Power of LP relaxations for Valued CSPs

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Simons Institute for the Theory of Computing
All Fools' Day 2016

What this talk is not about

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$$\{+, *\} \ \rightarrow \ \{\min, +\}$$

Outline

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- linear programming for optimal solutions

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- constraint satisfaction problems

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- unconditional characterisations

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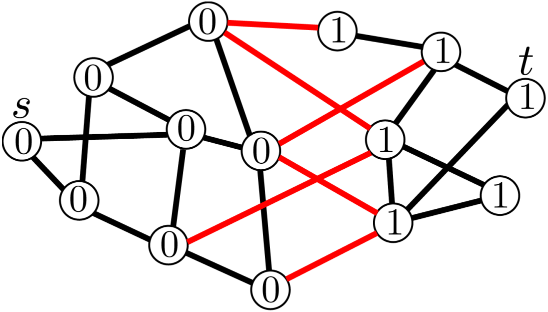
- constraint satisfaction problems
- unconditional characterisations
- complexity consequences

(s, t)-Min-Cut



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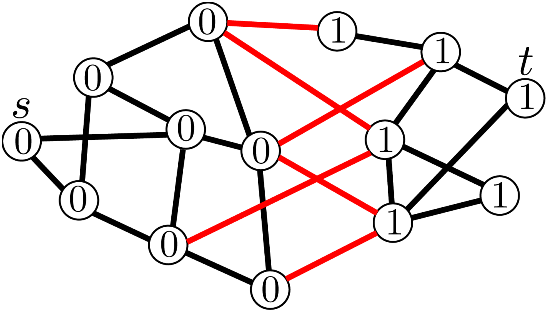

$$\min_{x_1 \in \{0,1\}, \ldots, x_n \in \{0,1\}} \left(\gamma_0(s) + \gamma_1(t) + \sum_{(i,j) \in E(G)} \phi(x_i, x_j) \right)$$

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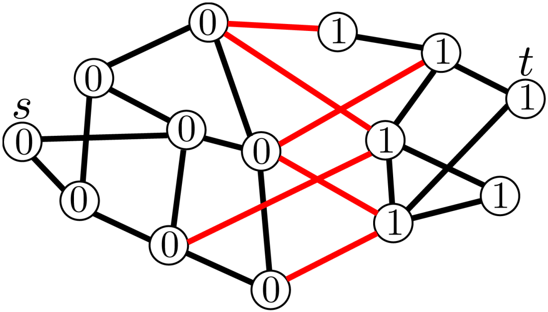

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x	$\gamma_d(x)$
d	0
$1-d$	∞

$\phi : \{0,1\}^2 \to \{0,1\}$

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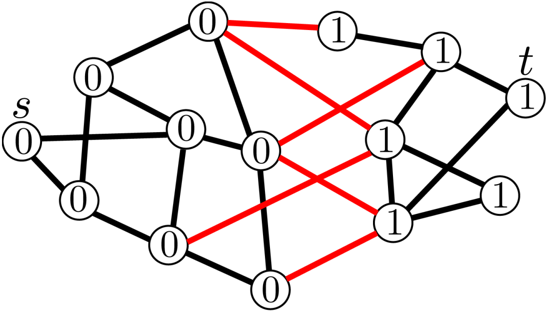

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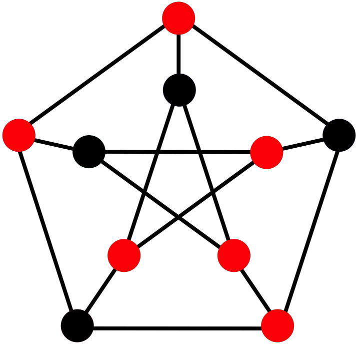
The natural LP relaxation solves it!

Vertex Cover



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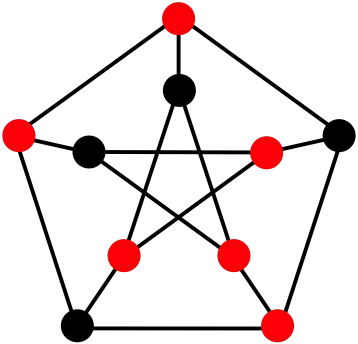

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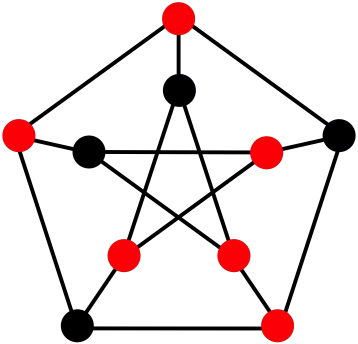

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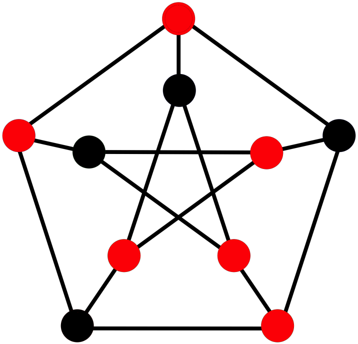

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The natural LP relaxation does not solve it!

Motivation

Why LP solves (s, t)-Min-Cut and not Vertex Cover?

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(apart from the obvious NP-completeness)

Valued Constraint Satisfaction Problem (VCSP)

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VCSP instance is given by $V = \{x_1, \ldots, x_n\}$, domain D, and

$$I(x_1, \ldots, x_n) \ = \ \phi_1(\mathbf{v}_1) + \ldots + \phi_q(\mathbf{v}_q)$$

where $\phi_i : D^{r_i} \to \overline{\mathbb{Q}}$ and $\mathbf{v}_i \subseteq V^{r_i}$. The goal is to find an assignment of labels from D to V minimising I.

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CSP	$\{0, \infty\}$
Min-CSP	$\{0, 1\}$
Weighted Min-CSP	$\{0, w_i\}$
Finite-Valued CSP	$\mathbb{Q}$
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Approximation CSP

- maximisation
- mostly $D = \{0, 1\}$
- mostly $\{0, 1\}$-valued
- "strict": $\{0, \infty\}$
- "generalized": $\mathbb{Q}$
- "mixed": $\{0, 1\}$ or $\{0, \infty\}$

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Which VCSPs are solved **exactly** by LP relaxations?

Basic LP Relaxation (BLP)

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$$\begin{aligned} \text{s.t.} \quad & \lambda_i(\sigma), \mu_j(a) \geq 0 && \forall i \in [q], j \in [n], \sigma : V_i \to D, a \in D \\ & \lambda_i(\sigma) = 0 && \forall i \in [q], \sigma : V_i \to D, \sigma \notin \mathtt{dom}\,\phi_i \\ & \sum_{a \in D} \mu_i(a) = 1 && \forall i \in [n] \end{aligned}$$

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BLP = SA(1,1)

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Let I be a VCSP instance and R_I its SA(k, ℓ) relaxation.

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SA(k, ℓ) works for I if $\texttt{Opt}(I) = \texttt{Opt}(R_I)$

Power of SA(k, ℓ) for VCSP(Γ)

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Which VCSPs are solved **exactly** by SA(k, ℓ)?

- VCSP(Γ) = VCSP instances with all functions from Γ, where (language) Γ is finite set of functions on fixed finite D
- Γ solved by SA(k, ℓ) if SA(k, ℓ) works for every $I \in$ VCSP(Γ)

Power of SA(k, ℓ) for VCSP(Γ)

Main Result

Let Γ be a (valued constraint) language on a fixed finite D.
Then Γ is solved by SA(k, ℓ) iff ...

Polymorphisms

m feasible solutions $\longrightarrow$ feasible solution

Polymorphisms

An m-ary operation $f : D^m \to D$ is a polymorphism of a function $\phi : D^r \to \overline{\mathbb{Q}}$ if $\mathtt{dom}\,\phi$ is closed under f:
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- projections (or dictators) are trivial polymorphisms of any ϕ
- any operation is a polymorphism of $\mathbb{Q}$-valued ϕ
- $\phi(x, y, z) = (\overline{x} \vee \overline{y} \vee z)$ has binary min as a polymorphism

Weighted Polymorphisms

probability distribution ω on m-ary polymorphisms with expected value of solution $\leq$ avg of m feasible solutions

Weighted Polymorphisms

A probability distribution ω on $\mathsf{Pol}^{(m)}(\phi)$ is a weighted polymorphism of ϕ if for all $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathtt{dom}\,\phi$:

$$\mathop{\mathbb{E}}_{f \sim \omega}\Big[\phi(f(\mathbf{x}_1, \ldots, \mathbf{x}_m))\Big] \;\leq\; \frac{1}{m}\Big[\phi(\mathbf{x}_1) + \cdots + \phi(\mathbf{x}_m)\Big]$$

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$\omega(\min) = \omega(\max) = \frac{1}{2}$

Power of BLP

- $\mathrm{supp}(\Gamma) = \{f \mid \omega(f) > 0 \text{ with } \omega \in \mathrm{wPol}(\Gamma)\}$

Power of BLP

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Theorem [*Thapper & Ž. FOCS'12*]

Let Γ be a valued constraint language. TFAE:

1. $\forall m \geq 2\, \exists m$-ary $f \in \mathrm{supp}(\Gamma)$ with f symmetric.
2. Γ is solved by BLP.

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$\forall \pi \in S_k : f(x_1, \ldots, x_m) = f(x_{\pi(1)}, \ldots, x_{\pi(m)})$

Semilattice Example

- $f : D^2 \to D$ is a semilattice operation if
  (i) $f(x, x) = x \quad \forall x \in D$
  (ii) $f(x, y) = f(y, x) \quad \forall x, y \in D$
  (iii) $f(x, f(y, z)) = f(f(x, y), z) \quad \forall x, y, z \in D$

  $f_m(x_1, \ldots, x_m) = f(x_1, f(x_2, \ldots, f(x_{m-1}, x_m) \ldots))$ symmetric

- $\exists f \in \mathrm{supp}(\Gamma)$ with f semilattice $\Rightarrow$ Γ solved by BLP

Submodularity and Friends

$\phi : \{0,1\}^r \to \mathbb{Q}$ is submodular if $\forall \mathbf{x}, \mathbf{y} \in \{0,1\}^r$:

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$\min, \wedge, \wedge_0, f$ are semilattice operations

Power of BLP

Theorem [*Thapper & Ž. FOCS'12*]

Let Γ be a valued constraint language. TFAE:

1. $\forall m \geq 2 \, \exists m$-ary $f \in \mathrm{supp}(\Gamma)$ with f symmetric.
2. Γ is solved by BLP.

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- implies tractability of generalisations of submodularity
- FPT algorithms [*Wahlström SODA'14*]

Does BLP solve all VCSP?

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No.

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No. Does it for some subclass of VCSP?

Finite-Valued CSP

Theorem [*Kolmogorov, Thapper, Ž. SICOMP'15*]

Let Γ be a $\mathbb{Q}$-valued constraint language. TFAE:

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Theorem [*Thapper & Ž. JACM'*16]

Let Γ be a $\mathbb{Q}$-valued constraint language on any finite domain. Then either Γ admits a binary symmetric wPol, or Γ is NP-hard.

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Γ can express binary ϕ with $\operatorname{argmin} \phi = \{(a, b), (b, a)\}$

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Let Γ be a $\mathbb{Q}$-valued constraint language on any finite domain.
Then either Γ admits a binary symmetric wPol, or Γ is NP-hard.

- {0,1}-valued functions on $|D| = 2$ [*Creignou JCSS'95*]
- {0,1}-valued functions on $|D| = 3$ [*Jonsson et al. SICOMP'06*]
- {0,1}-valued functions on $|D| = 4$ [*Jonsson et al. CP'11*]
- {0,1}-valued conservative functions [*Deineko et al. JACM'08*]
- functions on $|D| = 2$ [*Cohen et al. AIJ'06*]
- functions on $|D| = 3$ [*Huber et al. SICOMP'14*]
- conservative $\mathbb{Q}$-valued functions [*Kolmogorov & **Ž.** JACM'13*]
- min 0-extension problems [*Hirai SODA'13*]

Power of Sherali-Adams

- $\mathrm{supp}(\Gamma) = \{f \mid \omega(f) > 0 \text{ with } \omega \in \mathrm{wPol}(\Gamma)\}$

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Let Γ be a valued constraint language. TFAE:

1. $\forall m \geq 3\, \exists m$-ary $f \in \mathrm{supp}(\Gamma)$ with f weak near-unanimity.
2. Γ is solved by $\mathrm{SA}(k, \ell)$.
3. Γ is solved by $\mathrm{SA}(2, 3)$.

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$f(y, x, \ldots, x) = f(x, y, x, \ldots, x) = \ldots = f(x, \ldots, x, y)$

Examples of Previously Open Cases

- $\exists f \in \mathrm{supp}(\Gamma)$ with f majority $\Rightarrow \Gamma$ solved by SA(2, 3)

  proof: $f_m(x_1, \ldots, x_m) = f(x_1, x_2, x_3)$

  before: $\omega \in \mathrm{wPol}(\Gamma)$ where $\omega(Maj_1) = \omega(Maj_2) = \omega(Mn) = \frac{1}{3}$

- $\exists f \in \mathrm{supp}(\Gamma)$ with f tournament $\Rightarrow \Gamma$ solved by SA(2, 3)

  f tournament: $f(x, y) \in \{x, y\}$ and $f(x, y) = f(y, x)$

  proof: f 2-semilattice & WNU, generate f_m as for semilattice

  before: $\omega \in \mathrm{wPol}(\Gamma)$ where $\omega(f) = \omega(g) = \frac{1}{2}$

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VCSPs with an Injective Unary

Theorem [*Thapper & Ž.* '16+]

Let Γ be a language that can express a unary injective $\nu : D \to \mathbb{Q}$. Then either Γ is solved by SA$(2,3)$, or Γ is NP-hard.

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$\nu \in \text{VCSP}(\Gamma)$

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Γ can interpret something SA(2, 3) cannot solve

Corollary 1: Conservative VCSPs

- Γ conservative if Γ contains all $\{0, 1\}$-valued functions

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Corollary 1: Conservative VCSPs

- Γ conservative if Γ contains all $\{0,1\}$-valued functions

Theorem [*Thapper & **Ž**. '16+*]

Let Γ be a conservative language. Then either Γ is solved by SA(2, 3), or Γ is NP-hard.

- dichotomy known [*Kolmogorov & **Ž.** JACM'13*]
- simplifies both tractable and intractable parts
- new tractability criterion: majority in $\mathrm{supp}(\Gamma)$

Corollary 2: Minimum-Solution

- $\Gamma = \Delta \cup \{\nu\}$ Min-Sol if Δ relations on D and $\nu : D \to \mathbb{Q}$ injective

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Let Γ be a Min-Sol language on a finite domain D. Then either Γ is solved by SA(2, 3), or Γ is NP-hard.

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- Min-Sol (Min-Ones) on $|D| = 2$ [*Khanna et al. SICOMP'*01]
- Min-Sol on $|D| = 3$ [*Uppman ICALP'*13]
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- maximal and homogeneous Min-Sol [*Jonsson et al. SICOMP'*08]

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any Γ equivalent to $\Gamma' = \Delta' \cup \{\nu'\}$, where ν' is not necessarily injective

General Theme

- unconditional characterisations of power of LP relaxations
- universality of relaxations for classes of problems

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- universality of relaxations for classes of problems
- invariants preserved (by complexity and) by LP solvability