrees of contextuality for quantum states

Degrees of contextuality for quantum states

We can lift these concepts to define a novel way of classifying **quantum states** in terms of their **degree of contextuality**.

Degrees of contextuality for quantum states

We can lift these concepts to define a novel way of classifying **quantum states** in terms of their **degree of contextuality**.

In particular, we shall focus on n-qubit pure states. If we fix local observables for each party, such a state gives rise to a probability model as above.

Degrees of contextuality for quantum states

We can lift these concepts to define a novel way of classifying **quantum states** in terms of their **degree of contextuality**.

In particular, we shall focus on n-qubit pure states. If we fix local observables for each party, such a state gives rise to a probability model as above.

We can lift the properties of models to states.

Degrees of contextuality for quantum states

We can lift these concepts to define a novel way of classifying **quantum states** in terms of their **degree of contextuality**.

In particular, we shall focus on n-qubit pure states. If we fix local observables for each party, such a state gives rise to a probability model as above.

We can lift the properties of models to states.

- We say that a state is strongly contextual if for **some** choice of local observables for each party, the resulting empirical model is strongly contextual.

Degrees of contextuality for quantum states

We can lift these concepts to define a novel way of classifying **quantum states** in terms of their **degree of contextuality**.

In particular, we shall focus on *n*-qubit pure states. If we fix local observables for each party, such a state gives rise to a probability model as above.

We can lift the properties of models to states.

- We say that a state is strongly contextual if for **some** choice of local observables for each party, the resulting empirical model is strongly contextual.
- We can similarly define logical contextuality for states; we say that a state is logically contextual if for some choice of local observables, the resulting empirical model is logically contextual; while the state is **not** strongly contextual.

Degrees of contextuality for quantum states

We can lift these concepts to define a novel way of classifying **quantum states** in terms of their **degree of contextuality**.

In particular, we shall focus on n-qubit pure states. If we fix local observables for each party, such a state gives rise to a probability model as above.

We can lift the properties of models to states.

- We say that a state is strongly contextual if for **some** choice of local observables for each party, the resulting empirical model is strongly contextual.
- We can similarly define logical contextuality for states; we say that a state is logically contextual if for some choice of local observables, the resulting empirical model is logically contextual; while the state is **not** strongly contextual.
- Finally, a state is weakly contextual if it is contextual, but neither of the previous two cases apply.

The Characterization Problem

The Characterization Problem

This gives rise to a natural and challenging problem:

Problem

Characterize the multipartite states in terms of their maximum degree of contextuality.

The Characterization Problem

This gives rise to a natural and challenging problem:

Problem

Characterize the multipartite states in terms of their maximum degree of contextuality.

We believe that an answer to this problem will shed considerable light on the structure of multipartite states, not least because it will necessitate solving the following task:

> *Given a multipartite state, find local observables which witness its highest degree of contextuality.*

A Recent Result and a Question

SA, Carmen Constantin and Shenggang Ying. *Hardy is (almost) everywhere*. *Information and Computation* 2016. `arXiv:1506.01365`

A Recent Result and a Question

SA, Carmen Constantin and Shenggang Ying. *Hardy is (almost) everywhere*. *Information and Computation* 2016. `arXiv:1506.01365`

This paper provides an algorithm which given an n-qubit entangled state, constructs $n + 2$ local observables leading to a logically contextual model.

A Recent Result and a Question

SA, Carmen Constantin and Shenggang Ying. *Hardy is (almost) everywhere*. *Information and Computation* 2016. `arXiv:1506.01365`

This paper provides an algorithm which given an n-qubit entangled state, constructs $n+2$ local observables leading to a logically contextual model.

Proof of correctness is non-trivial.

A Recent Result and a Question

SA, Carmen Constantin and Shenggang Ying. *Hardy is (almost) everywhere*. *Information and Computation* 2016. `arXiv:1506.01365`

This paper provides an algorithm which given an *n*-qubit entangled state, constructs $n+2$ local observables leading to a logically contextual model.

Proof of correctness is non-trivial.

This leads us on to the **main question** which is the natural next challenge:

> For which quantum states can we find local observables which give rise to a strongly contextual empirical model?

A Recent Result and a Question

SA, Carmen Constantin and Shenggang Ying. *Hardy is (almost) everywhere*. *Information and Computation* 2016. `arXiv:1506.01365`

This paper provides an algorithm which given an n-qubit entangled state, constructs $n + 2$ local observables leading to a logically contextual model.

Proof of correctness is non-trivial.

This leads us on to the **main question** which is the natural next challenge:

For which quantum states can we find local observables which give rise to a strongly contextual empirical model?

This question remains open, and appears difficult!