Counting Matrix Partitions of Graphs

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Joint work with
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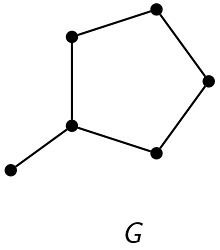
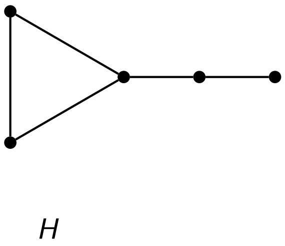
Graph homomorphisms

A homomorphism from G to H is a partition of $V(G)$ such that

- each part is labeled with a vertex of H;
- G has no edges between parts u and v if $uv \notin E(H)$.
- G has no edges within part u if $uu \notin E(H)$.





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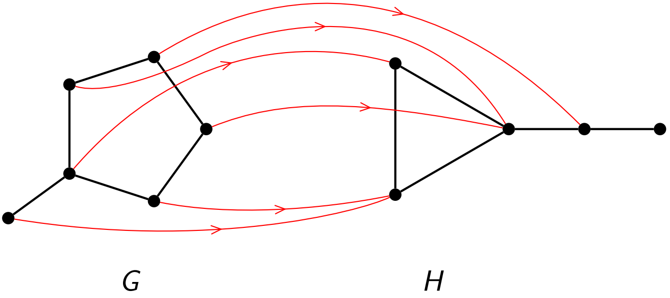

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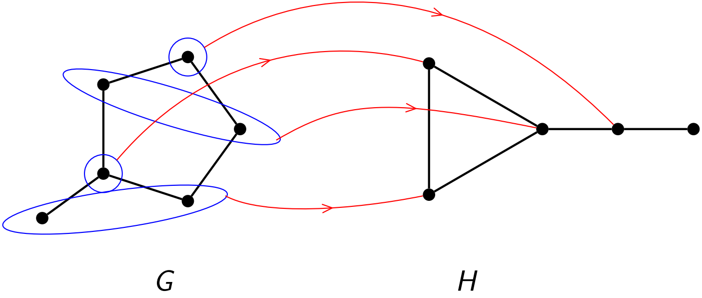

Complexity of counting homomorphisms

Theorem (Dyer–Greenhill)

If every component of H is a clique with a self-loop on every vertex, or is a complete bipartite graph, counting homomorphisms to H is in FP; otherwise, it is $\#\mathrm{P}$-complete.

Note: easy cases are near-trivial.

Homomorphisms as CSPs

For any target graph H, we can define a constraint language Γ_H:

- domain is $V(H)$;
- single binary relation $R = E(H)$.

A homomorphism $G \to H$ is a satisfying assignment to the instance of $\mathrm{CSP}(\Gamma_H)$ with:

- variables $V(G)$;
- constraints $\{\langle R, u, v\rangle \mid uv \in E(G)\}$.

Matrix partitions

Let M be a symmetric $k \times k$ matrix over $\{0, 1, *\}$.

An M-partition of G is a partition $V_1, \dots, V_k$ of $V(G)$ such that:

- G has no edges between V_i and V_j if $M_{i,j} = 0$;
- G has every possible edge between V_i and V_j if $M_{i,j} = 1$;
- if $M_{i,j} = *$, there is no constraint on the edges between V_i and V_j.

V_i is an independent set if $M_{i,i} = 0$ and a clique if $M_{i,i} = 1$.

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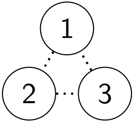
Homomorphism problems are M-partition problems where M is a $\{0, *\}$-matrix.

Examples

3-colouring



$\left(\quad \right)$

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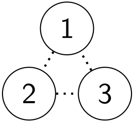

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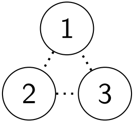

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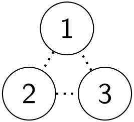

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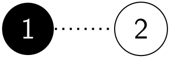

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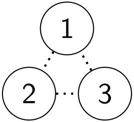

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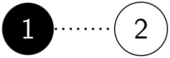

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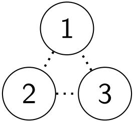

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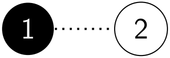

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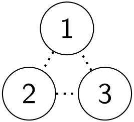

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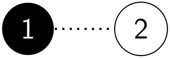

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Clique cutsets

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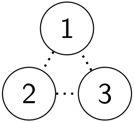

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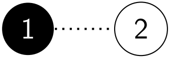

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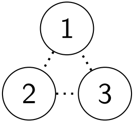

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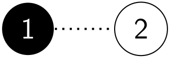

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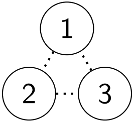

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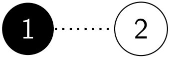

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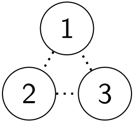

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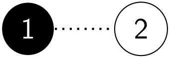

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An encoding as a CSP

For a $k \times k$ symmetric $\{0, 1, *\}$-matrix M, define Γ_M:

- domain $\{1, \ldots, k\}$;
- binary relation $R^+ = \{(i, j) \mid M_{i,j} \in \{1, *\}\}$;
- binary relation $R^- = \{(i, j) \mid M_{i,j} \in \{0, *\}\}$.

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- variables $V(G)$;
- constraints $\{\langle R^+, u, v\rangle \mid uv \in E(G)\} \cup \{\langle R^-, u, v\rangle \mid u \neq v \text{ and } uv \notin E(G)\}$.

#CSP dichotomy

Theorem (Bulatov, Dyer–Richerby)

For every constraint language Γ, $\#\mathrm{CSP}(\Gamma)$ *is either in* FP *or is* $\#\mathrm{P}$*-complete. Further, deciding which of the two cases holds is in* NP.

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This doesn't resolve the complexity of counting M-partitions because there are instances of $\mathrm{CSP}(\Gamma_M)$ that don't correspond to M-partitions problems.

Maybe these instances are the hard ones?

List M-partitions

Most work has been on the "list" version of the problem.

Instance: a graph G and a function $\lambda\colon V(G) \to \mathcal{P}(\{1, \ldots, k\})$.

Question: how many M-partitions σ of G, with $\sigma(v) \in \lambda(v)$ for all $v \in V(G)$?

(We say that such a partition σ *respects* the list function λ.)

Complexity of list M-partitions

Theorem (Feder, Hell, 2006)

For every fixed M, LIST-M-PARTITIONS *is either in Time* $\left[n^{O(\log n)}\right]$ *or is* NP-*complete.*

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Theorem (this talk)

For every symmetric $M \in \{0, 1, *\}^{k \times k}$, if M has a "derectangularizing sequence", then #LIST-M-PARTITIONS is $\#\mathrm{P}$-complete; otherwise, it is in FP.

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For the non-list version, we know what happens for small matrices: Feder, Hell, Klein, Motwani (decision); Hell, Hermann, Nevisi, and Dyer, Goldberg, Richerby (counting).

Parameterization

For any subset-closed $\mathcal{L} \subseteq \mathcal{P}(\{1, \ldots, k\})$, $\#\mathcal{L}$-M-PARTITIONS is "$\#$LIST-M-PARTITIONS but you're only allowed elements of $\mathcal{L}$ as lists."

Allows recursive definition of algorithms.

Take $\mathcal{L} = \mathcal{P}(\{1, \ldots, k\})$ to recover $\#$LIST-M-PARTITIONS.

From now on, write $D = \{1, \ldots, k\}$.

Purity

A matrix M is *pure* if it has no 0s or no 1s.

A set $\mathcal{L} \subseteq \mathcal{P}(D)$ is M-purifying if all submatrices $M|_{X \times Y}$ $(X, Y \in \mathcal{L})$ are pure.

$$M = \begin{pmatrix} * & * & 0 \\ * & 1 & * \\ 0 & * & * \end{pmatrix} \qquad \mathcal{L} = \{\{2\}, \{1, 3\}\}$$

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Derectangularizing sequences

For $X, Y \subseteq D$, $R_{X,Y} = \{(i,j) \in X \times Y \mid M_{i,j} = *\}$.

Definition

$D_1, \dots, D_\ell$ with each $D_i \subseteq D$ is a *derectangularizing sequence* for M if

- $\{D_1, \dots, D_\ell\}$ is M-purifying;
- $R_{D_1,D_2} \circ R_{D_2,D_3} \circ \cdots \circ R_{D_{\ell-1},D_\ell}$ is not rectangular.

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Dichotomy for purifying lists (I)

If $\mathcal{L}$ is subset-closed and M-purifying, then $\#\mathcal{L}$-M-PARTITIONS is polytime-equivalent to $\#\mathrm{CSP}(\Gamma)$, where

$$\Gamma = \mathcal{P}(D) \cup \left\{ R_{X,Y} \mid X, Y \in \mathcal{L} \right\}.$$

Theorem

For any symmetric matrix $M \in \{0, 1, *\}^{k \times k}$ and any subset-closed, M-purifying set $\mathcal{L}$, $\#\mathcal{L}$-M-PARTITIONS is $\#\mathrm{P}$-complete if $\mathcal{L}$ contains a derectangularizing sequence for M and in FP, otherwise.

Dichotomy for purifying lists (II)

A useful trick: let $\overline{M}$ be M with 0s and 1s switched and $\overline{G}$ be the complement of G.

There's a one-to-one correspondence between M partitions of G and $\overline{M}$-partitions of $\overline{G}$. So all pure matrices are essentially homomorphism problems.

Allows translation of $\#\mathcal{L}$-M-PARTITIONS instances to $\#\mathrm{CSP}(\Gamma_M)$ instances.

If there's no derectangularizing sequence, solve these using arc-consistency.

Dichotomy for purifying lists (III)

Existence of a derectangularizing sequence corresponds directly to $\#\mathrm{P}$-completeness of $\#\mathrm{CSP}(\Gamma_M)$.

By equivalence of $\#\mathrm{CSP}(\Gamma_M)$ and $\#\mathcal{L}$-M-PARTITIONS, the partitions problem is hard, too.

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But what if $\mathcal{L}$ isn't M-purifying?

Especially, what if $\mathcal{P}(D)$ isn't M-purifying – the most interesting case!

Purification

For each $\mathcal{L}$, M, we give a *purification algorithm*.

Input: a graph G and an assignment of lists $\lambda\colon V(G) \to \mathcal{L}$

Output: a sequence $\lambda_1, \ldots, \lambda_t\colon V(G) \to \mathcal{L}$

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3. every M-partition of G that respects λ respects exactly one of the λ_i.

Purification runs in polynomial time if there's no derectangularizing sequence $D_1, D_2 \subseteq \mathrm{img}\,(\lambda)$.

Dichotomy

Theorem

For any symmetric matrix $M \in \{0, 1, *\}^{k \times k}$ *and any subset-closed set of allowable lists* $\mathcal{L} \subseteq \mathcal{P}(\{1, \ldots, k\})$*, the problem* $\#\mathcal{L}$*-M-*PARTITIONS *is* $\#\mathrm{P}$*-complete if* $\mathcal{L}$ *contains a derectangularizing sequence for M and is in* FP*, otherwise.*

For $\#$LIST-M-PARTITIONS, take $\mathcal{L} = \mathcal{P}(\{1, \ldots, k\})$.