Finite And Algorithmic Model Theory I: Definability and Undefinability

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Finite and Algorithmic Model Theory

In the 1980s, the term *finite model theory* came to be used to describe the study of the expressive power of logics (from first-order to second-order logic and in between), on the class of all finite structures.

The motivation for the study is that problems in computer science (especially in *complexity theory* and *database theory*) are naturally expressed as questions about the expressive power of logics.
And, the structures involved in computation are finite.

A wide range of techniques, many of them *algorithmic*, for studying expressive power were developed.

Many of these techniques have been extended to the study of structures that are not necessarily finite but admit a *finite, alogrithmic* description.

Model Theoretic Questions

The kind of questions we are interested in are about the *expressive power* of logics. Given a formula φ, its class of models is the collection of *finite* relational structures $\mathbb{A}$ in which it is true.

$$\mathrm{Mod}(\varphi) = \{\mathbb{A} \mid \mathbb{A} \models \varphi\}$$

What classes of structures are definable in a given logic $\mathcal{L}$?

How do syntactic restrictions on φ relate to semantic restrictions on $Mod(\varphi)$?

How does the computational complexity of $Mod(\varphi)$ relate to the syntactic complexity of φ?

Expressive Power of Logics

We are interested in the *expressive power* of logics on finite structures.

We consider finite structures in a *relational vocabulary*.

> *A finite set* A*, with relations* $R_1, \ldots, R_m$ *and constants* $c_1, \ldots, c_n$.

A *property* of finite structures is any *isomorphism-closed* class of structures.

For a logic $\mathcal{L}$, we ask for which properties P, there is a sentence φ of the language such that

$$\mathbb{A} \in P \quad \text{if, and only if,} \quad \mathbb{A} \models \varphi.$$

First-Order Logic

As an example, consider *coloured graphs*, i.e. structures in a vocabulary with one binary relation E, some number of unary relations $C_1, \ldots, C_n$, and possibly some constant symbols.

Formulas of *first-order logic* are given by the following rules

terms – c, x

atomic formulae – $E(t_1, t_2), t_1 = t_2, C_i(t)$

boolean operations – $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$

first-order quantifiers – $\exists x \varphi$, $\forall x \varphi$

Example - Vertex Cover

For each k, we can write a *first-order formula* in the language of graphs which says that there is a vertex cover of size at most k.

$$\exists x_1 \cdots \exists x_k (\forall y \forall z (E(y,z) \Rightarrow (\bigvee_{1 \leq i \leq k} y = x_i \vee \bigvee_{1 \leq i \leq k} z = x_i)$$

Here, quantifiers range over vertices of the graph.

Example - 3-Colourability

3-colourability of graphs can be expressed by a formula when we allow quantification over *sets of vertices*.

$$\begin{aligned}
\exists R \subseteq V\, \exists B \subseteq V\, \exists G \subseteq V \quad & \\
& \forall x(Rx \lor Bx \lor Gx) \land \\
& \forall x(\neg(Rx \land Bx) \land \neg(Bx \land Gx) \land \neg(Rx \land Gx)) \land \\
& \forall x \forall y(Exy \to (\neg(Rx \land Ry) \land \\
& \qquad\qquad\qquad \neg(Bx \land By) \land \\
& \qquad\qquad\qquad \neg(Gx \land Gy)))
\end{aligned}$$

Compactness and Completeness

The *Compactness Theorem* fails if we restrict ourselves to finite structures.

The *Completeness Theorem* also fails:

> **Theorem (Trakhtenbrot 1950)**
> *The set of finitely valid sentences is not recursively enumerable.*

Various preservation theorems *(Łoś-Tarski, Lyndon)* fail when restricted to finite structures.

The finitary analogues of *Craig Interpolation Theorem* and the *Beth Definability Theorem* also fail.

Tools for Finite Model Theory

It seems that the class of finite structures is not well-behaved for the study of definability.

What *tools and methods* are available to study the expressive power of logic in the finite?

- *Ehrenfeucht-Fraïssé Games* and related *model-comparison games*;
- *Locality Theorems* (reviewed in this lecture);
- *Automata-based methods* (in the next lecture);
- *Complexity* (in later lectures);
- *Asymptotic Combinatorics* (in the guest lecture by Albert Atserias).

Elementary Equivalence

On finite structures, the elementary equivalence relation is trivial:

$\mathbb{A} \equiv \mathbb{B}$ if, and only if, $\mathbb{A} \cong \mathbb{B}$

Given a structure $\mathbb{A}$ with n elements, we construct a sentence

$$\varphi_{\mathbb{A}} = \exists x_1 \ldots \exists x_n \psi \wedge \forall y \bigvee_{1 \leq i \leq n} y = x_i$$

where, $\psi(x_1, \ldots, x_n)$ is the conjunction of all atomic and negated atomic formulas that hold in $\mathbb{A}$.

Then, if $\mathbb{B} \models \varphi_{\mathbb{A}}$, $\mathbb{A} \cong \mathbb{B}$.

Theories vs. Sentences

First order logic can make all the distinctions that are there to be made between finite structures.

Any isomorphism closed class of finite structures S can be defined by a *first-order theory*:

$$\{\neg\varphi_{\mathbb{A}} \mid \mathbb{A} \not\in S\}.$$

To understand the limits on the expressive power of *first-order sentences*, we need to consider coarser equivalence relations than $\equiv$.

We will also be interested in the expressive power of logics extending first-order logic. This amounts to studying theories satisfying a weaker *axiomatisibality* requirement than *finite axiomatisability*.

Quantifier Rank

The *quantifier rank* of a formula φ, written $\text{qr}(\varphi)$ is defined inductively as follows:

1. if φ is atomic then $\text{qr}(\varphi) = 0$,
2. if $\varphi = \neg\psi$ then $\text{qr}(\varphi) = \text{qr}(\psi)$,
3. if $\varphi = \psi_1 \vee \psi_2$ or $\varphi = \psi_1 \wedge \psi_2$ then $\text{qr}(\varphi) = \max(\text{qr}(\psi_1), \text{qr}(\psi_2))$.
4. if $\varphi = \exists x\psi$ or $\varphi = \forall x\psi$ then $\text{qr}(\varphi) = \text{qr}(\psi) + 1$

Note: For the rest of this lecture, we assume that our signature consists only of relation and constant symbols.
With this proviso, it is easily proved that in a finite vocabulary, for each q, there are (up to logical equivalence) only finitely many sentences φ with $\text{qr}(\varphi) \leq q$.

Finitary Elementary Equivalence

For two structures $\mathbb{A}$ and $\mathbb{B}$, we say $\mathbb{A} \equiv_p \mathbb{B}$ if for any sentence φ with $\mathrm{qr}(\varphi) \leq p$,

$$\mathbb{A} \models \varphi \text{ if, and only if, } \mathbb{B} \models \varphi.$$

Key fact:

> *a class of structures S is definable by a first order sentence if, and only if, S is closed under the relation $\equiv_p$ for some p.*

In a *finite* relational vocabulary, for any structure $\mathbb{A}$ there is a sentence $\theta^p_{\mathbb{A}}$ such that

$$\mathbb{B} \models \theta^p_{\mathbb{A}} \quad \text{if, and only if,} \quad \mathbb{A} \equiv_p \mathbb{B}$$

Partial Isomorphisms

The equivalence relations $\equiv_p$ can be characterised in terms of sequences of partial isomorphisms

(Fraïssé 1954)

or two player games.

(Ehrenfeucht 1961)

A *partial isomorphism* is an injective partial function f from $\mathbb{A}$ to $\mathbb{B}$ such that:

- for any constant c: $f(c^{\mathbb{A}}) = c^{\mathbb{B}}$; and
- for any tuple $\mathbf{a}$ of elements of $\mathbb{A}$ such that all elements of $\mathbf{a}$ are in $\mathrm{dom}(f)$ and any relation R we have

$$R^{\mathbb{A}}(\mathbf{a}) \quad \Leftrightarrow \quad R^{\mathbb{B}}(f(\mathbf{a}))$$

Ehrenfeucht-Fraïssé Game

The p-round Ehrenfeucht game on structures $\mathbb{A}$ and $\mathbb{B}$ proceeds as follows:

- There are two players called *Spoiler* and *Duplicator*
- At the ith round, *Spoiler* chooses one of the structures (say $\mathbb{B}$) and one of the elements of that structure (say b_i).
- *Duplicator* must respond with an element of the other structure (say a_i).
- If, after p rounds, the map $a_i \mapsto b_i$ is a partial isomorphism, then *Duplicator* has won the game, otherwise *Spoiler* has won.

Theorem (Fraïssé 1954; Ehrenfeucht 1961)
Duplicator has a strategy for winning the p-round Ehrenfeucht game on $\mathbb{A}$ and $\mathbb{B}$ if, and only if, $\mathbb{A} \equiv_p \mathbb{B}$.

Proof by Example

Suppose $\mathbb{A} \not\equiv_3 \mathbb{B}$, in particular, suppose $\theta(x, y, z)$ is quantifier free, such that:

$$\mathbb{A} \models \exists x \forall y \exists z \theta \quad \text{and} \quad \mathbb{B} \models \forall x \exists y \forall z \neg\theta$$

round 1: Spoiler chooses $a_1 \in A$ *such that* $\mathbb{A} \models \forall y \exists z \theta[a_1]$.
Duplicator responds with $b_1 \in B$.

round 2: Spoiler chooses $b_2 \in B$ *such that* $\mathbb{B} \models \forall z \neg\theta[b_1, b_2]$.
Duplicator responds with $a_2 \in A$.

round 3: Spoiler chooses $a_3 \in A$ *such that* $\mathbb{A} \models \theta[a_1, a_2, a_3]$.
Duplicator responds with $b_3 \in B$.

Spoiler wins, since $\mathbb{B} \not\models \theta[b_1, b_2, b_3]$.

Using Games

To show that a class of structures S is not definable in FO, we find, for every p, a pair of structures $\mathbb{A}_p$ and $\mathbb{B}_p$ such that

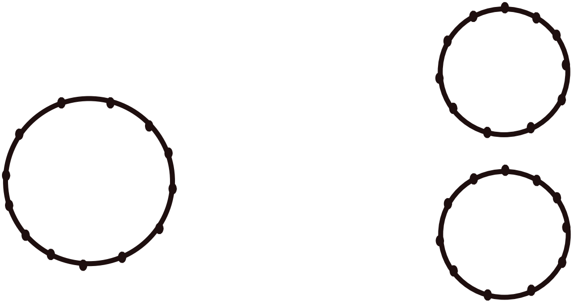
- $\mathbb{A}_p \in S$, $\mathbb{B}_p \in \overline{S}$; and
- *Duplicator* wins a p-round game on $\mathbb{A}_p$ and $\mathbb{B}_p$.

Example:
C_n—a cycle of length n.
Duplicator wins the p-round game on $C_{2^p} \oplus C_{2^p}$ and $C_{2^{p+1}}$.

- 2-Colourability is not definable in FO.
- Even cardinality is not definable in FO.
- Connectivity is not definable in FO.

Using Games

An illustration of the game for undefinability of *connectivity* and *2-colourability*.



Duplicator's strategy is to ensure that after r moves, the distance between corresponding pairs of pebbles is either *equal* or $\geq 2^{p-r}$.

Stratifying Elementary Equivalence

In order to study the expressive power of *first-order logic* on finite structures, we consider one stratification of elementary equivalence:

$$\mathbb{A} \equiv_p \mathbb{B}$$

if $\mathbb{A}$ and $\mathbb{B}$ cannot be distinguished by any sentence with *quantifier rank* at most p.

An alternative stratification that is useful in studying *fixed-point logics* is based on the number of variables.

$$\mathbb{A} \equiv^k \mathbb{B}$$

if $\mathbb{A}$ and $\mathbb{B}$ cannot be distinguished by any sentence with at most k *distinct variables*.

Finite Variable Logic

We write L^k for the first order formulas using only the variables $x_1, \ldots, x_k$.

A first order formula φ is equivalent to one of L^k if no sub-formula of φ contains more than k *free* variables.

$$\mathbb{A} \equiv^k \mathbb{B}$$

denotes that $\mathbb{A}$ and $\mathbb{B}$ agree on all sentences of L^k.

For any k, $\quad \mathbb{A} \equiv^k \mathbb{B} \quad \Rightarrow \quad \mathbb{A} \equiv_k \mathbb{B}$

However, for any p, there are $\mathbb{A}$ and $\mathbb{B}$ such that

$$\mathbb{A} \equiv_p \mathbb{B} \quad \text{and} \quad \mathbb{A} \not\equiv^2 \mathbb{B}.$$

Examples

Connectivity and *2-colourability* are axiomatizable in L^k (for $k \geq 3$).
Even cardinality is *not*.

Connectivity in L^4:

$$\text{path}_{\leq l}(x, y) := \exists z_1(E(x, z_1) \wedge \exists z_2(E(z_1, z_2) \wedge \exists z_1(E(z_2, z_1) \wedge \cdots E(z_i, y))))$$

$$\text{disconnect}_l := \forall x, y(\text{path}_{\leq l+1}(x, y) \Rightarrow \text{path}_{\leq l}(x, y)) \wedge \exists x, y \neg \text{path}_{\leq l}(x, y)$$

Connectivity is then axiomatized by the set

$$\{\neg \text{disconnect}_l \mid l \in \mathbb{N}\}$$

Definability and Invariance

A class of structures is closed under $\equiv_p$ (for some p) if, and only if, it is *defined* by a FO sentence.

A class of finite structures is closed under $\equiv^k$ if, and only if, it is axiomatizable in L^k (possibly by an infinite collection of sentences).

In a *finite, relational* vocabulary, there are only finitely many sentences of quantifier rank at most p.
Thus, the relation $\equiv_p$ has only finitely many equivalence classes.

The relation $\equiv^k$ has infintiely many classes for all $k \geq 2$.

Pebble Games

The k-pebble game is played on two structures $\mathbb{A}$ and $\mathbb{B}$, by two players—*Spoiler* and *Duplicator*—using k pairs of pebbles $\{(a_1, b_1), \ldots, (a_k, b_k)\}$.

Spoiler moves by picking a pebble and placing it on an element (a_i on an element of $\mathbb{A}$ or b_i on an element of $\mathbb{B}$).

Duplicator responds by picking the matching pebble and placing it on an element of the other structure

Spoiler wins at any stage if the partial map from $\mathbb{A}$ to $\mathbb{B}$ defined by the pebble pairs is not a partial isomorphism

If Duplicator has a winning strategy for p moves, then $\mathbb{A}$ and $\mathbb{B}$ agree on all sentences of L^k of quantifier rank at most p.
(Barwise)

Using Pebble Games

To show that a class of structures S is not definable in first-order logic:
$\forall k \; \forall p \; \exists \mathbb{A}, \mathbb{B} \; (\mathbb{A} \in S \land \mathbb{B} \notin S \land \mathbb{A} \equiv_p^k \mathbb{B})$

To show that S is not axiomatisable with a finite number of variables:
$\forall k \; \exists \mathbb{A}, \mathbb{B} \; \forall p \; (\mathbb{A} \in S \land \mathbb{B} \notin S \land \mathbb{A} \equiv_p^k \mathbb{B})$

Evenness

Evenness is not axiomatizable with a finite number of variables.

> *for every k, there are structures $\mathbb{A}_k$ and $\mathbb{B}_k$ such that $\mathbb{A}_k$ has an even number of elements, $\mathbb{B}_k$ has an odd number of elements and*
>
> $$\mathbb{A} \equiv^k \mathbb{B}.$$

It is easily seen that *Duplicator* has a strategy to play forever when one structure is a set containing k elements (and no other relations) and the other structure has $k+1$ elements.

Matching

Take $K_{k,k}$—the complete bipartite graph on two sets of k vertices.
and $K_{k,k+1}$—the complete bipartite graph on two sets, one of k vertices, the other of $k+1$.

These two graphs are $\equiv^k$ equivalent, yet one has a perfect matching, and the other does not. One contains a Hamiltonian cycle, the other does not.

Gaifman Graphs and Neighbourhoods

On a structure $\mathbb{A}$, define the binary relation:

> $E(a_1, a_2)$ *if, and only if, there is some relation* R *and some tuple* $\mathbf{a}$ *containing both* a_1 *and* a_2 *with* $R(\mathbf{a})$.

The graph $G\mathbb{A} = (A, E)$ is called the *Gaifman graph* of $\mathbb{A}$.

$dist(a, b)$ — the distance between a and b in the graph (A, E).

$\mathrm{Nbd}_r^{\mathbb{A}}(a)$ — the substructure of $\mathbb{A}$ given by the set:

$$\{b \mid dist(a, b) \leq r\}$$

Hanf Locality Theorem

We say $\mathbb{A}$ and $\mathbb{B}$ are *Hanf equivalent* with radius r ($\mathbb{A} \simeq_r \mathbb{B}$) if, for every $a \in A$ the two sets

$$\{a' \in A \mid \mathrm{Nbd}_r^{\mathbb{A}}(a) \cong \mathrm{Nbd}_r^{\mathbb{A}}(a')\} \quad \text{and} \quad \{b \in B \mid \mathrm{Nbd}_r^{\mathbb{A}}(a) \cong \mathrm{Nbd}_r^{\mathbb{B}}(b)\}$$

have the same cardinality.
and, similarly for every $b \in B$.

Theorem (Hanf)
For every vocabulary σ and every p there is $r \leq 3^p$ such that for any σ-structures $\mathbb{A}$ and $\mathbb{B}$: if $\mathbb{A} \simeq_r \mathbb{B}$ then $\mathbb{A} \equiv_p \mathbb{B}$.

In other words, if $r \geq 3^p$, the equivalence relation $\simeq_r$ is a refinement of $\equiv_p$.

Proving Hanf Locality

Duplicator's strategy is to maintain the following condition:
After k moves, if $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ have been selected, then

$$\bigcup_i \mathrm{Nbd}^{\mathbb{A}}_{3^{p-k}}(a_i) \cong \bigcup_i \mathrm{Nbd}^{\mathbb{B}}_{3^{p-k}}(b_i)$$

If *Spoiler* plays on a within distance $2 \cdot 3^{p-k-1}$ of a previously chosen point, play according to the isomorphism, otherwise, find b such that

$$\mathrm{Nbd}_{3^{p-k-1}}(a) \cong \mathrm{Nbd}_{3^{p-k-1}}(b)$$

and b is not within distance $2 \cdot 3^{p-k-1}$ of a previously chosen point.
Such a b is guaranteed by $\simeq_r$.