On the Power of Holographic Algorithms with Matchgates

Jin-Yi Cai
University of Wisconsin, Madison

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Joint work with Zhiguo Fu

Three Frameworks for Counting Problems

The following three frameworks are in increasing order of strength.

1. **Graph Homomorphisms**
2. **Constraint Satisfaction Problems (#CSP)**
3. **Holant Problems**

In each framework, there has been remarkable progress in the classification program.

Graph Homomorphism

L. Lovász: Operations with structures, Acta Math. Hung. 18 (1967), 321-328.

`http://www.cs.elte.hu/~lovasz/hom-paper.html`

Graphs and Homomorphisms

Pavol Hell and Jaroslav Nešetřil

Decision Dichotomy

Graph Homomorphisms

Given a graph $G = (V, E)$.

Consider all vertex assignments $\xi : V \to [q] = \{1, 2, \ldots, q\}$.

Suppose there is a binary constraint function $\mathbf{A} = (A_{i,j}) \in \mathbb{C}^{q \times q}$ assigned to each edge. For each $(u, v) \in E$, an assignment ξ gives an evaluation $\prod_{(u,v)\in E} A_{\xi(u),\xi(v)}$.

Then the partition function of Graph Homomorphism is

$$Z_{\mathbf{A}}(G) = \sum_{\xi:V\to[q]} \prod_{(u,v)\in E} A_{\xi(u),\xi(v)}.$$

Counting Problems Expressed as Graph Homomorphisms

INDEPENDENT SET

k-COLORING

VERTEX COVER

EVEN-ODD INDUCED SUBGRAPHS

$$\mathbf{H} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$Z_{\mathbf{H}}(G)$ computes the number of induced subgraphs of G with an even (or odd) number of edges.

$$\Big(2^n - Z_{\mathbf{H}}(G)\Big) \Big/ 2$$

is the number of induced subgraphs of G with an odd number of edges.

Dichotomy Theorem for Graph Homomorphism

After results by Dyer, Greenhill, Bulatov, Grohe, Goldberg, Jerrum, Thurley, . . .

Theorem 0.1 (C. Xi Chen and Pinyan Lu)**.** *There is a complexity dichotomy for* $Z_{\mathbf{A}}(\cdot)$*:*

For any symmetric complex valued matrix $\mathbf{A} \in \mathbb{C}^{q \times q}$*, the problem of computing* $Z_{\mathbf{A}}(G)$*, for any input* G*, is either in P or #P-hard.*

Given $\mathbf{A}$*, whether* $Z_{\mathbf{A}}(\cdot)$ *is in P or #P-hard can be decided in polynomial time in the size of* $\mathbf{A}$*.*

SIAM J. Comput. 42(3): 924-1029 (2013) (106 pages)

Counting CSP (#CSP)

- **Let $\mathcal{F} = \{f_1, \ldots, f_h\}$ be a finite set of constraint functions:**

$$f_i : [q]^{r_i} \to \mathbb{C}.$$

- **An instance of #CSP$(\mathcal{F})$ consists of variables $x_1, \ldots, x_n$ over $[q]$ and a finite sequence of constraint functions from $\mathcal{F}$, each applied to a sequence of these variables. It defines a new n-ary function F: for any assignment $\mathbf{x} = (x_1, \ldots, x_n) \in [q]^n$, $F(\mathbf{x})$ is the product of the constraint function evaluations.**

- **Given an input instance F, compute the partition function:**

$$\sum_{\mathbf{x} \in [q]^n} F(\mathbf{x})$$

This can be viewed in terms of a bipartite graph.

Dichotomy Theorem for #CSP

After results by Creignou, Hermann, Goldberg, Jerrum, Paterson, Bulatov, Dalmau, Dyer, Richerby, Jalsenius, C., Chen, Lu, Xia ....

Theorem 0.2 (C. and Chen)**.** *For any domain $[q]$ and any complex-valued constraint function set $\mathcal{F}$, #CSP($\mathcal{F}$) is either solvable in polynomial time (if $\mathcal{F}$ satisfies some tractability conditions), or else it is #P-hard (if $\mathcal{F}$ fails these conditions).*

It is not known whether it is decidable to classify a given constraint function set $\mathcal{F}$.

The strongest decidable dichotomy criterion is for non-negative valued $\mathcal{F}$.

Signatures of Affine Type

Definition 0.3. *A constraint function* $f(x_1, \ldots, x_n)$ *of arity* n *is of affine type if it has the form*

$$\lambda \cdot \chi_{AX=0} \cdot \mathfrak{i}^{Q(X)},$$

where $\lambda \in \mathbb{C}$, $X = (x_1, x_2, \ldots, x_n, 1)$, A *is a matrix over* $\mathbb{Z}_2$, $Q(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_4[x_1, x_2, \ldots, x_n]$ *is a quadratic (total degree at most 2) multilinear polynomial with the additional requirement that the coefficients of all cross terms are even, and* χ *is a 0-1 indicator function such that* $\chi_{AX=0}$ *is* 1 *iff* $AX = 0$. *We use* $\mathscr{A}$ *to denote the set of all affine signatures.*

Explicit List of Symmetric functions in $\mathscr{A}$

1. $[1, 0, \ldots, 0, \pm 1]$**;**

2. $[1, 0, \ldots, 0, \pm \mathfrak{i}]$**;**

3. $[1, 0, 1, 0, \ldots, 0$ **or** $1]$**;**

4. $[1, -\mathfrak{i}, 1, -\mathfrak{i}, \ldots, (-\mathfrak{i})$ **or** $1]$**;**

5. $[0, 1, 0, 1, \ldots, 0$ **or** $1]$**;**

6. $[1, \mathfrak{i}, 1, \mathfrak{i}, \ldots, \mathfrak{i}$ **or** $1]$**;**

7. $[1, 0, -1, 0, 1, 0, -1, 0, \ldots, 0$ **or** 1 **or** $(-1)]$**;**

8. $[1, 1, -1, -1, 1, 1, -1, -1, \ldots, 1$ **or** $(-1)]$**;**

9. $[0, 1, 0, -1, 0, 1, 0, -1, \ldots, 0$ **or** 1 **or** $(-1)]$**;**

10. $[1, -1, -1, 1, 1, -1, -1, 1, \ldots, 1$ **or** $(-1)]$**.**

Signatures of Product Type

Definition 0.4. *A constraint function on a set of variables X is of* product type *if it can be expressed as a product of unary functions $u(x_i)$, binary equality functions $(x_i = x_j)$, and binary disequality functions $(x_i \neq x_j)$, on variables of X. We use $\mathscr{P}$ to denote the set of product-type functions.*

A Concrete #CSP Dichotomy

Over the Boolean domain, for any set of complex-valued constraint functions $\mathscr{F}$ there is an explicit dichotomy criterion.

Theorem 0.5. *Suppose $\mathscr{F}$ is a set of functions mapping Boolean inputs to complex numbers. If $\mathscr{F} \subseteq \mathscr{A}$ or $\mathscr{F} \subseteq \mathscr{P}$, then #CSP($\mathscr{F}$) is computable in polynomial time. Otherwise, #CSP($\mathscr{F}$) is* #P*-hard.*

My main discussion today is what happens when we add Valiant's holographic algorithms.

A Concrete #CSP Dichotomy

Over the Boolean domain, for any set of complex-valued constraint functions $\mathscr{F}$ there is an explicit dichotomy criterion.

Theorem 0.6. *Suppose $\mathscr{F}$ is a set of functions mapping Boolean inputs to complex numbers. If $\mathscr{F} \subseteq \mathscr{A}$ or $\mathscr{F} \subseteq \mathscr{P}$, then #CSP($\mathscr{F}$) is computable in polynomial time. Otherwise, #CSP($\mathscr{F}$) is* #P*-hard.*

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Do-Nothing reductions to FKT.

Holant

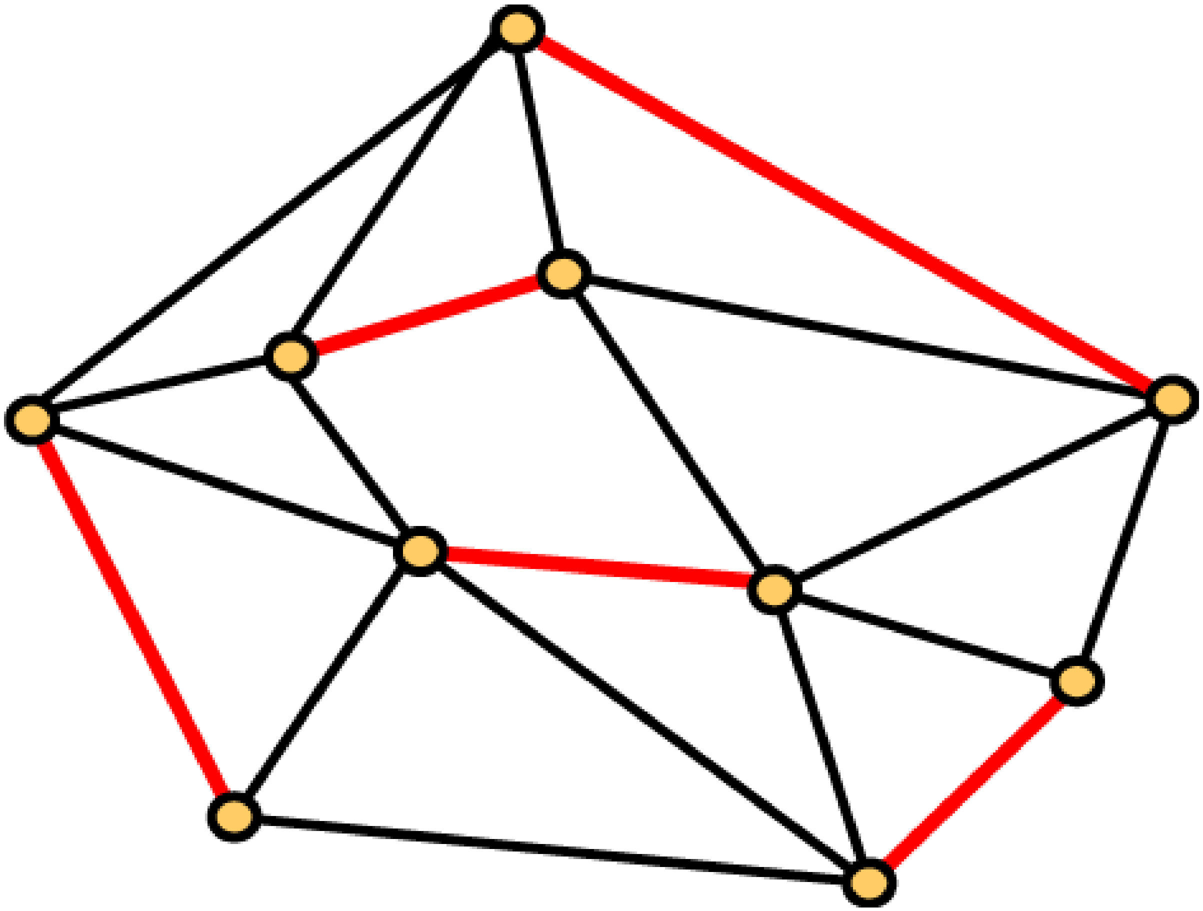
A signature grid $\Omega = (G, \mathcal{F}, \pi)$ **is a tuple, where** $G = (V, E)$ **is a graph,** π **labels each** $v \in V$ **with a function** $f_v \in \mathcal{F}$**, and** f_v **maps** $\{0,1\}^{\deg(v)}$ **to** $\mathbb{C}$**.**

$$\text{Holant}_\Omega = \sum_{\sigma: E \to \{0,1\}} \prod_{v \in V} f_v \left(\sigma \mid_{E(v)}\right).$$

where

- $E(v)$ **denotes the incident edges of** v
- $\sigma \mid_{E(v)}$ **denotes the restriction of** σ **to** $E(v)$**.**

Perfect Matchings



Matching as Holant

$$\text{Holant}_\Omega = \sum_{\sigma:E\to\{0,1\}} \prod_{v\in V} f_v\left(\sigma\mid_{E(v)}\right).$$

The problem of counting PERFECT MATCHINGS **on G corresponds to attaching the Exact-One function at every vertex of G.**

The problem of counting all MATCHINGS **on G is to attach the At-Most-One function at every vertex of G.**

FKT Algorithm

The Fisher-Kasteleyn-Temperley (FKT) algorithm is a classical gem that counts perfect matchings over planar graphs in P.

For almost 50 years, FKT stood as *the* P-time algorithm for any counting problem over planar graphs that is #P-hard over general graphs.

In 2001 Valiant introduced matchgates and holographic algorithms. This novel technique extended the reach of the FKT algorithm.

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Do-Nothing reductions to FKT.

Can Holographic Algorithms Solve NP-hard Problems?

... the situation with the $P = NP$ question is not dissimilar to that of other unresolved enumerative conjectures in mathematics. The possibility that accidental or freak objects in the enumeration exist cannot be discounted if the objects in the enumeration have not been studied systematically.

—**Leslie Valiant**

Indeed, if any freak object exists in this framework, it would collapse #P to P.

Getting Acquainted

Consider the constraint function

$$f : \{0,1\}^4 \to \mathbb{C},$$

where if the input (x_1, x_2, x_3, x_4) **has Hamming weight** w**, then** $f(x_1, x_2, x_3, x_4) = 3, 0, 1, 0, 3$**, if** $w = 0, 1, 2, 3, 4$**, resp.**

We denote this function by $f = [3, 0, 1, 0, 3]$**.**

What is the counting problem defined by the Holant sum?

$$\text{Holant}_\Omega = \sum_{\sigma : E \to \{0,1\}} \prod_{v \in V} f\left(\sigma\mid_{E(v)}\right).$$

What's that problem?

On 4-regular graphs, $\sum_\sigma \prod_{v \in V} f\left(\sigma \mid_{E(v)}\right)$ is a sum over all 0-1 edge assignments σ of products of local evaluations.

We only sum over assignments which assign an even number of 1's to the incident edges of each vertex, since

$$f = [3, 0, 1, 0, 3]$$

Thus $f = 0$ for $w = 1$ and 3.

Then each vertex contributes a factor 3 if the 4 incident edges are assigned all 0 or all 1, and contributes a factor 1 if exactly two incident edges are assigned 1.

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Before anyone thinks that this problem is artificial, let's consider a holographic transformation.

An Equivalent Bipartite Formulation

Let

$$I(G) = (E(G), V(G), \{(e,v) \mid v \textbf{ is incident to } e \textbf{ in } G\})$$

be the edge-vertex incidence graph of G**.**

Holant $(=_2 \mid f)$ **on** $I(G)$**:**

Each $e \in E(G)$ **is attached** $=_2$ **(binary** Equality**).**

The truth table of $=_2$ **is** $(1, 0, 0, 1)$ **indexed by** $\{0, 1\}^2$**.**

A Holographic Transformation

Apply

$$Z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ \mathfrak{i} & -\mathfrak{i} \end{bmatrix},$$

to

$$\text{Holant}\,(=_2 \mid f) = \text{Holant}\,\big((=_2)Z^{\otimes 2} \mid (Z^{-1})^{\otimes 4} f\big)$$

Here $(=_2)Z^{\otimes 2}$ **is a row vector indexed by** $\{0,1\}^2$ **denoting the transformed function under** Z **from** $(=_2) = (1,0,0,1)$**, and** $(Z^{-1})^{\otimes 4} f$ **is the column vector indexed by** $\{0,1\}^4$ **denoting the transformed function under** Z^{-1} **from** f**.**

A Holographic Transformation

$Z = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$ **transforms** $=_2$ **to the binary** DISEQUALITY**:**

$$\begin{aligned}
(=_2)Z^{\otimes 2} &= (1\ 0\ 0\ 1)Z^{\otimes 2} \\
&= \left\{(1\ 0)^{\otimes 2} + (0\ 1)^{\otimes 2}\right\} Z^{\otimes 2} \\
&= \tfrac{1}{2}\left\{(1\ 1)^{\otimes 2} + (i\ \ -i)^{\otimes 2}\right\} \\
&= (0\ 1\ 1\ 0) \\
&= [0, 1, 0] \\
&= (\neq_2).
\end{aligned}$$

A Holographic Transformation

Let

$$\hat{f} = [0, 0, 1, 0, 0]$$

be the Exact-Two **function on** $\{0,1\}^4$**.**

Consider $Z^{\otimes 4}\hat{f}$**, where**

$$Z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ \mathfrak{i} & -\mathfrak{i} \end{bmatrix},$$

$$Z^{\otimes 4} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

$$= \quad \frac{1}{4} \left\{ \begin{bmatrix} 1 \\ \mathfrak{i} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ \mathfrak{i} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -\mathfrak{i} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -\mathfrak{i} \end{bmatrix} + \begin{bmatrix} 1 \\ \mathfrak{i} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -\mathfrak{i} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ \mathfrak{i} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -\mathfrak{i} \end{bmatrix} + \cdots + \begin{bmatrix} 1 \\ -\mathfrak{i} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -\mathfrak{i} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ \mathfrak{i} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ \mathfrak{i} \end{bmatrix} \right\}$$

$$= \quad \frac{1}{2}\,[3, 0, 1, 0, 3] = \frac{1}{2}\, f$$

Hence $(Z^{-1})^{\otimes 4} f = 2\hat{f}$**.**

A Holographic Transformation

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$$= \quad \frac{1}{2} [3, 0, 1, 0, 3] = \frac{1}{2} f$$

Hence $(Z^{-1})^{\otimes 4} f = 2\hat{f}$**.**

What's Natural and What's Artificial?

$$\text{Holant}\,(=_2 \mid f) = \text{Holant}\,\big((=_2)Z^{\otimes 2} \mid (Z^{-1})^{\otimes 4} f\big) = \text{Holant}\,(\neq_2 \mid 2[0,0,1,0,0])$$

Hence, up to a global constant factor of 2^n on a graph with n vertices, the Holant problem with $[3,0,1,0,3]$ is exactly the same as $\text{Holant}\,(\neq_2 \mid [0,0,1,0,0])$.

A moment's reflection shows that $\text{Holant}\,(\neq_2 \mid [0,0,1,0,0])$ is counting the number of Eulerian orientations on 4-regular graphs, an eminently natural problem!

Our goal is to classify all of them.

Matchgate Signatures

A planar matchgate $\Gamma = (G, X)$ **is a weighted graph** $G = (V, E, W)$ **with a planar embedding, having external nodes, placed on the outer face.**

Define $\mathrm{PerfMatch}(G) = \sum_M \prod_{(i,j) \in M} w_{ij}$**, where the sum is over all perfect matchings** M**.**

A matchgate Γ **is assigned a Matchgate Signature**

$$G = (G^S),$$

where

$$G^S = \mathrm{PerfMatch}(G - S).$$

We denote the class of *matchgate signatures* by $\mathscr{M}$**.**

Main Theorem

Theorem 0.7 (C., Zhiguo Fu)**.** *For any set of constraint functions $\mathcal{F}$ over Boolean variables, each taking complex values and not necessarily symmetric, #CSP($\mathcal{F}$) belongs to exactly one of three categories according to $\mathcal{F}$: (1) It is P-time solvable; (2) It is P-time solvable over planar graphs but #P-hard over general graphs; (3) It is #P-hard over planar graphs. Moreover, category (2) consists precisely of those problems that are holographically reducible to the FKT algorithm.*

The tractability criterion for (2) is

$$\mathcal{F} \subseteq \mathscr{A}, \quad \text{or} \quad \mathcal{F} \subseteq \mathscr{P}, \quad \text{or} \quad \mathcal{F} \subseteq \widehat{\mathscr{M}}.$$

`http://arxiv.org/abs/1603.07046`

Proof is 94 pages.

Symmetric version over $\mathbb{R}$ by C., Pinyan Lu, Mingji Xia.

Symmetric version over $\mathbb{C}$ by Heng Huo, Tyson Williams.

The Universality Claim

The claim that holographic reductions followed by the FKT are universal (for all counting problems in #CSP on Boolean variables that are #P-hard in general but solvable in P over planar structure) is not self-evident.

In fact such a sweeping claim should invite skepticism.

Moreover, for Holant problems, the corresponding universality statement is false [C., Heng Guo, Tyson Williams, Zhiguo Fu, in FOCS 2015].

#CSP and Holant Problems

It turns out that the class of Holant problems is more than just a separate framework providing a cautionary reference to our Main Theorem.

They form the main arena we carry out the proof.

A basic idea is a holographic transformation between the #CSP setting and the Holant setting.

It is similar to the Fourier transform. Certain properties are easier to handle in one setting while others are easier after a transform. We will go back and forth.

#CSP and Holant Problems

Define the Hadamard transformation

$$H_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Define $\widehat{\mathcal{F}} = H_2 \mathcal{F}$**.**

$$\text{Pl-\#CSP}(\mathcal{F}) \equiv_T \text{Pl-Holant}(\widehat{\mathcal{EQ}}, \widehat{\mathcal{F}}), \tag{1}$$

Outline of the Proof

If $\mathcal{F} \subseteq \mathscr{A}$**, or** $\mathcal{F} \subseteq \mathscr{P}$**, or** $\mathcal{F} \subseteq \widehat{\mathscr{M}}$**, then** $\text{Pl-\#CSP}(\mathcal{F})$ **is computable in polynomial time.**

Otherwise, we want to show that $\text{Pl-\#CSP}(\mathcal{F})$ **is** $\#\text{P}$**-hard.**

In the $\text{Pl-Holant}(\widehat{\mathcal{EQ}}, \widehat{\mathcal{F}})$ **setting, the tractability condition is expressed as** $\widehat{\mathcal{F}} \subseteq \mathscr{A}$**, or** $\widehat{\mathcal{F}} \subseteq \widehat{\mathscr{P}}$**, or** $\widehat{\mathcal{F}} \subseteq \mathscr{M}$**.**

Outline of the Proof, continued.

$\mathscr{A}$ **is invariant under the transformation, i.e.,** $\widehat{\mathscr{A}} = \mathscr{A}$**.**

$\widehat{\mathscr{P}}$ **is more difficult to reason about than** $\mathscr{P}$**, while** $\mathscr{M}$ **is easier than** $\widehat{\mathscr{M}}$ **to handle.**

The former suggests that we carry our proof in the Pl-#CSP **framework, while the latter suggests the opposite, that we do so in the** Pl-Holant **framework instead.**

Outline of the Proof, continued.

One necessary condition for $\mathscr{M}$ is the Parity Condition. If some signature in $\widehat{\mathcal{F}}$ violates the Parity Condition, then we have eliminated one possibility $\widehat{\mathcal{F}} \subseteq \mathscr{M}$. In this case we prove in the Pl-#CSP **framework, and avoid discussing $\widehat{\mathscr{M}}$.**

If $\widehat{\mathcal{F}}$ satisfies the Parity Condition, then we have the lucky situation that $\mathcal{F} \cap \mathscr{P} \subseteq \mathscr{A}$. This is equivalent to $\widehat{\mathcal{F}} \cap \widehat{\mathscr{P}} \subseteq \mathscr{A}$, and therefore $\widehat{\mathcal{F}} \subseteq \widehat{\mathscr{P}}$ already implies $\widehat{\mathcal{F}} \subseteq \mathscr{A}$, so we do not need to specifically discuss the tractability condition $\widehat{\mathcal{F}} \subseteq \widehat{\mathscr{P}}$, avoidig the irksome class $\widehat{\mathscr{P}}$.

Outline of the Proof, continued.

The first main case is some $f \in \widehat{\mathcal{F}}$ fails the Parity Condition. We can construct a unary signature $[1, w]$ with $w \neq 0$ in the Holant framework $\text{Pl-Holant}(\widehat{\mathcal{EQ}}, \widehat{\mathcal{F}})$. Any signature that violates the Parity Condition is a witness that $\widehat{\mathcal{F}} \not\subseteq \mathscr{M}$, or equivalently $\mathcal{F} \not\subseteq \widehat{\mathscr{M}}$.

If $\mathcal{F} \subseteq \mathscr{A}$ or $\mathcal{F} \subseteq \mathscr{P}$, then the problem $\text{Pl-\#CSP}(\mathcal{F})$ is in P.

Otherwise, there exist some signatures $f, g \in \mathcal{F}$ such that $f \notin \mathscr{A}$ and $g \notin \mathscr{P}$.

We would like to construct some *symmetric* signatures from these that are also $\notin \mathscr{A}$ and $\notin \mathscr{P}$, respectively, and then apply the symmetric dichotomy theorem.

Outline of the Proof, continued.

The second main case is when all signatures in $\widehat{\mathcal{F}}$ satisfy the Parity Condition.

In this case, if $\widehat{\mathcal{F}} \subseteq \mathscr{A}$, or $\widehat{\mathcal{F}} \subseteq \widehat{\mathscr{P}}$, or $\widehat{\mathcal{F}} \subseteq \mathscr{M}$, then the problem is tractable in P.

Due to the Parity Condition, there are really only two kinds of containment here, $\widehat{\mathcal{F}} \subseteq \mathscr{A}$ or $\widehat{\mathcal{F}} \subseteq \mathscr{M}$; the containment $\widehat{\mathcal{F}} \subseteq \widehat{\mathscr{P}}$ is subsumed by $\widehat{\mathcal{F}} \subseteq \mathscr{A}$.

Therefore we want to prove that if $\widehat{\mathcal{F}} \not\subseteq \mathscr{A}$ and $\widehat{\mathcal{F}} \not\subseteq \mathscr{M}$, then $\text{Pl-Holant}(\widehat{\mathcal{EQ}}, \widehat{\mathcal{F}})$ is $\#\text{P}$-hard.

Outline of the Proof, continued.

A natural idea is to construct non-affine and non-matchgate *symmetric* signatures from any such asymmetric signatures, and then we can apply the known dichotomy for symmetric signatures. But this idea does not work.

For a given non-matchgate signature, we first construct a non-matchgate signature f of arity 4. Then we can construct either the crossover function $\mathfrak{X}$ or $(=_4)$ from f.

With $\mathfrak{X}$, we can finish the proof by the non-planar #CSP dichotomy Theorem.

The Crossover Function

The crossover function $\mathfrak{X}$ is a constraint function of arity 4 that satisfies $\mathfrak{X}_{0000} = \mathfrak{X}_{1111} = \mathfrak{X}_{0101} = \mathfrak{X}_{1010} = 1$ and $\mathfrak{X}_\alpha = 0$ for all other $\alpha \in \{0,1\}^4$.

The signature matrix of $\mathfrak{X}$ is

$$M_{x_1x_2,x_4x_3}(\mathfrak{X}) = \begin{bmatrix} \mathfrak{X}_{0000} & \mathfrak{X}_{0010} & \mathfrak{X}_{0001} & \mathfrak{X}_{0011} \\ \mathfrak{X}_{0100} & \mathfrak{X}_{0110} & \mathfrak{X}_{0101} & \mathfrak{X}_{0111} \\ \mathfrak{X}_{1000} & \mathfrak{X}_{1010} & \mathfrak{X}_{1001} & \mathfrak{X}_{1011} \\ \mathfrak{X}_{1100} & \mathfrak{X}_{1110} & \mathfrak{X}_{1101} & \mathfrak{X}_{1111} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Outline of the Proof, continued.

If we have $(=_4)$**, then we can get all** $\mathcal{EQ}_2$**. This implies that**

$$\text{Pl-Holant}(\widehat{\mathcal{EQ}}, \widehat{\mathcal{F}}) \equiv_{\mathrm{T}} \text{Pl-Holant}(\mathcal{EQ}_2, \widehat{\mathcal{EQ}}, \widehat{\mathcal{F}}) \equiv_{\mathrm{T}} \text{Pl-\#CSP}^2(\widehat{\mathcal{EQ}}, \widehat{\mathcal{F}}), \quad (2)$$

where $\text{Pl-\#CSP}^2$ **is a special kind of #CSP problems where every variable appears an even number of times.**

Outline of the Proof, continued.

Now comes a "cognitive dissonance". By (2), what used to be the "right-hand-side" in the equivalence $\text{Pl-\#CSP}(\mathcal{F}) \equiv_T \text{Pl-Holant}(\widehat{\mathcal{EQ}}, \widehat{\mathcal{F}})$ **will be treated as a** $\text{Pl-\#CSP}^2$ **problem with function set** $\widehat{\mathcal{EQ}} \cup \widehat{\mathcal{F}}$**.**

$$\text{Pl-\#CSP}(\mathcal{F}) \equiv_T \text{Pl-Holant}(\widehat{\mathcal{EQ}}, \widehat{\mathcal{F}}) \equiv_T \text{Pl-Holant}(\mathcal{EQ}_2, \widehat{\mathcal{EQ}}, \widehat{\mathcal{F}})$$

$$|||_T$$

$$\text{Pl-\#CSP}^2(\widehat{\mathcal{EQ}}, \widehat{\mathcal{F}})$$

Outline of the Proof, continued.

A Pl-#CSP^2 **problem is more in line with a** Pl-#CSP **problem. For** Pl-#CSP^2 **problems over symmetric signatures, a known dichotomy theorem says that there are five tractability classes** $\mathscr{P}, \mathscr{A}, \mathscr{A}^\dagger, \widehat{\mathscr{M}}$ **and** $\widehat{\mathscr{M}}^\dagger$**. But now we will apply these on the "dual side"** $\widehat{\mathcal{EQ}} \cup \widehat{\mathcal{F}}$**, instead of the "primal side"** $\mathcal{F}$**.**

... **a lot of difficulties are glossed over.**

A Taste of the Dichotomy

$$M_{x_1x_2,x_4x_3}(f)$$

$$= \begin{bmatrix} f_{0000} & f_{0010} & f_{0001} & f_{0011} \\ f_{0100} & f_{0110} & f_{0101} & f_{0111} \\ f_{1000} & f_{1010} & f_{1001} & f_{1011} \\ f_{1100} & f_{1110} & f_{1101} & f_{1111} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \in \mathscr{A} \cap \mathscr{M}$$

$$f = \chi_{[x_1+x_2+x_3+x_4=0]}(-1)^{x_1+x_2+x_2x_3}$$

A Taste of the Dichotomy

$$M_{x_1x_2,x_4x_3}(f) \quad = \quad \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \in \mathscr{A} \setminus \mathscr{M}$$

$$M_{x_1x_2,x_4x_3}(f) \quad = \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \in \mathscr{M} \setminus \mathscr{A}$$

A Taste of Proof Ideas

Lemma 0.8. *Suppose all signatures in $\widehat{\mathcal{F}}$ take values in $\{0, 1, -1\}$ and satisfy the Parity Condition. If $\exists f \in \widehat{\mathcal{F}} \setminus \mathscr{A}$ of arity $n \geq 3$, then $\exists g \notin \mathscr{A}$ of arity $< n$, such that*

$$\text{Pl-Holant}(\widehat{\mathcal{EQ}}, g, [1, 0, -1], \widehat{\mathcal{F}}) \leq_{\text{T}} \text{Pl-Holant}(\widehat{\mathcal{EQ}}, [1, 0, -1], \widehat{\mathcal{F}}).$$

Furthermore, if f satisfies the even Parity Condition, so does g.

A Taste of Proof Ideas, continued

By induction on n.

0. A simple proof shows that we may assume $f_{00\ldots0} = 1$ and f satisfies the even parity.

1. Use Tableau Calculus we show that either supp(f) is an affine subspace over $\mathbb{Z}_2$, or we are done.

So assume that supp(f) is affine and $\dim(\mathrm{supp}(f)) = k \le n$.

Let $Y = \{y_1, y_2, \ldots, y_k\} \subseteq \{x_1, x_2, \ldots, x_n\}$ be a set of free variables.

2. If $k \le 2$, prove directly.

So assume $k \ge 3$. On $\mathrm{supp}(f)$ we denote the "compressed signature" of f by $\underline{f}(y_1, y_2, \ldots, y_k)$.

I will sketch a proof when $\underline{f}$ has a special form.

A Taste of Proof Ideas, continued

Since $\underline{f}$ **takes values in** $\{1, -1\}$**,** $\exists$ **a unique multilinear polynomial** $P(y_1, y_2, \ldots, y_k) \in \mathbb{Z}_2[Y]$ **such that**

$$\underline{f}(y_1, y_2, \ldots, y_k) = (-1)^{P(y_1, y_2, \ldots, y_k)}.$$

$$f \in \mathscr{A} \iff \underline{f} \in \mathscr{A} \iff \deg(P) \leq 2.$$

3. Using $[1, 0] \in \widehat{\mathcal{E}\mathcal{Q}}$**, we can inductively reduce the proof to the case when**

$$P(y_1, y_2, \ldots, y_k) = Q(y_1, y_2, \ldots, y_k) + a y_1 y_2 \cdots y_k,$$

where $\deg(Q) \leq 2$ **and** $a = 0, 1$**. We want to show that** $a = 0$**.**

For a contradiction suppose $P = Q + y_1 y_2 \cdots y_k$**.**

A Taste of Proof Ideas, continued

Connecting one variable of $[1, 0, -1]$ **to** y_i **of** f**, we get** f' **such that**

$$f'(x_1, x_2, \ldots, x_n) = (-1)^{y_i} f(x_1, x_2, \ldots, x_n).$$

This implies that f' **has the same support of** f **and**

$$\underline{f'}(y_1, y_2, \ldots, y_k) = (-1)^{y_i + Q(y_1, y_2, \ldots, y_k) + y_1 y_2 \cdots y_k},$$

where $\underline{f'}$ **is the compressed signature of** f' **for** Y**.**

Thus $f' \notin \mathscr{A}$**. This implies that we can add a linear term to** $P(y_1, y_2, \ldots, y_k)$ **freely.**

A Taste of Proof Ideas, continued

Connect all variables of f except for y_1 to $n-1$ variables of

$$\frac{1}{2}\{[1,1]^{\otimes n}+[1,-1]^{\otimes n}\}=[1,0,1,\ldots,0\ (\text{or }1)]\in\widehat{\mathcal{EQ}}$$

to get the binary signature $f^*=[f^*_{00},0,f^*_{11}]$.

$$\begin{aligned} f^*_{00} &= \sum_{y_2,y_3,\ldots,y_k\in\{0,1\}} \underline{f}^{y_1=0}(y_2,y_3,\ldots,y_k), \\ f^*_{11} &= \sum_{y_2,y_3,\ldots,y_k\in\{0,1\}} \underline{f}^{y_1=1}(y_2,y_3,\ldots,y_k). \end{aligned} \tag{3}$$

A Taste of Proof Ideas, continued

We will sketch the proof for the special case where the coefficient of $y_i y_j$ in $Q(y_1, y_2, \ldots, y_k)$ is nonzero for all $1 \leq i < j \leq k$.

If $k = 3$, we may assume that

$$Q(y_1, y_2, y_3) = y_1 + y_2 + y_3 + y_1 y_2 + y_1 y_3 + y_2 y_3.$$

Then we have

$$M_{y_1, y_2 y_3}(\underline{f}) = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix}.$$

Thus $f^* = [-2, 0, -4] \notin \mathscr{A}$, since its nonzero terms have unequal norms.

A Taste of Proof Ideas, continued

For $k \geq 4$, we may assume that $Q(y_1, y_2, \ldots, y_k)$ has no linear terms since we can add linear terms freely.

We can calculate that

$$\begin{aligned}
\underline{f}^{y_1=0} &= [1, 1, -1, -1, \ldots, (-1)^{\frac{(k-1)(k-2)}{2}}] \\
&= \frac{1}{1+\mathfrak{i}} \left\{ [1, \mathfrak{i}]^{\otimes k-1} + \mathfrak{i}[1, -\mathfrak{i}]^{\otimes k-1} \right\} \in \mathscr{A}, \\
\underline{f}^{y_1=1} &= [1, -1, -1, 1, \ldots, (-1)^{\frac{k(k-1)}{2}}] + \textit{error term} \\
&= \frac{1}{1-\mathfrak{i}} \left\{ [1, \mathfrak{i}]^{\otimes k-1} - \mathfrak{i}[1, -\mathfrak{i}]^{\otimes k-1} \right\} - 2(-1)^{\frac{k(k-1)}{2}} [0, 1]^{\otimes k-1}.
\end{aligned}$$

A Taste of Proof Ideas, continued

$$
\begin{aligned}
f^*_{00} &= \sum_{\beta \in \{0,1\}^{k-1}} (\underline{f}^{y_1=0})_\beta \\
&= 2^{\frac{k}{2}} \cos((k-2)\pi/4), \\
f^*_{11} &= \sum_{\beta \in \{0,1\}^{k-1}} (\underline{f}^{y_1=1})_\beta \\
&= -2^{\frac{k}{2}} \sin((k-2)\pi/4) - 2(-1)^{\frac{k(k-1)}{2}}.
\end{aligned}
$$

A Taste of Proof Ideas, continued

For $k \equiv 1 \mod 2$**,** $|f^*_{00}| = 2^{\frac{k-1}{2}}$**, and** $|f^*_{11}| = 2^{\frac{k-1}{2}} \pm 2$ **(since** $k \geq 5$**), we have** $f^*_{11} f^*_{00} \neq 0$ **and** $|f^*_{11}| \neq |f^*_{00}|$**. Thus** $f^* \notin \mathscr{A}$**.**

For $k \equiv 2 \mod 4$**,** $|f^*_{00}| = 2^{\frac{k}{2}} \geq 4$ **since** $k \geq 4$**, and** $|f^*_{11}| = 2$**, so** $f^* \notin \mathscr{A}$**.**

For $k \equiv 0 \mod 4$**,** $f^*_{00} = 0$**,** $|f^*_{11}| = 2^{\frac{k}{2}} \pm 2 \neq 0$**. So** $f^* = f^*_{11}[0,1]^{\otimes 2}$**. By** $[1, 0, -1]$**,** $[0,1]^{\otimes 2}$ **and** $f \notin \mathscr{A}$**, we can get a binary signature that is not in** $\mathscr{A}$**.**

This completes the proof of the special case.

Open Problems

- **Tableau Calculus versus Clone Theory.**
- **Holant problems for asymmetric signatures.**
- **Planar Holant problems for asymmetric signatures.**
- **Higher Domain Problems.**

Some References

Holographic Algorithm with Matchgates Is Universal for Planar #CSP Over Boolean Domain

C., Zhiguo Fu

`http://arxiv.org/abs/1603.07046`

THANK YOU!