The Difficulty of Counting:

Dichotomy Theorems for Generalized Chromatic Polynomials.

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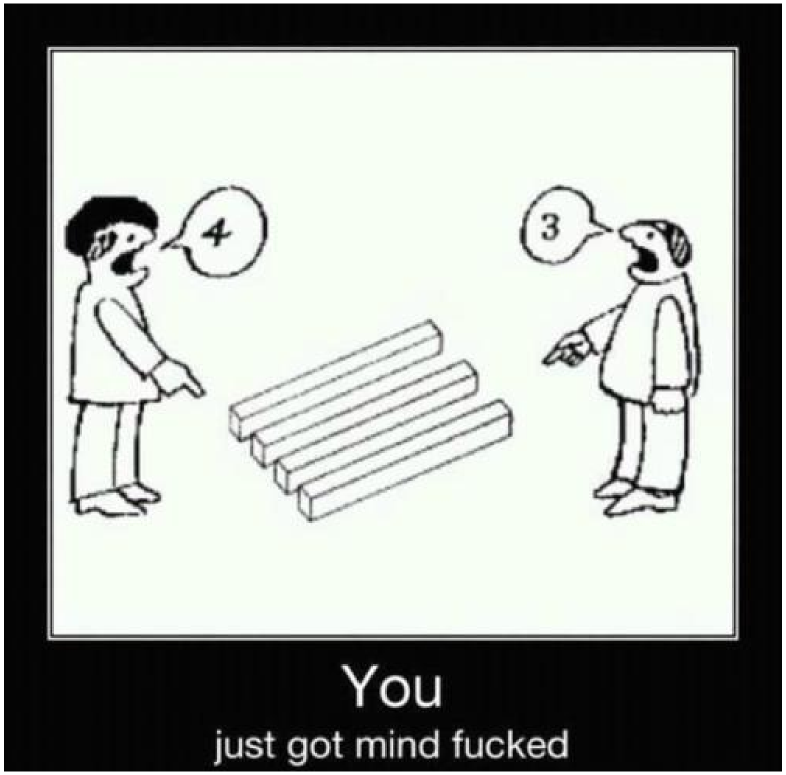
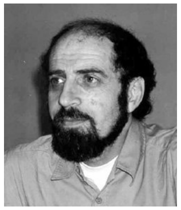
Partially joint work with
A. Goodall, M. Hermann, T. Kotek, S. Noble and E.V. Ravve.

Graph polynomial project:
http://www.cs.technion.ac.il/∼janos/RESEARCH/gp-homepage.html

The Difficulty of Counting



Intriguing Graph Polynomials


Boris Zilber and JAM, September 2004, in Oxford

BZ: What are you studying nowadays?

JAM: Graph polynomials.

BZ: Uh? What? Can you give examples ....

JAM: Matching polynomial, chromatic polynomial, Tutte polynomial, characteristic polynomial, ... (detailed definitions) ...

BZ: I know these! They all occur as growth polynomials in my work on $\aleph_0$-categorical, ω-stable models!

JAM: ??? Let's see!

A detailed exposition of the work which resulted from this conversation can be found in:

Model Theoretic Methods in Finite Combinatorics

M. Grohe and J.A. Makowsky, eds.,
Contemporary Mathematics, vol. 558 (2011), pp. 207-242
American Mathematical Society, 2011

Especially the papers

- **On Counting Generalized Colorings**
  T. Kotek, J. A. Makowsky, and **B. Zilber**

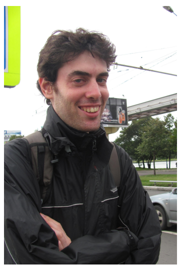
- *Application of Logic to Combinatorial Sequences and Their Recurrence Relations*
  E. Fischer, T. Kotek, and J. A. Makowsky

- *Counting Homomorphisms and Partition Functions*
  M. Grohe and M. Thurley

Related Ph.D. Theses under my supervision

-  I. Averbouch,

  Completeness and Universality Properties of Graph Invariants and Graph Polynomials
  Technion - Israel Institute of Technology, Haifa, Israel, 2011

-  T. Kotek,

  Definability of combinatorial functions,
  Technion - Israel Institute of Technology, Haifa, Israel, 2012

Outline of this talk

- One, two, many ”chromatic” polynomials
- Computing graph polynomials over the real or complex numbers
  Turing vs Blum-Shub-Smale computability
- Difficult Point Property (DPP)
- Univariate case
- Conclusion and many open problems

Dagstuhl, January 2013

Earlier versions of this talk can be found at

- J.A. Makowsky and T. Kotek and E.V. Ravve,
  A Computational Framework for the Study of Partition Functions and Graph Polynomials,
  Proceedings of the 12th Asian Logic Conference '11,
  World Scientific, 2013, pp 210-230.

- Talk given at the Dagstuhl Seminar 13031
  Computational Counting
  January 13-18, 2013
  Organizers:
  Peter Bürgisser, Leslie Ann Goldberg, Mark R. Jerrum, Pascal Koiran

In this talk we examine our earlier **premature conjectures**,
give counter-examples,
and propose revised **Open Problems**

Back to outline

One, two, many chromatic polynomials

The prototype: The chromatic polynomial

Let G be a graph and $[k] = \{1, 2, \ldots, k\}$. We think of $[k]$ as colors.

A function $c : V(G) \to [k]$ is a proper coloring of G with at most k colors, if for each $i \in [k]$ the set $c^{-1}(i)$ is an independent set in G.

We denote by $\chi(G; k)$ the number of proper colorings of G with k colors.

Birkhoff (1912) showed that $\chi(G; k)$ is a polynomial in $\mathbb{Z}[k]$. Furthermore, he showed that for the edgeless graph with n vertices, $E_n = ([n], \emptyset)$ we have that $\chi(E_n; k) = k^n$ and

$$\chi(G_{/e}; k) = \chi(G; k) + \chi(G_{\backslash e}; k)$$

where $e \in E(G)$ and $G_{/e}$ is obtained by deleting the edge e and $G_{\backslash e}$ is obtained by contracting the edge e.

This shows that $\chi(G; k)$ is a polynomial counting colorings and that $\chi(G; k)$ has a recursive definition,

and can be **extended** to a polynomial $\chi(G; X) \in \mathbb{R}[X]$.

Deletion and contraction

- Proving that $\chi(G; X)$ is a graph polynomial using

  deletion and **contraction**

  is very **elegant**, and led to the theory of **graph minors**.

- However, this proof is **restricted** to graph polynomials which are **related to the Tutte polynomial via the recipe theorem**.

- B. Zilber's observation, stripped to its basics, shows that counting other graph colorings of a graph G with k colors leads, for each graph G, to a polynomial in k.

The elementary generic proof

THEOREM:

For every graph G, the counting function $\chi(G, k)$ is a polynomial in k of the form

$$\sum_{j=0}^{|V(G)|} c(G, j)\binom{k}{j}$$

where $c(G, j)$ is the number of proper k-colorings with a fixed set of j colors.

Polynomials in $\mathbb{Z}[k]$ with monomials of the form $\binom{k}{j}$
are sometimes called **Newton polynomials**.

Proof

A **proper** coloring uses at most $N = |V(G)|$ of the k colors.

For any $j \leq N$, let $c(G, j)$ be the number of **proper** colorings, with a fixed set of j colors, which are **proper** colorings and use all j of the colors.

We observe:

(A): Every permutation of the set of colors used gives also a **proper** coloring.

(B): Colors not used do not have an effect on being a **proper** coloring.

Therefore, given k colors, the number of vertex colorings that use exactly j of the k colors is the product of $c(G, j)$ and the binomial coefficient $\binom{k}{j}$. So

$$\chi(G, k) = \sum_{j \leq N} c(G, j) \binom{k}{j}$$

The right side here is a polynomial in k, because each of the binomial coefficients is. We also use that for $k \leq j$ we have $\binom{k}{j} = 0$. Q.E.D.

Variations on coloring, I

Using properties **(A)** and **(B)** we get variations of chromatic polynomials:

We can count other coloring functions.

- **proper k-edge-colorings**:

  $f_E : E(G) \to [k]$ such that if $(e, f) \in E(G)$ have a common vertex then $f_E(e) \neq f_E(f)$.

  $\chi_e(G, k)$ denotes the number of k- edge-colorings

- **Total colorings**

  $f_V : V \to [k_V]$, $f_E : E \to [k_E]$ and $f = f_V \cup f_E$,
  with f_V a proper vertex coloring and f_E a proper edge coloring.

- **Connected components**

  $f_V : V \to [k_V]$, If $(u, v) \in E$ then $f_V(u) = f_V(v)$.

- **Hypergraph colorings**

  Vitaly I. Voloshin, Coloring Mixed Hypergraphs: Theory, Algorithms and Applications, Fields Institute Monographs, AMS 2002

Variations on coloring, II

Let $f : V(G) \rightarrow [k]$ be a function, such that Φ is one of the properties below and $\chi_\Phi(G, k)$ denotes the number of such colorings with atmost k colors.

For **(*)**, $\chi_\Phi(G, k)$ satisfies **(A)** and **(B)**, hence is a polynomial in k, for **(-)**, it is not.

- **complete:** f is a **proper** coloring such that **every pair of colors occurs along some edge**.
  F. Harary and S. Hedetniemi and G. Prins, An interpolation theorem for graphical homomorphisms, Portugal. Math., 26 (1967), 453-462.

* **harmonious:** f is a **proper** coloring such that **every pair of colors occurs at most once along some edge**.
  J.E. Hopcroft and M.S. Krishnamoorthy, On the harmonious coloring of graphs, SIAM J. Algebraic Discrete Methods, 4 (1983), 306-311.

* **convex:** Every monochromatic set induces a **connected graph**.
  S. Moran and S. Snir, Efficient approximation of convex recolorings, Journal of Computer and System Sciences, 73.7 (2007), 1078-1089

Variations on coloring, III

More coloring polynomials in $\mathbb{Z}[k]$:

* **injective:** f is injectiv on the neighborhood of every vertex.
  G. Hahn and J. Kratochvil and J. Siran and D. Sotteau, On the injective chromatic number of graphs, Discrete mathematics, 256.1-2, (2002), 179-192.

* **path-rainbow:** Let $f : E \to [k]$ be an edge-coloring. f is **path-rainbow** if between any two vertices $u, v \in V$ there as a path where all the edges have different colors.

  Rainbow colorings of various kinds arise in computational biology

  Rainbow connection in graphs, G. Chartrand and G.L. Johns and K. McKeon A and P. Zhang, Mathematica Bohemica, 133.1, (2008), 85-98.

* **monochromatic components:** Let $f : V \to [k]$ be an vertex-coloring and $t \in \mathbb{N}$. f is an mcc_t-coloring of G with k colors, if all the connected components of a monochromatic set have size at most t. N. Alon, G. Ding, B. Oporowski, and D. Vertigan. Partitioning into graphs with only small components. Journal of Combinatorial Theory, Series B, 87:231–243, 2003.

Variations on coloring, IV

Let $\mathcal{P}$ be any graphs property and let $n \in \mathbb{N}$.

We can define coloring functions $f : V \to [k]$ by requiring that the union of any n color classes induces a graph in $\mathcal{P}$.

- For $n = 1$ and $\mathcal{P}$ the empty graphs $G = (V, \emptyset)$ we get the proper colorings.
- For $n = 1$ and $\mathcal{P}$ the connected graphs we get the convex colorings.
- For $n = 1$ and $\mathcal{P}$ the graphs which are disjoint unions of graphs of size at most t, we get the mcc_t-colorings.
- For $n = 2$ and $\mathcal{P}$ the acyclic graphs we get the **acyclic colorings**, introduced in: B. Grunbaum, Acyclic colorings of planar graphs, Israel J. Math. 14 (1973), 390-412 and further studied in N.Alon , C. Mcdiarmid, B. Reed, Acyclic coloring of graphs, Random Structures & Algorithms 2.3 (1991) 277-288.

Theorem: Let $\chi_{\mathcal{P},n}(G, k)$ be the number of colorings of G with k colors such that the union of any n color classes induces a graph in $\mathcal{P}$.

Then $\chi_{\mathcal{P},n}(G, k)$ is a polynomial in k.

Back to outline

Equivalence of graph polynomials

with E.V. Ravve

Two graph polynomials $P(G;X)$ and $Q(G;X)$ are **d.p.-equivalent** if for all graphs G_1, G_2 we have

$$P(G_1, X) = P(G_2, X) \text{ iff } Q(G_1, X) = Q(G_2, X)$$

Theorem:(JAM and E.V. Ravve)

Let $\mathcal{P}$ and $\mathcal{Q}$ be two graph properties.

$\chi_{\mathcal{P},1}(G,k)$ is d.p.-equivalent to $\chi_{\mathcal{Q},1}(G,k)$ iff $\mathcal{P} = \mathcal{Q}$ or $\mathcal{P} = \bar{\mathcal{Q}}$.

Conclusion: There uncountably many generalized chromatic polynomials.

Back to outline

Skip multivariate and go to DPP, Skip multivariate and go to BSS,

Variations on colorings, V: Two kinds of colors.

Let $G = (V, E)$.
Here we look at two disjoint color sets $A = [k_1]$ and $B = [k_1 + k_2] - [k_1]$.
The colors in A are called **proper** colors.
Our coloring is a function $f : V \rightarrow [k_1 + k_2] = [k]$ such that

- If $(u, v) \in E$ and $f(u) \in A$ and $f(v) \in A$ then $f(u) \neq f(v)$.
- We count the number of colorings with $k = k_1 + k_2$ colors such that k_1 colors are in A, i.e., **proper**.

Theorem 1 (K. Dohmen, A. Pönitz and P. Tittman, 2003)
This gives us a polynomial $P(G, k_1, k)$ in k_1 and k.

Back to outline

Computing chromatic polynomials

Turing computability vs Computability over $\mathbb{R}$ or $\mathbb{C}$.

An in-depth discussion can be found in:

- J.A. Makowsky and T. Kotek and E.V. Ravve,

  A Computational Framework for the Study of Partition Functions and Graph Polynomials,

  Proceedings of the 12th Asian Logic Conference '11,
  World Scientific, 2013, pp. 210-230

Evaluations of graph polynomials, I

Let $P(G; \bar{X})$ be a graph polynomial in the indeterminates $X_1, \ldots, X_n$.

Let $\mathcal{R}$ be a subfield of the complex numbers $\mathbb{C}$.

For $\bar{a} \in \mathcal{R}^n$, $P(-; \bar{a})$ is a graph invariant taking values in $\mathcal{R}$.

We could restrict the graphs to be from a class (graph property) $\mathcal{C}$ of graphs.

What is the complexity of computing $P(-; \bar{a})$ for graphs from $\mathcal{C}$?

- If for all graphs $G \in \mathcal{C}$ the value of $P(-; \bar{a})$ is a graph invariant taking values in $\mathbb{N}$, we can work in the Turing model of computation.
- Otherwise we identify the graph G with its adjacency matrix M_G, and we work in the Blum-Schub-Smale (BSS) model of computation.

Our goal

We want to discuss and extend the classical result of

F. Jaeger and D.L. Vertigan and D.J.A. Welsh

on the complexity of evaluations of the Tutte polynomial. They show:

- either evaluation at a point $(a, b) \in \mathbb{C}^2$ is polynomial time computable in the Turing model, and a and b are integers,
- or some $\sharp\mathbf{P}$-complete problem is reducible to the evaluation at $(a, b) \in \mathbb{C}^2$.
- To stay in the Turing model of computation, they assume that (a, b) is in some finite dimensional extension of the field $\mathbb{Q}$.

The proof of the second part is a hybrid statement:
The reduction is more naturally placed in the BSS model of computation,
However, $\sharp\mathbf{P}$-completeness has no suitable counterpart in the BSS model.

Pseudo-$\sharp\mathbf{P}$-completeness

Authors often use implicitly the following **hybrid** definition:

The evaluation of $P(-,\bar{a})$ with $\bar{a} \in \mathbb{C}^m$ is (pseudo)-$\sharp\mathbf{P}$ hard (or complete) if

- There is $\bar{c} \in \mathbb{C}^m$ with $P(-,\bar{c}) \in \sharp\mathbf{P}$ and $\sharp\mathbf{P}$-complete in the Turing model, and
- $P(-,\bar{c}) \leq_{algebraically} P(-,\bar{a})$,

  where the reduction $\leq_{algebraically}$ is given
  no precise definition, but can be given easily subject to minor variations.

It seems to us **more natural** to work **entirely**
in the BSS model of computation.
How exactly ?

Evaluations of graph polynomials, II

- A graph invariant or graph parameter is a function $f : \bigcup_n \{0,1\}^{n \times n} \rightarrow \mathcal{R}$ which is invariant under permutations of columns and rows of the input adjacency matrix.

- A graph transformation is a function $T : \bigcup_n \{0,1\}^{n \times n} \rightarrow \bigcup_n \{0,1\}^{n \times n}$ which is invariant under permutations of columns and rows of the input adjacency matrix.

- The BSS-P-time computable functions over $\mathcal{R}$, $P_{\mathcal{R}}$, are the functions $f : \{0,1\}^{n \times n} \rightarrow \mathcal{R}$ BSS-computable in time $O(n^c)$ for some fixed $c \in \mathbb{N}$.

- Let f_1, f_2 be graph invariants. f_1 is BSS-P-time reducible to f_2, $f_1 \leq_P f_2$ if there are BSS-P-time computable functions T and F such that

  (i) T is a graph transformation ;

  (ii) For all graphs G with adjacency matrix M_G we have
  $$f_1(M_G) = F(f_2(T(M_G)))$$

- two graph invaraints f_1, f_2 are BSS-P-time equivalent, $f_1 \sim_{BSS-P} f_2$, if $f_1 \leq_{BSS_P} f_2$ and $f_2 \leq_{BSS_P} f_1$.

Evaluations of graph polynomials, III: Degrees and Cones

What are difficult graph parameters in the BSS-model?

Let g, g' be a graph parameters computable in exponential time in the BSS-model, i.e., $g, g' \in EXP_{BSS}$.

BSS-Degrees We denote by $[g]_{BSS}$ and $[g]_T$ the equivalence class (BSS-degree) of all graph parameters $g' \in EXP_{BSS}$ under the equivalence relation $\sim_{BSS-P}$.

BSS-Cones We denote by $< g >_{BSS}$ the class (BSS-cone) $\{g' \in EXP_{BSS} : g \leq_{BSS-P} g'\}$.

NP-completeness There are BSS-NP-complete problems, and instead of specifing them, we consider NP to be a degree (which may vary with the choice of the Ring $\mathcal{R}$).

NP-hardness The cone of an NP-complete problem forms the NP-hard problems.

Evaluations of graph polynomials, IV

We work in BSS model over $\mathcal{R}$.

We define

$$\text{EASY}_{BSS}(P,\mathcal{C}) = \{\bar{a} \in \mathcal{R}^n : P(-;\bar{a}) \text{ is BSS-P-time computable } \}$$

and

$$\text{HARD}_{BSS}(P,\mathcal{C}) = \{\bar{a} \in \mathcal{R}^n : P(-;\bar{a}) \text{ is BSS-NP-hard } \}$$

We use $\text{EASY}_{BSS}(P)$ and $\text{HARD}_{BSS}(P)$ if $\mathcal{C}$ is the class of all finite graphs.

How can we describe $\text{EASY}(P,\mathcal{C})$ and $\text{HARD}(P,\mathcal{C})$?

Counting Complexity Classes in the BSS-model, I

K. Meer (2000) introduced an analogue of $\sharp\mathbf{P}$ for discrete counting in the BSS model which can easily be extended to the BSS model over $\mathbb{C}$.

- A *counting function over* $\mathbb{C}$ is a functions $f : \mathbb{C}^\infty \to \mathbb{N} \cup \{\infty\}$.
- For a complexity class of functions FC we denote by FC^{count} the class of counting functions in FC.
- A function f is in $\sharp\mathbf{P}_{\mathbb{C}}$ if there exists a polynomial time BSS-machine over $\mathbb{C}$ and a polynomial q such that

$$f(y) = |\{z \in \mathbb{C}^{q(size(y))} : M(y,z) \text{ accepts } \}|$$

- For every sub-field $\mathcal{R}$ of $\mathbb{C}$ we have $\mathsf{FP}_{\mathcal{R}}^{count} \subseteq \sharp\mathbf{P}_{\mathcal{R}} \subseteq \mathsf{FE}_{\mathcal{R}}$.

Typical examples over the reals $\mathbb{R}$ are counting zeroes of multivariate polynomials of degree at most 4 ($\sharp 4-\mathsf{FEAS}$) or counting the number of sign changes of a sequence of real numbers ($\sharp$SC).

Counting Complexity Classes in the BSS-model, II

Proposition: Over $\mathbb{C}$ the number of k-colorings of a graph is in $\sharp\mathbf{P}_{\mathbb{C}}$.

To see this, we associate with a graph $G = (V, E)$ with $V = \{1, \ldots, n\} = [n]$ the following set $\mathcal{E}^k_{color}(G)$ of equations:

(i) $x_i^k - 1 = 0, i \in [n]$

(ii) $\sum_{d=0}^{k-1} x_i^{k-1-d} x_j^d, (i,j) \in E.$

Clearly, $\mathcal{E}^k_{color}(G)$ has at most k^n many complex solutions.

D.A. Beyer in his Ph.D. thesis (1982) observed that a graph G is k-colorable iff $\mathcal{E}^k_{color}(G)$ has a complex solution, and each solution corresponds exactly to proper k-coloring of G.

In S. Margulies' Ph.D. thesis (2008) it is shown:

Theorem: Every decision problem in $\mathbf{NP}$ (in the Turing model) can be encoded as solvability problem of sets of equations over $\mathbb{C}$.

Using the fact that $\sharp$SAT is $\sharp\mathbf{P}$ complete we get:

Theorem: Every function $f \in \sharp\mathbf{P}$ (in the Turing model) has an encoding in $\sharp\mathbf{P}_{\mathbb{C}}$ (in the BSS model).

The difficult counting hypothesis (**DCH**)

- In the literature there are no **explicit graph parameters** which are $\sharp\mathbf{P}_{\mathbb{C}}$-complete problems.
- Counting the number of colorings is not known to be $\mathbf{NP}_{\mathbb{C}}$-hard in the BSS-model. On the other hand, it would be **truly surprising** if counting the number of k-colorings were in $\mathsf{FP}_{\mathbb{C}}^{count}$.

Hence we formulate two complexity hypotheses for the BSS model over $\mathbb{C}$:

SDHC: Strong difficult counting hypothesis
Every counting function in $f \in \sharp\mathbf{P}_{\mathbb{C}}$ with discrete input which is $\sharp\mathbf{P}$-hard in the Turing model is $\mathbf{NP}_{\mathbb{C}}$-hard in the BSS model over $\mathbb{C}$.

WDCH: Weak difficult counting hypothesis
A counting function in $f \in \sharp\mathbf{P}_{\mathbb{C}}$ which is $\sharp\mathbf{P}$-hard in the Turing model cannot be in $\mathsf{FP}_{\mathbb{C}}^{count}$.

If $\mathbf{P}_{\mathbb{C}} \neq \mathbf{NP}_{\mathbb{C}}$ then **SDCH** implies **WDCH**.

Back to outline

Computing the chromatic polynomial and the Tutte polynomial, revisited

The complexity of the chromatic polynomial, I

Theorem:

- $\chi(G, 0)$, $\chi(G, 1)$ and $\chi(G, 2)$ are P-time computable (Folklore)
- $\chi(G, 3)$ is $\sharp\mathbf{P}$-complete (Valiant 1979).
- $\chi(G, -1)$ is $\sharp\mathbf{P}$-complete (Linial 1986).

Question:

What is the complexity of computing $\chi(G, \lambda)$ for

$\lambda = \lambda_0 \in \mathbb{Q}$

or even for

$\lambda = \lambda_0 \in \mathbb{C}$?

The complexity of the chromatic polynomial, II
Linial's Trick

Let $G_1 \bowtie G_2$ denote the join of two graphs.

We observe that

$$\chi(G \bowtie K_n, \lambda) = (\lambda)^{\underline{n}} \cdot \chi(G, \lambda - n) \qquad (\star)$$

Hence we get

(i) $\chi(G \bowtie K_1, 4) = 4 \cdot \chi(G, 3)$

(ii) $\chi(G \bowtie K_n, 3 + n) = (n + 3)^{\underline{n}} \cdot \chi(G, 3)$
hence for $n \in \mathbb{N}$ with $n \geq 3$ it is **$\sharp$P-complete**.

This works in the Turing model of computation

for λ in some Turing-computable field extending $\mathbb{Q}$.

The complexity of the chromatic polynomial, III

If we have have an oracle for some $q \in \mathbb{Q} - \mathbb{N}$ which allows us to compute $\chi(G, q)$ we can compute $\chi(G, q')$ for any $q' \in \mathbb{Q}$ as follows:

Algorithm $A(q, q', \mid V(G) \mid)$:

(i) Given G the degree of $\chi(G, q)$ is at most $n = \mid V(G) \mid$.

(ii) Use the oracle and $(\star)$ to compute $n + 1$ values of $\chi(G, \lambda)$.

(iii) Using Lagrange interpolation we can compute $\chi(G, q')$ in polynomial time.

We note that this algorithm is purely algebraic and works for all graphs G, $q \in (F) - \mathbb{N}$ and $q' \in F$ for any field F extending $\mathbb{Q}$.

Hence we get that for all $q_1, q_2 \in \mathbb{C} - \mathbb{N}$ the graph parameters are polynomially reducible to each other.

Furthermore, for $3 \leq i \leq j \in \mathbb{N}$, $\chi(G, i)$ is reducible to $\chi(G, j)$.

This works in the BSS-model of computation.

The complexity of the chromatic polynomial, IV

We summarize the situation for the chromatic polynomial as follows:

(i) $\text{EASY}_{BSS}(\chi) = \{0, 1, 2\}$ and $\text{HARD}_{BSS}(\chi) = \mathbb{C} - \{0, 1, 2\}$.

(ii) $\text{HARD}_{BSS}(\chi)$ can be split into two sets:

(ii.a) $\text{HARD}_{\sharp\mathbf{P}}(\chi)$: the graph parameters which are **counting functions** in $\sharp\mathbf{P}$ in the sense of Valiant, with $\chi(-, 3) \leq_P \chi(-, j)$ for $j \in \mathbb{N}$ and $3 \leq j$.
All graph parameters in $\text{HARD}_{\sharp\mathbf{P}}(\chi)$ are $\sharp\mathbf{P}$-complete in the Turing model.

(ii.b) $\text{HARD}_{BSS-NP}(\chi)$: the graph parameters which are not **counting functions**.
In the BSS model they are all **polynomially reducible to each other**, and all graph parameters in $\text{HARD}_{\sharp\mathbf{P}}(\chi)$ are P-reducible to each of the graph parameters in $\text{HARD}_{BSS}(\chi)$.

(ii.c) In the BSS-model the graph parameter $\chi(-, 3)$ is P-reducible to all the parameters in $\text{HARD}_{BSS}(\chi)$.

(ii.d) Inside $\text{HARD}_{BSS}(\chi)$ we have:

$$\chi(-, 3) \leq_{BSS_P} \chi(-, 4) \leq_{BSS_P} \ldots \chi(-, j) \ldots \leq_{BSS-P} \chi(-, a) \sim_{BSS_P} \chi(-, -1)$$

with $j \in \mathbb{N} - \{0, 1, 2\}$ and $a \in \mathbb{C} - \mathbb{N}$.

The complexity of the chromatic polynomial, V

We have a **Dichotomy Theorem** for the evaluations of $\chi(-, \lambda)$:

(i) $\text{EASY}_{BSS}(\chi) = \{0, 1, 2\}$

Over $\mathbb{C}$ this is a quasi-algebraic set (a finite boolean combination of algebraic sets) of dimension 0.

(ii) All graph parameters in $\text{HARD}_{BSS}(\chi)$

are at least as difficult as $\chi(-, 3)$

(via BSS-P-reductions)

This is a quasi-algebraic set of dimension 1.

Evaluating the Tutte polynomial (Jaeger, Vertigan, Welsh)

The Tutte polynomial $T(G, X, Y)$ is a bivariate polynomial

and $\chi(G, \lambda) \leq_P T(G, 1-\lambda, 0)$.

We have the following **Dichotomy Theorem**:

(i) $\text{EASY}_{BSS}(T) = \{(x, y) \in \mathbb{C}^2 : (x-1)(y-1) = 1\} \cup \text{Except}$, with
$\text{Except} = \{(0,0), (1,1), (-1,-1), (0,-1), (-1,0), (i,-i), (-i,i), (j,j^2), (j^2,j)\}$
and $j = e^{\frac{2\pi i}{3}}$
Over $\mathbb{C}$ this is a quasi-algebraic set of dimension 1.

(ii) All graph parameters in $\text{HARD}_{BSS}(T)$ are at least as hard as $T(G, 1-\lambda, 0)$.
This is a quasi-algebraic set of dimension 2.

The proof and its generalizations

Variations on Linial's Trick

- ($\star$) is replaced by two (or more) operations:

  stretching and **thickening**.

- **Lagrange interpolation** is done on a **grid**.

- There are considerable **technical challenges** in **details** of the proof for the **Tutte polynomial**.

- Allthough in all **successfull generalizations** to other cases, the same general outline of the proof is always similar, substantial challenges in the details have to be overcome.

How hard is $\sharp$3COL $= \chi(-, 3)$?

- In the Turing model $\chi(-, 3)$ and $\chi(-, -1)^2$ are both in $\sharp\mathbf{P}$ and $\chi(-, 3) \leq_P \chi(-, -1)$.
  As $\chi(-, 3)$ is $\sharp\mathbf{P}$-complete, they are both $\sharp\mathbf{P}$-complete.

  In BSS **this does not work!**

- For $\mathbb{C}$, Malajovich and Meer (2001) proved an analogue of Ladner's Theorem for the BSS-model over $\mathbb{C}$:

  Assuming that $\mathbf{P}_{\mathbb{C}} \neq \mathbf{NP}_{\mathbb{C}}$ there are infinitely many different BSS-degrees between them.

- Although the problem $\chi(-, 3) \neq 0$? is in $\mathbf{NP}_{\mathbb{C}}$ we do not know whether there is $a \in \mathbb{C} - \mathbb{N}$ for which computing $\chi(-, a)$ is really harder!

- In particular, we know that $\chi(a, 3) \leq_{BSS-P} \chi(-, -1)$,

  but we do not know whether $\chi(a, -1) \leq_{BSS-P} \chi(-, 3)$

Skip problems with hybrid
Back to outline

Problems with hybrid complexity, I

Let f_1, f_2 be two graph parameters taking values in $\mathbb{N}$

as a subset of the ring $\mathcal{R}$.

We have **two kind of reductions**:

- **T-P-time Turing reductions** (via oracles) in the Turing model.
  $f_1 \leq_{T-P} f_2$ iff f_1 can be computed in T-P-Time using f_2 as an oracle.
- **BSS-P-time reductions** over the ring $\mathcal{R}$.
  $f_1 \leq_{BSS-P} f_2$ iff f_1 can be computed in BSS-P-Time using f_2 as an oracle.
- In the Turing model there is a natural class of problems $\sharp\mathbf{P}$ for counting, problems which contains many evaluation of graph polynomials.
  However, $\sharp\mathbf{P}$ is **NOT CLOSED** under T-P-reductions.
- In the BSS model no corresponding class seems to **accomodate graph polynomials**.

Problems with hybrid complexity, II

- In 2013, T. Kotek, JAM, E. Ravve proposed a new candidate, the **class** SOLEVAL$_{\mathcal{R}}$ **of evaluations of** SOL-**polynomials**, the graph polynomials definable in Second Order Logic as described by T. Kotek, JAM, and B. Zilber (2008, 2011).
- The **main problem with hybrid complexity** is the apparent incompatibility of the two notions of polynomial reductions, $f_1 \leq_{T-P} f_2$ and $f_1 \leq_{BSS-P} f_2$ even in the case where f_1 and f_2 are both in $\sharp\mathbf{P}$.
- The number of 3-colorings of a graph, ♯3COL, and the number of acyclic orientations ♯ACYCLOR are T-P-equivalent, and $\sharp\mathbf{P}$-complete in the Turing model.
- In the BSS model we have ♯3COL $\leq_{BSS-P}$ ♯ACYCLOR, but it is open whether ♯ACYCLOR $\leq_{BSS-P}$ ♯3COL holds.

The difficult point properties (DPP)

Difficult Point Property, I

Given a graph polynomial $P(G, \bar{X})$ in n indeterminates $X_1, \ldots, X_n$

we are interested in the set $\text{HARD}_{BSS}(P)$.

(i) We say that P has the **weak difficult point property (WDPP)**
if $\text{HARD}_{BSS}(P) \neq \emptyset$ then
there is a quasi-algebraic subset $D \subset \mathbb{C}^n$ of co-dimension $\leq n-1$
such that $\mathbb{C}^n - D \subseteq \text{HARD}_{BSS}(P)$.

(ii) We say that P has the **strong difficult point property (SDPP)**
if $\text{HARD}_{BSS}(P) \neq \emptyset$ then
there is a quasi-algebraic subset $D \subset \mathbb{C}^n$ of co-dimension $\leq n-1$
such that $\mathbb{C}^n - D = \text{HARD}_{BSS}(P) \neq \emptyset$ and $D = \text{EASY}_{BSS}(P)$.

In both cases $\text{EASY}_{BSS}(P)$ is of dimension $\leq n-1$, and for almost all points $\bar{a} \in \mathbb{C}^n$ the evaluation of $P(-, \bar{a})$ is BSS-NP-hard.

$\chi(G; \lambda)$ and $T(G; X, Y)$ both have the SDPP.

Difficult Point Property, II

We compare WDPP and SDPP to Dichtomy Properties.

(i) We say that P has the **dichotomy property (DiP)** if
$\text{HARD}_{BSS}(P) \cup \text{EASY}_{BSS}(P) = \mathbb{C}^n$.
Clearly, if $\mathbf{P}_{\mathbb{C}} \neq \mathbf{NP}_{\mathbb{C}}$, $\text{HARD}_{BSS}(P) \cap \text{EASY}_{BSS}(P) = \emptyset$.

(ii) WDPP is **not** a dichtomy property, but **SDPP a dichotomy property**.

(iii) The two versions of DPP have a quantitative aspect:

$\text{EASY}_{BSS}(P)$ **is small**.

Graph polynomials with the DPP, I

SDPP: The Tutte polynomial (our paradigma).

SDPP: the cover polynomial $C(G, x, y)$ introduced
by Chung and Graham (1995)
by Bläser, Dell 2007, Bläser, Dell, Fouz 2011

SDPP: the bivariate matching polynomial for multigraphs,
by Averbouch and JAM, 2007

WDPP: the Bollobás-Riordan polynomial, generalizing the Tutte polynomial and introduced by Bollobás and Riordan (1999),
by Bläser, Dell and JAM 2008, 2010.

WDPP: the interlace polynomial (aka Martin polynomial) introduced by Martin (1977) and independently by Arratia, Bollobás and Sorkin (2000),
by Bläser and Hoffmann, 2007, 2008

Skip partition functions

Back to outline

Partition functions as graph polynomials

- Let $A \in \mathbb{C}^{n \times n}$ a symmetric and G be a graph. Let

$$Z_A(G) = \sum_{\sigma : V(G) \rightarrow [n]} \prod_{(v,w) \in E(G)} A_{\sigma(v),\sigma(w)}$$

  Z_A is called a partition function.

- Let $\mathbf{X}$ be the matrix $(X_{i,j})_{i,j \leq n}$ of indeterminates.
  Then $Z_{\mathbf{X}}$ is a graph polynomial in n^2 indeterminates,
  Z_A is an evaluation of $Z_{\mathbf{X}}$, and $Z_{\mathbf{X}}$ is MSOL-definable.

Partition functions have the SDPP

- J. Cai, X. Chen and P. Lu (2010),
  building on A. Bulatov and M. Grohe (2005),
  proved a dichotomy theorem for Z_{X} where $\mathcal{R} = \mathbb{C}$.

- Analyzing their proofs reveals:
  Z_{X} satisfies the SDPP for $\mathcal{R} = \mathbb{C}$.

- There are various generalizations of this to Hermitian matrices,
  M. Thurley (2009),
  and beyond.

Skip conjectures

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The DPP conjectures for graph polynomials definable in SOL or MSOL, I

The graph polynomials discussed in the literature all are definable in the formalism of **SOL-definable** graph polynomials.

Actually, **many** of the prominent graph polynomials are **MSOL-definable** using an ordering on the vertices or edges.
Among them the chromatic polynomial, the Tutte poynomials, the matching polynomials, etc.

Here we only need these definability criterion to formulate our conjectures.

Details can be found in

- J.A. Makowsky,
  From a Zoo to a Zoology: Towards a general theory of graph polynomials,
  Theory of Computing Systems, vol. 43 (2008), pp. 542-562.

- T. Kotek, J.A. Makowsky and B. Zilber,
  On counting generalized colorings
  Contemporary Mathematics, vol. 558 (2011), pp. 207-242

The DPP conjectures for graph polynomials definable in SOL or MSOL II

Let P be anaSOL-definable graph polynomial in n indeterminates.

Assume that for some $\bar{a} \in \mathbb{C}^n$ evaluation of $P(-, \bar{a})$ is BSS-NP-hard over $\mathbb{C}$.

Weak DPP Conjecture: Then P has the WDPP.

Strong DPP Conjecture: Then P has the SDPP.

DPP for univariate graph polynomials, I

In

J.A. Makowsky and T. Kotek and E.V. Ravve,
A Computational Framework for the Study of
Partition Functions and Graph Polynomials,
Proceedings of the 12th Asian Logic Conference '11, (2013), pp. 210-230

we (frivolously) conjectured that,
for a **very large class** of graph polynomials,
some form of DPP would hold.

It turns out we were overreaching.

We took the **very large class** to be
all SOL-definable graph polynomials.

DPP for univariate graph polynomials, II

We now discuss this for **univariate** graph polynomials:

What we really had in mind was to

analyze the possible distribution

of the evaluation points of graph polynomials

which are **easy**,

i.e in $\mathbf{FP}$ or in $\mathbf{FP}_{\mathcal{R}}$ for $\mathcal{R} = \mathbb{R}$ or $\mathcal{R} = \mathbb{C}$

DPP for univariate graph polynomials, III

Let $P(G;X)$ be a univariate graph polynomial.
To analize the situation one may need the following:

(i) Find a point $a \in \mathbb{N}$ for which $P(-;a) \in \sharp\mathbf{P}$ and is $\sharp\mathbf{P}$-complete.

(ii) Find a way apply a generalization of **Linial's trick**.

When Can we prove the same for generalized univariate graph polynomials?

- Proper edge colorings and total (vertex and edge) colorings;
- Connected components colorings.
- Convex colorings.
- Complete colorings and harmonious colorings.
- $mcc(t)$-colorings.
- etc.

? ? ? ? ?

Proper edge colorings $\chi_{edge}(G; X)$, I

Surprisingly, the complexity of counting proper edge colorings was proven $\sharp\mathbf{P}$-hard only recently:

Theorem: (J. Y. Cai, H. Guo, T. Williams, 2014):

- $\sharp$-EdgeColoring is $\sharp\mathbf{P}$-hard over planar r-regular graphs for all $k \geq r \geq 3$.
- It is trivially tractable when $k \geq r \geq 3$ does not hold.

J. Y. Cai, H. Guo, T. Williams
The complexity of counting edge colorings and a dichotomy
for some higher domain Holant problems,
FOCS 2014 (full paper on arXiv http://arxiv.org/pdf/1404.4020.pdf, 75 pages)

Problem: Find an elementary proof of the complexity result.

Proper edge colorings $\chi_{edge}(G;X)$, II

Furthermore, we have

$$\chi_{edge}(G \bowtie K_2; X + |V(G)| + 1) = \chi_{edge}(G;X) \cdot (|V(G)| + 1)!$$

This gives us that **SDPP** holds.

The same holds for **Total (vertex and edge) colorings**.

Connected components

Here we look at colorings where neighboring vertices must have the same color:

$\chi_{connected}(G;k)$ is the number of these colorings with at most k colors.

- $\chi_{connected}(G;m) = m^{k(G}$ where $k(G)$ is the number of connected components of G.
- Clearly, $\chi_{connected}(G;X)$ is easily computable for all $X = a$ with $a \in \mathbb{R}$ or oany other field.

This gives us that **SDPP** holds in a trivial way
(as there are no difficult points).

Convex colorings

Joint work with A. Goodall and S. Noble

Recall: A convex (vertex) coloring with k colors is **convex** if every monochromatic set of vertices inudces a connected graph.

Theorem:

- The problem of counting the number of colorings of the vertices of a graph with at most two colours, such that the color classes induce connected subgraphs is $\sharp\mathbf{P}$-complete.

  A. Goodall and S. Noble, 2008 (`http://arxiv.org/pdf/1404.4020.pdf`)

- $\chi_{convex}(G \sqcup K_1; X+1) = X \cdot \chi_{convex}(G; X)$

- Computing $\chi_{convex}(G; 0)$ and $\chi_{convex}(G; 1)$ is easy.

This gives us that **SDPP** holds.

Note that, for $\chi_{connected}(G; k)$, where each color class is a connected component of G, we have $\chi_{connected}(G; k) = \binom{X}{k(G)}$, which is easy to compute.

Complete and harmonious colorings

Joint work with T. Kotek

Recall that a coloring is

- **complete** if **every pair of colors occurs along some edge**.
- **harmonious** if **every pair of colors occurs at most once along some edge**.
- $\chi_{complete}(G; k)$ is not a polynomial in k.

The exact complexity for fixed k seemingly is open......

Harmonious colorings, continued.

Proposition: For every $k \in \mathbb{N}$ $\chi_{harm}(-;k)$ is easy to compute for $k \in \mathbb{N}$, because there are only finitely many graphs without isolated vertices which admit a harmonious coloring with k-colors.

However, this is not uniform: For each k a different polynomial time Turing machine is used.

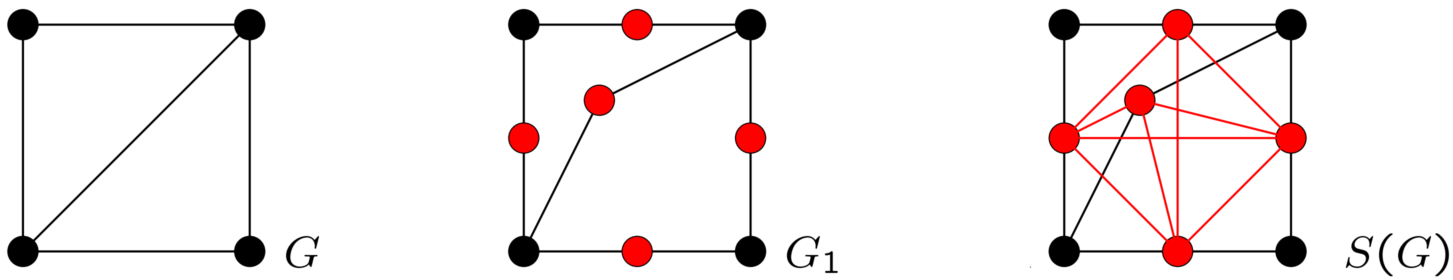
Theorem: For each $x \in \mathbb{C} - \mathbb{N}$ the evaluation of $\chi_{harm}(G;x)$ is $\sharp\mathbf{P}$-hard.

This gives us that **SDPP does not** hold
for **harmonious** colorings.

Skip proof

Harmonious colorings, proof, I

We show that for each $x \in \mathbb{C} - \mathbb{N}$ the evaluation of $\chi_{harm}(G; x)$ is $\sharp\mathbf{P}$-hard.



We add a **red** vertex on each edge of G (making two **black** edges out of it) and then add **red** edges such that the **red** vertices form a clique.

First we note that

$$\chi_{har}(S(G); k+e) = \chi(G; k) \cdot \binom{k+e}{e} e!$$

where $e = |E(G)|$ and $\chi(G; k)$ is the chromatic polynomial.

Harmonious colorings, proof, II

- Now for $k = a$ we have

$$\frac{\chi_{har}(S(G); a)}{\binom{a}{e} e!} = \chi(G; a - e)$$

- It remains to be shown that

$$\chi(G; a - e)$$

is is $\sharp\mathbf{P}$-hard for every $a \in \mathbb{C} - \mathbb{N}$.

- We use Linial's Trick:

Let $v = |V(G)|$ and $|E(G \bowtie K_1)| = e + v$:

$$\chi(G \bowtie K_1; a - (e + v) + 1) = (a - (e + v) + 1) \cdot \chi(G, a - (e + v))$$

Which can be used for every $a \in \mathbb{C} - \mathbb{N}$.

Harmonious colorings, analysis

- It was shown by T. Kotek and JAM (CSL-2012), that $\chi_{har}(G; k)$ is not MSOL-definable.
- There are infinitely many easy points.
- The easy points form a discrete subset of $\mathbb{C}$.
- The easy points are exactly $\mathbb{N}$.

How shall we formulate a new version of DPP?

For univariate generalized chromatic polynomials
the set of easy points is either
(a) $\mathbb{C}$, or (b) $\mathbb{N}$, or (c) a finite subset of $\mathbb{N}$.

$mcc(t)$-colorings

Joint work with Miki Hermann

Let $t \in \mathbb{N}$.
Recall that a coloring $f : V(G) \to [k]$ is an $mcc(t)$ **-coloring** with k colors, if the connected components of each color class have size at most t.
Let $\chi_{mcc(t)}(G; k)$ be the corresponding graph polynomial.

Theorem: Computing $\chi_{mcc(t)}(G; 2)$ is $\sharp\mathbf{P}$-hard.

Proof: Reduction to $\sharp$NAE3SAT.

$\sharp$NAE3SAT is $\sharp\mathbf{P}$-complete by

Creignou, Nadia, and Miki Hermann.
"Complexity of generalized satisfiability counting problems."
Information and Computation 125.1 (1996): 1-12.

We don't know how to use a version of Linial's Trick.

Open Problem: Is the set of easy points finite ?

Back to outline

More generalized chromatic polynomials

Let $f : V(G) \to [k]$ be a coloring of the vertices of $G = (V(G), E(G))$.

(i) f is t-improper if for every $i \in [k]$ the counter-image $[f^{-1}(i)]$ induces a graph of maximal degree t..

(ii) f is H-free if for every $i \in [k]$ the counter-image $[f^{-1}(i)]$ induces an H-free graph.

(iii) f is acyclic if for every $i, j \in [k]$ the union $[f^{-1}(i)] \cup [f^{-1}(i)]$ induces an acyclic graph.

By Kotek, JAM, Zilber (2008), for all the above properties, counting the number of colorings is a polynomial in k.

More graph polynomials with the DPP

T. Kotek and JAM (2011) have shown

SDPP: The graph polynomial for t-improper colorings (for multigraphs).

SDPP: The bivariate chromatic polynomial introduced by Döhmen, Pönitz and Tittman in 2003.

WDPP: The graph polynomial for acyclic colorings.

C. Hoffmann's PhD thesis (written under M. Bläser, 2010) contains a general sufficient criterion which allows to establish the WDPP for a wide class of (mostly non-prominent) graph polynomials.

A good test problem: H-free colorings, I

We look at the generalized chromatic polynomial $\chi_{H-free}(G;k)$, which, for $k \in \mathbb{N}$ counts the number of H-free colorings of G.

- For $H = K_2$, $\chi_{H-free}(G;k) = \chi(G;k)$, and we have the SDPP.
- For $H = K_3$, $\chi_{H-free}(G;k)$ counts the triangle free-colorings.

A good test problem: H-free colorings, II

- From [ABCM98] it follows that $\chi_{H-free}(G;k)$ is #P-hard for every $k \geq 3$ and H of size at least 2.

  D. Achlioptas, J. Brown, D. Corneil, and M. Molloy. The existence of uniquely -G colourable graphs. *Discrete Mathematics*, 179(1-3):1–11, 1998.

- In [Achlioptas97] it is shown that computing $\chi_{H-free}(G;2)$ is NP-hard for every H of size at most 2.

  D. Achlioptas. The complexity of G-free colourability. *DMATH: Discrete Mathematics*, 165, 1997.

- **Characterize H for which $\chi_{H-free}(G;k)$ satisfies the SDPP (WDPP).**

Back to outline

What did we learn ?

- There are more things in heaven and earth, Horatio,
  Than are dreamt of in your philosophy.
  - Hamlet (1.5.167-8), Hamlet to Horatio

- There are more graph polynomials in heaven and earth,
  George David Birkhoff,
  Than are dreamt of in your mathematics.

- **What we don't understand:**
  How are the difficulties of different evaluations related?

And the problems ?

- Study the complexity of generalized chromatic polynomials.
- Study the complexity of graph polynomials defined as generating functions

$$\sum_{A \subseteq V(G)} X^{|A|}$$

where $G[A] \in \mathcal{P}$
$G[A]$ is the induced subgraph generated by A in G, and $\mathcal{P}$ is any graph property.

- Find criteria on graph polynomials which imply versions of DPP

Thank you for your attention !