Logic and Databases

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Logic and Databases are inextricably intertwined.

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Logic and Databases

- Extensive interaction between logic and databases during the past 45 years.

- Logic provides both a unifying framework and a set of tools for formalizing and studying data management tasks.

- The interaction between logic and databases is a prime example of
  - Logic **in** Computer Science

    but also
  - Logic **from** Computer Science

Logic and Databases

Two main uses of logic in databases:

- Logic is used as a database query language to express questions asked against databases.

- Logic is used as a specification language to express integrity constraints in databases.

We will discuss both of these uses with emphasis on the first.

Thematic Roadmap

- Logic and Database Query Languages
  - Relational Algebra and Relational Calculus
  - Conjunctive queries and their variants
  - Datalog
- Query Evaluation, Query Containment, Query Equivalence
  - Decidability and Complexity
- Other Aspects of Conjunctive Query Evaluation
- Alternative Semantics of Queries
  - Bag Databases: Semantics and Conjunctive Query Containment
  - Probabilistic Databases: Semantics and Dichotomy Theorems for Conjunctive Query Evaluation
  - Inconsistent Databases: Semantics and Dichotomy Theorems
- Guest Lecture on Data Provenance by Val Tannen

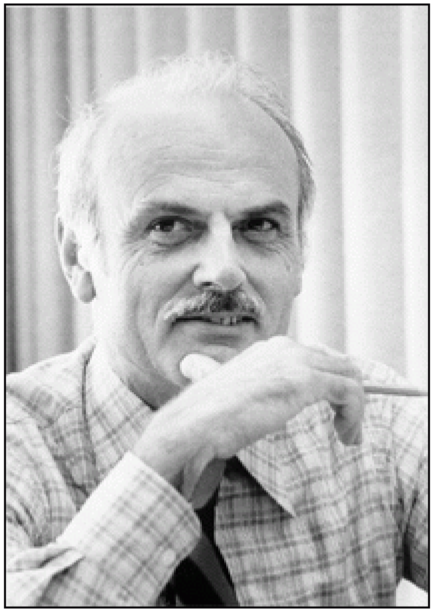
Relational Databases: How it all got started

- The history of relational databases is the history of a scientific and technological revolution.

- The scientific revolution started in 1970 by Edgar (Ted) F. Codd at the IBM San Jose Research Laboratory (now the IBM Almaden Research Center)

- Codd introduced the relational data model and two database query languages: relational algebra and relational calculus.
  - "A relational model for data for large shared data banks", CACM, 1970.
  - "Relational completeness of data base sublanguages", in: Database Systems, ed. by R. Rustin, 1972.

Edgar F. Codd, 1923-2003



The Relational Data Model (E.F. Codd – 1970)

- The Relational Data Model uses the mathematical concept of a relation as the formalism for describing and representing data.
- Question: What is a relation?
- Answer:
  - Formally, a relation is a subset of a cartesian product of sets.
  - Informally, a relation is a "table" with rows and columns.

CHECKING Table

branch-name	**account-no**	**customer-name**	**balance**
Orsay	10991-06284	Abiteboul	$13,567.53
Hawthorne	10992-35671	Hull	$21,245.75
…	…	…	…

Relational Database Schemas

- A k-ary relation schema **R**$(A_1,A_2,\ldots,A_K)$ is a set $\{A_1,A_2,\ldots,A_k\}$ of k attributes.

  CHECKING(branch-name, account-no, customer-name, balance)

  - Thus, a k-ary relation schema is a "blueprint" for k-ary relations.
  - It is a k-ary relation symbol in logic with names for the positions.
- An instance of a relation schema is a relation conforming to the schema (arities match; also, in DBMS, data types of attributes match).
- A relational database schema is a set of relation schemas $\mathbf{R_i}(A_1,A_2,\ldots,A_{k_i})$, for $1\leq i\leq m$.
- A relational database instance of a relational schema is a set of relations R_i each of which is an instance of the relation schema $\mathbf{R_i}$, $1\leq i\leq m$.

Relational Structures vs. Relational Databases

- Relational Structure

$$\mathbf{A} = (A, R_1,\ldots,R_m)$$

  - A is the **universe** of **A**
  - $R_1,\ldots,R_m$ are the relations of **A**

- Relational Database

$$\mathbf{D} = (R_1,\ldots,R_m)$$

- Thus, a relational database can be thought of as a relational structure **without** its universe.
  - And this causes some problems down the road …

Query Languages for the Relational Data Model

Codd introduced two different query languages for the relational data model:

- Relational Algebra, which is a procedural language.
  - It is an algebraic formalism in which queries are expressed by applying a sequence of operations to relations.

- Relational Calculus, which is a declarative language.
  - It is a logical formalism in which queries are expressed as formulas of first-order logic.

Codd's Theorem: Relational Algebra and Relational Calculus are "essentially equivalent" in terms of expressive power.
(but what does this really mean?)

The Five Basic Operations of Relational Algebra

- Group I: Three standard set-theoretic binary operations:
  - Union
  - Difference
  - Cartesian Product.
- Group II. Two special unary operations on relations:
  - Projection
  - Selection.
- Relational Algebra consists of all expressions obtained by combining these five basic operations in syntactically correct ways.

More on the Syntax of the Projection Operation

- Projection Operation:
  - Syntax: $\pi_{i_1,\ldots,i_m}(R)$, where R is of arity k, and $i_1, \ldots. i_m$ are distinct integers from 1 up to k.
  - Semantics:

$\pi_{i_1,\ldots,i_m}(R) =$

$\{(a_1,\ldots,a_m)$: there is a tuple $(b_1,\ldots,b_k)$ in R such that $a_1 = b_{i_1}, \ldots, a_m = b_{i_m}\}$

- Example: If R is R(A,B,C,D), then

$\pi_{3,1}(R) = \{(c,a)$: there are b,d such that $(a,b,c,d) \in R\} =$

$\pi_{C,A}(R)$

The Selection Operation

- Selection is a family of unary operations of the form

$$\sigma_{\Theta}(R),$$

where R is a relation and Θ is a condition that can be applied as a test to each row of R.

- When a selection operation is applied to R, it returns the subset of R consisting of all rows that satisfy the condition Θ

- A condition in the selection operation is an expression built up from:
  - Comparison operators =, <, >, ≠, ≤, ≥ applied to operands that are constants or attribute names or component numbers.
    - These are the basic (atomic) clauses of the conditions.
  - Boolean combinations ($\wedge$, $\vee$, $\neg$) of basic clauses.

Relational Algebra

- Definition: A relational algebra expression is a string obtained from relation schemas using union, difference, cartesian product, projection, and selection.

- Context-free grammar for relational algebra expressions:

$$E := R, S, \ldots \mid (E_1 \cup E_2) \mid (E_1 - E_2) \mid (E_1 \times E_2) \mid \pi_L(E) \mid \sigma_\Theta(E),$$

where

- R, S, … are relation schemas
- L is a list of attributes
- Θ is a condition.

Strength from Unity and Combination

- By itself, each basic relational algebra operation has limited expressive power, as it carries out a specific and rather simple task.

- When used in combination, however, the five relational algebra operations can express interesting and, quite often, rather complex queries.

- Derived relational algebra operations are operations on relations that are expressible via a relational algebra expression (built from the five basic operators).

Natural Join

- Definition: Let A1, …, Ak be the common attributes of two relation schemas R and S. Then

  $$R \bowtie S = \pi_{<list>} (\sigma_{R.A1=S.A1 \wedge \ldots \wedge R.A1 = S.Ak} (R \times S)),$$

  where <list> contains all attributes of $R \times S$, except for S.A1, …, S.Ak (in other words, duplicate columns are eliminated).

- Example: Given
  TEACHES(fac-name,course,term) and
  ENROLLS(stud-name,course,term),
  we want to obtain
  TAUGHT-BY(stud-name,course,term,fac-name)
  Then
  TAUGHT-BY = ENROLLS $\bowtie$ TEACHES

Independence of the Basic Operations

- Question: Are all five basic relational algebra operations really needed? Can one of them be expressed in terms of the other four?

- Theorem: Each of the five basic relational algebra operations is independent of the other four, that is, it cannot be expressed by a relational algebra expression that involves only the other four.

Proof Idea: For each relational algebra operation, we need to discover a property that is possessed by that operation, but is not possessed by any relational algebra expression that involves only the other four operations.

SQL vs. Relational Algebra

SQL	**Relational Algebra**
SELECT	Projection π
FROM	Cartesian Product $\times$
WHERE	Selection σ

Semantics of SQL via interpretation to Relational Algebra

SELECT $R_{i1}.A1, \ldots, R_{im}.A.m$
FROM $R_1, \ldots, R_K$
WHERE Ψ

$$= \pi_{R_{i1}.A1, \ldots, R_{im}.A.m} (\sigma_{\Psi} (R_1 \times \ldots \times R_K))$$

Relational Calculus

- In addition to relational algebra, Codd introduced relational calculus.
- Relational calculus is a declarative database query language based on first-order logic.
- Relational calculus comes into two different flavors:
  - Tuple relational calculus
  - Domain relational calculus.

  We will focus on domain relational calculus.
  There is an easy translation between these two formalisms.
- Codd's main technical result is that relational algebra and relational calculus have "essentially" the same expressive power.

Relational Calculus (FO Logic for Databases)

- First-order variables: x, y, z, …, x_1, …,x_k,…
  - They range over values that may occur in tables.
- Relation symbols: R, S, T, … of specified arities (names of relations)
- Atomic (Basic) Formulas:
  - $R(x_1,\ldots,x_k)$, where R is a k-ary relation symbol (alternatively, $(x_1,\ldots,x_k) \in R$; the variables need not be distinct)
  - (x op y), where op is one of $=, \neq, <, >, \leq, \geq$
  - (x op c), where c is a constant and op is one of $=, \neq, <, >, \leq, \geq$.
- Relational Calculus Formulas:
  - Every atomic formula is a relational calculus formula.
  - If φ and ψ are relational calculus formulas, then so are:
    - $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $\neg \psi$, $(\varphi \rightarrow \psi)$ (propositional connectives)
    - $(\exists\ x\ \varphi)$ (existential quantification)
    - $(\forall\ x\ \varphi)$ (universal quantification).

Relational Calculus as a Query Language

Definition:

- A relational calculus expression is an expression of the form
$$\{ (x_1,\ldots,x_k):\ \varphi(x_1,\ldots x_k) \},$$
where $\varphi(x_1,\ldots,x_k)$ is a relational calculus formula with $x_1,\ldots,x_k$ as its free variables.
- When applied to a relational database D, this relational calculus expression returns the k-ary relation that consists of all k-tuples $(a_1,\ldots,a_k)$ that make the formula "true" on D.

Example: The relational calculus expression
$$\{ (x,y):\ \exists z(E(x,z) \wedge E(z,y)) \}$$
returns the set P of all pairs of nodes (a,b) that are connected via a path of length 2.

Relational Algebra vs. Relational Calculus

Codd's Theorem (informal statement):
Relational Algebra and Relational Calculus have "essentially" the same expressive power, i.e., they can express the same queries.

Note: It is **not** true that for every relational calculus expression φ, there is an equivalent relational algebra expression E.

Examples:

- $\{ (x_1,\ldots,x_k): \neg R(x_1,\ldots,x_k) \}$
- $\{ x: \forall y,z \text{ ENROLLS}(x,y,z) \}$,
  where ENROLLS(s-name,course,term)

From Relational Calculus to Relational Algebra

Note: The previous relational calculus expression may produce different answers when we consider different domains over which the variables are interpreted.

Example: If the variables $x_1,\ldots,x_k$ range over a domain D, then

$$\{(x_1,\ldots,x_k)\colon \neg R(x_1,\ldots,x_k)\} = D^k - R.$$

Fact:

- The relational calculus expression $\{ (x_1,\ldots,x_k)\colon \neg R(x_1,\ldots,x_k) \}$ is **not** "domain independent".
- The relational calculus expression $\{(x_1,\ldots,x_k)\colon S(x_1,..,x_k) \wedge \neg R(x_1,\ldots,x_k)\}$ is "domain independent".

Active Domain and Active Domain Interpretation

Definition:

- The active domain adom(D) of a relational database instance D is the set of all values that occur in the relations of D.

- Let $\varphi(x_1,\ldots,x_k)$ be a relational calculus formula and let D be a relational database instance. Then

$$\varphi^{adom}(D)$$

is the result of evaluating $\varphi(x_1,\ldots,x_k)$ over adom(D) and D, i.e.,

  - all variables and quantifiers are assumed to range over adom(D);
  - the relation symbols in φ are interpreted by the relations in D.

Equivalence of Relational Algebra and Calculus

Theorem: If q is a k-ary query, then the following statements are equivalent:

1. There is a relational algebra expression E such that q(D) = E(D), for every database instance D (in other words, q is expressible in relational algebra).

2. There is a relational calculus formula ψ such that q(D) = ψ^{adom} (D) (in other words, q is expressible in relational calculus under the active domain interpretation).

Equivalence of Relational Algebra and Calculus

Proof (Sketch):

1. $\Rightarrow$ 2. By a straightforward induction on the construction of relational algebra expressions.

Note: Projection π is simulated using $\exists$

2. $\Rightarrow$ 1.

- Show first that for every relational database schema **S**, there is a relational algebra expression E such that for every database instance D, we have that adom(D) = E(D).

- Use the above fact and induction on the construction of relational calculus formulas to obtain a translation of relational calculus under the active domain interpretation to relational algebra.

Equivalence of Relational Algebra and Calculus

- In this translation, the most interesting part is the simulation of the universal quantifier $\forall$ in relational algebra.
  - It uses the logical equivalence $\forall y \psi \equiv \neg \exists y \neg \psi$
- As an illustration, consider $\forall y R(x,y)$.
  - $\forall y R(x,y) \equiv \neg \exists y \neg R(x,y)$
  - $adom(D) = \pi_1(R) \cup \pi_2(R)$

Rel.Calc. formula φ	Relational Algebra Expression for φ^{adom}
$\neg R(x,y)$	$(\pi_1(R) \cup \pi_2(R)) \times (\pi_1(R) \cup \pi_2(R)) - R$
$\exists y \neg R(x,y)$	$\pi_1((\pi_1(R) \cup \pi_2(R)) \times (\pi_1(R) \cup \pi_2(R)) - R)$
$\neg \exists y \neg R(x,y)$	$(\pi_1(R) \cup \pi_2(R)) - (\pi_1((\pi_1(R) \cup \pi_2(R)) \times (\pi_1(R) \cup \pi_2(R)) - R))$

Queries

Definition: Let **S** be a relational database schema.

- A k-ary query on **S** is a function q defined on database instances over S such that if D is a database instance over S, then q(D) is a k-ary relation on adom(D) that is invariant under isomorphisms (i.e., if h: D $\rightarrow$ F is an isomorphism, then q(F) = h(q(D)).

- A Boolean query on **S** is a function q defined on database instances over S such that if D is a database instance over S, then q(D) = 0 or q(D) = 1, and q(D) is invariant under isomorphisms.

Example: The following are Boolean queries on graphs:
- Given a graph E (binary relation), is the diameter of E at most 3?
- Given a graph E (binary relation), is E connected?

Fundamental Algorithmic Problems about Queries

- The Query Evaluation Problem: Given a query q and a database instance D, find q(D).

- The Query Equivalence Problem: Given two queries q and q' of the same arity, is it the case that q $\equiv$ q' ?
(i.e., is it the case that, for every database instance D, we have that q(D) = q'(D)?)

- The Query Containment Problem: Given two queries q and q' of the same arity, is it the case that q $\subseteq$ q' ?
(i.e., is it the case that, for every database instance D, we have that q(D) $\subseteq$ q'(D)?)

Fundamental Algorithmic Problems about Queries

- The Query Evaluation Problem is the main problem in query processing.
- The Query Equivalence Problem underlies query processing and optimization, as we often need to transform a given query to an equivalent one.
- The Query Containment Problem and Query Equivalence Problem are closely related to each other:
  - $q \equiv q'$ if and only if $q \subseteq q'$ and $q' \subseteq q$.
  - $q \subseteq q'$ if and only if $q \equiv q \wedge q'$.

Undecidability of Equivalence and Containment

Theorem: The Query Equivalence Problem for relational calculus queries is undecidable.

Proof: Use Trakhtenbrot's Theorem (1949):

The Finite Validity Problem is undecidable.

- Finite Validity Problem $\preccurlyeq$ Query Equivalence Problem
  - If ψ^* is a fixed finitely valid relational calculus sentence, then for every relational calculus sentence φ, we have that

$$\varphi \text{ is finitely valid} \Leftrightarrow \varphi \equiv \psi^*.$$

Corollary: The Query Containment Problem for relational calculus queries in undecidable.

Proof: Query Equivalence $\preccurlyeq$ Query Containment, since

$$q \equiv q' \Leftrightarrow q \subseteq q' \text{ and } q' \subseteq q.$$

Complexity of the Query Evaluation Problem

The Query Evaluation Problem for Relational Calculus:
Given a relational calculus formula φ and a database instance D, find $\varphi^{adom}(D)$.

Theorem: The Query Evaluation Problem for Relational Calculus is PSPACE-complete.

Proof: We need to show that

- This problem is in PSPACE.
- This problem is PSPACE-hard.

We start with the second task.

Complexity of the Query Evaluation Problem

Theorem: The Query Evaluation Problem for Relational Calculus is PSPACE-hard.

Proof: QBF – Quantified Boolean Formulas

Show that

QBF $\preccurlyeq_p$ Query Evaluation for Relational Calculus

Given QBF $\forall x_1 \exists x_2 \ldots. \forall x_k \psi$

- Let V and P be two unary relation symbols
- Obtain ψ^* from ψ by replacing x_i by $P(x_i)$, and $\neg x_i$ by $\neg P(x_i)$
- Let D be the database instance with V = {0,1}, P={1}.
- Then the following statements are equivalent:
  - $\forall x_1 \exists x_2 \ldots. \forall x_k \psi$ is true
  - $\forall x_1 (V(x_1) \rightarrow \exists x_2 (V(x_2) \wedge (\ldots \forall x_k(V(x_k) \rightarrow \psi^*))\ldots)$ is true on D.

Complexity of the Query Evaluation Problem

- Theorem: The Query Evaluation Problem for Relational Calculus is in PSPACE.

  Proof (Hint): Let φ be a relational calculus formula $\forall x_1 \exists x_2 \ldots \forall x_m \psi$ and let I be a database instance.

  - Exponential Time Algorithm: We can find $\varphi^{adom}(D)$, by exhaustively cycling over all possible interpretations of the x_i's.

    This runs in time $O(n^m)$, where $n = |D|$ (size of D).
  - A more careful analysis shows that this algorithm can be implemented in $O(m \cdot \log n)$-space.
    - Use m blocks of memory, each holding one of the n elements of adom(I) written in binary (so $O(\log n)$ space is used in each block).
    - Maintain also m counters in binary to keep track of the number of elements examined.

$\forall x_1$	$\exists x_2$	…	$\forall x_m$
a_1 in adom(I) written in binary	a_2 in adom(I) written in binary	…	a_m in adom(I) written in binary

Complexity of the Query Evaluation Problem

- Corollary: The Query Evaluation Problem for Relational Algebra is PSPACE-complete.

  Proof:

  The translation of relational calculus to relational algebra yields a polynomial-time reduction of the Query Evaluation Problem for Relational Calculus to the Query Evaluation Problem for Relational Algebra.

Summary

- The Query Evaluation Problem for Relational Calculus is PSPACE-complete.

- The Query Equivalence Problem for Relational Calculus is undecidable.

- The Query Containment Problem for Relational Calculus is undecidable.

The Query Evaluation Problem Revisited

- Since the Query Evaluation Problem for Relational Calculus is PSPACE-hard, there are **no** polynomial-time algorithms for this problem, unless PSPACE = P (which is considered highly unlikely).
- Let's take another look at the exponential-time algorithm for this problem:
  - Let φ be a relational calculus formula $\forall x_1 \exists x_2 \ldots \forall x_m \psi$ and let D be a database instance.
  - Exponential Time Algorithm: We can find $\varphi^{adom}(D)$, by exhaustively cycling over all possible interpretations of the x_i's. This runs in time $O(n^m)$, where $n = |D|$).
  - So, the running time is $O(|D|^{|\varphi|})$, where $|D|$ is the size of D and $|\varphi|$ is the size of the relational calculus formula φ.
  - This tells that the source of exponentiality is the formula size.

The Query Evaluation Problem Revisited

- Theorem: Let φ be a fixed relational calculus formula. Then the following problem is solvable in polynomial time: given a database instance D, find $\varphi^{adom}(D)$. In fact, this problem is in LOGSPACE.

  Proof:

  Let φ be a fixed relational calculus formula $\forall x_1 \exists x_2 \ldots \forall x_m \psi$
  - The previous algorithm has running time $O(|D|^{|\varphi|})$, which is a polynomial, since now $|\varphi|$ is a constant.
  - Moreover, the algorithm can now be implemented using logarithmic-space only, since we need only maintain a constant number of memory blocks, each of logarithmic size

$\forall x_1$	$\exists x_2$	…	$\forall x_m$
a_1 in adom(I) written in binary	a_2 in adom(I) written in binary	…	a_m in adom(I) written in binary

Vardi's Taxonomy of the Query Evaluation Problem

M.Y Vardi, "The Complexity of Relational Query Languages", 1982

- Definition: Let L be a database query language.
  - The combined complexity of L is the decision problem: given an L-sentence and a database instance D, is φ true on D? (does D satisfy φ?) (in symbols, does $D \models \varphi$?)
  - The data complexity of L is the family of the following decision problems P_φ, where φ is an L-sentence: given a database instance D, does $D \models \varphi$?
  - The query complexity of L is the family of the following decision problems P_D, where D is a database instance: given an L-sentence φ, does $D \models \varphi$?

Vardi's Taxonomy of the Query Evaluation Problem

Definition: Let L be a database query language and let C be a computational complexity class.

- The data complexity of L is in C if for each L-sentence φ, the decision problem P_φ is in C.
- The data complexity of L is C-complete if it is in C and there is an L-sentence φ such that the decision problem P_φ is C-complete.
- The query complexity of L is in C if for every database D, the decision problem P_D is in C.
- The query complexity of L is C-complete if it is in C and there is a database D such that the decision problem P_D is C-complete.

Vardi's Taxonomy of the Query Evaluation Problem

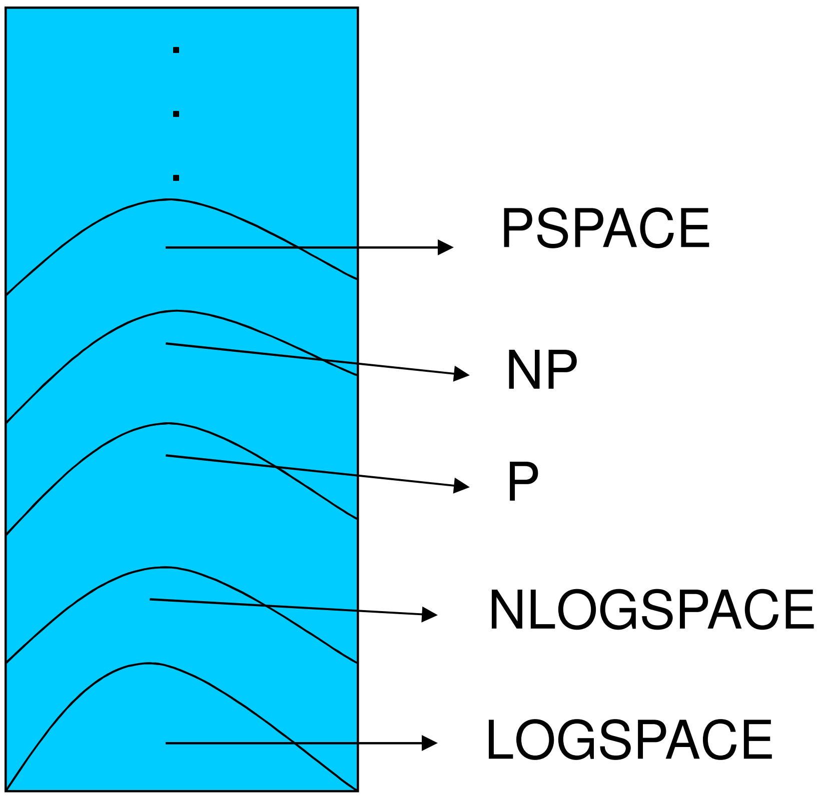
Vardi's "empirical" discovery:

For most query languages L:

- The data complexity of L is of lower complexity than both the combined complexity of L and the query complexity of L.
- The query complexity of L can be as hard as the combined complexity of L.

Taxonomy of the Query Evaluation Problem for Relational Calculus

Complexity Classes



The Query Evaluation Problem for Relational Calculus

Problem	**Complexity**
Combined Complexity	PSPACE-complete
Query Complexity	- Is in PSPACE - It can be PSPACE-complete
Data Complexity	In LOGSPACE

Summary

- Relational Algebra and Relational Calculus have "essentially" the same expressive power.
- The Query Equivalence Problem for Relational Calculus in undecidable.
- The Query Containment Problem for Relational Calculus is undecidable.
- The Query Evaluation Problem for Relational Calculus is PSPACE-complete (combined / query complexity).

Sublanguages of Relational Calculus

- Question: Are there interesting sublanguages of relational calculus for which the Query Containment Problem and the Query Evaluation Problem are "easier" than the full relational calculus?

- Answer:
  - Yes, the language of conjunctive queries is such a sublanguage.
  - Moreover, conjunctive queries are the most frequently asked queries against relational databases.

Conjunctive Queries

- Definition: A conjunctive query is a query expressible by a relational calculus formula in prenex normal form built from atomic formulas $R(y_1,\ldots,y_n)$, and $\wedge$ and $\exists$ only.

$$\{ (x_1,\ldots,x_k): \exists z_1 \ldots \exists z_m \chi(x_1, \ldots,x_k, z_1,\ldots,z_k) \},$$

where $\chi(x_1, \ldots,x_k, z_1,\ldots,z_k)$ is a conjunction of atomic formulas of the form $R(y_1,\ldots,y_m)$.

  - Equivalently, a conjunctive query is a query expressible by a relational algebra expression of the form
    $\pi_X(\sigma_\Theta(R_1\times \ldots\times R_n))$, where
    Θ is a conjunction of equality atomic formulas (equijoin).
  - Equivalently, a conjunctive query is a query expressible by an SQL expression of the form
    SELECT <list of attributes>
    FROM <list of relation names>
    WHERE <conjunction of equalities>

Conjunctive Queries

- Definition**:** A conjunctive query is a query expressible by a relational calculus formula in prenex normal form built from atomic formulas $R(y_1,\ldots,y_n)$, and $\wedge$ and $\exists$ only.

  $$\{(x_1,\ldots,x_k)\colon \exists z_1 \ldots \exists z_m\, \chi(x_1, \ldots,x_k, z_1,\ldots,z_k)\}$$

  - A conjunctive query can be written as a logic-programming rule:

    $$Q(x_1,\ldots,x_k) \text{ :-- } R_1(\mathbf{u}_1), \ldots, R_n(\mathbf{u}_n), \text{ where}$$

    - Each variable x_i occurs in the right-hand side of the rule.
    - Each $\mathbf{u}_i$ is a tuple of variables (not necessarily distinct)
    - The variables occurring in the right-hand side (the body), but not in the left-hand side (the head) of the rule are existentially quantified (but the quantifiers are not displayed).
    - "," stands for conjunction.

Examples of Conjunctive Queries

– Path of Length 2: (Binary query)

$$\{(x,y): \exists z\ (E(x,z) \wedge E(z,y))\}$$

- As a relational algebra expression,

$$\pi_{1,4}(\sigma_{\$2 = \$3}\ (E \times E))$$

- As a rule:

q(x,y) :-- E(x,z), E(z,y)

– Cycle of Length 3: (Boolean query)

$$\exists x \exists y \exists z(E(x,y) \wedge E(y,z) \wedge E(z,x))$$

- As a rule (the head has no variables)
  – Q :-- E(x,z), E(z,y), E(z,x)

Conjunctive Queries

- Every natural join is a conjunctive query with no existentially quantified variables
  P(A,B,C), R(B,C,D) two relation symbols
  - $P \bowtie R = \{(x,y,z,w):\ P(x,y,z) \wedge R(y,z,w)\}$
  - q(x,y,z,w) :-- P(x,y,z), R(y,z,w)
    (no variables are existentially quantified)
  - SELECT P.A, P.B, P.C, R.D
    FROM P, R
    WHERE P.B = R.B AND P.C = R.C
- Conjunctive queries are also known as SPJ-queries (SELECT-PROJECT-JOIN queries)

Conjunctive Query Evaluation and Containment

- Definition: Two fundamental problems about CQs
  - Conjunctive Query Evaluation (CQE):
    Given a conjunctive query q and an instance D, find q(D).

  - Conjunctive Query Containment (CQC):
    - Given two k-ary conjunctive queries q_1 and q_2, is it true that $q_1 \subseteq q_2$? (i.e., for every instance D, we have that $q_1(D) \subseteq q_2(D)$)
    - Given two Boolean conjunctive queries q_1 and q_2, is it true that $q_1 \models q_2$? (that is, for all D, if $D \models q_1$, then $D \models q_2$)?

    CQC is logical implication.

CQE vs. CQC

- Recall that for relational calculus queries:
  - The Query Evaluation Problem is PSPACE-complete (combined complexity).
  - The Query Containment Problem is undecidable.

- Theorem: Chandra & Merlin, 1977
  - CQE and CQC are the "same" problem.
  - Moreover, each is an NP-complete problem.

- Question: What is the common link?
- Answer: The Homomorphism Problem

Homomorphisms

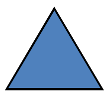
- Definition: Let D and F be two database instances over the same relational schema S.
  A homomorphism h: D $\rightarrow$ F is a function h: adom(D) $\rightarrow$ adom(F) such that for every relational symbol P of S and every $(a_1,\ldots,a_m)$, we have that

$$\text{if } (a_1,\ldots,a_m) \in P^D \text{, then } (h(a_1), .., h(a_m)) \in P^F.$$

- Note: The concept of homomorphism is a relaxation of the concept of isomorphism, since every isomorphism is also a homomorphism, but not vice versa.

- Example:

  A graph G = (V,E) is 3-colorable
  if and only if
  there is a homomorphism h: G $\rightarrow$ K_3



The Homomorphism Problem

- Definition: The Homomorphism Problem
  Given two database instances D and F, is there a homomorphism h: D $\rightarrow$ F?

- Notation: D $\rightarrow$ F denotes that a homomorphism from D to F exists.

- Theorem: The Homomorphism Problem is NP-complete.

  Proof: Easy reduction from 3-Colorabilty

  G is 3-colorable if and only if G $\rightarrow$ K_3.

- Exercise:
  Formulate 3SAT as a special case of the Homomorphism Problem.

The Homomorphism Problem

- Note: The Homomorphism Problem is a fundamental algorithmic problem:
  - Satisfiability can be viewed as a special case of it.
  - k-Colorability can be viewed as a special case of it.
  - Many AI problems, such as planning, can be viewed as a special case of it.
  - In fact, every constraint satisfaction problem can be viewed as a special case of the Homomorphism Problem (Feder and Vardi – 1993).

Homomorphism Problem & Conjunctive Queries

- Theorem: Chandra & Merlin, 1977
  - CQE and CQC are the "same" problem.
  - Moreover, each is an NP-complete problem.

- Question: What is the common link?
- Answer:
  - Both CQE and CQC are "equivalent" to the Homomorphism Problem.
  - The link is established by bringing into the picture
    - Canonical conjunctive queries and
    - Canonical database instances.